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# ON CONSERVATIVE AND COERCIVE SM-METHODS

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**Abstract.** We study the space $\mathcal{C}_e$ of double sequences $(x_{kl})$, satisfying $\lim_l \overline{\lim_k} |x_{kl} - a| = 0$ for some number $a$. In this note, using gliding hump arguments, we give necessary and sufficient conditions for a 3-dimensional matrix (i.e. SM-method) to transform every convergent or bounded sequence $(x_k)$ into the space $\mathcal{C}_e$ or $\mathcal{C}_{be}$, the space of elements in $\mathcal{C}_e$ with bounded columns.

**Key words:** summability, SM-methods, gliding hump method, theorems of Toeplitz–Silverman type.

## 1. INTRODUCTION AND PRELIMINARIES

The best known and well-studied convergence notion for double sequence spaces is Pringsheim convergence. A double sequence $(x_{kl})$ of complex (or real) numbers is said to *converge to the limit $a$ in the sense of Pringsheim* if

$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N} : \; k, l > N \;\Rightarrow | \, x_{kl} - a \, | < \varepsilon.$$

In case of this convergence the row-index $k$ and the column-index $l$ tend independently to infinity.

Boos et al. [1] considered a more general notion of convergence, where, in contrast to Pringsheim's notion of convergence, the row-index $k$ depends on the column-index $l$ in tending to infinity. The space of all double sequences converging in this way is denoted by $\mathcal{C}_e$. More precisely,

$$\begin{aligned}\mathcal{C}_e := & \Big\{ x \in \Omega \,|\, \exists a \in \mathbb{K} \, \forall \varepsilon > 0 \, \exists l_0 \in \mathbb{N} \, \forall l \geq l_0 \, \exists k_l \in \mathbb{N} : \\ & \qquad\qquad\qquad\qquad k \geq k_l \;\Rightarrow\; |x_{kl} - a| \leq \varepsilon \Big\} \\ = & \Big\{ x \in \Omega \,|\, \exists a \in \mathbb{K} : \; \lim_l \overline{\lim_k} |x_{kl} - a| = 0 \Big\},\end{aligned}$$

where $\Omega$ denotes the linear space of all complex (or real) double sequences and $\mathbb{K}$ is the field of all complex (or real) numbers. In more detail the paper [1] deals with the subspace

$$\mathcal{C}_{be} := \left\{x \in \mathcal{C}_e |\ \forall l \in \mathbb{N}:\ (x_{kl})_k \in \mathfrak{m}\right\}$$

of $\mathcal{C}_e$, where $\mathfrak{m}$ is the space of all bounded sequences. Note that in [1] the notation $\widehat{\mathcal{C}}$ was used instead of $\mathcal{C}_{be}$.

We refer the reader to [2,3] for the basic terminology and notation concerning the theory of locally convex spaces and sequence spaces.

We call linear subspaces of $\Omega$ *double sequence spaces*. Let $\mathcal{V}$ be a space of double sequences converging with respect to a linear notion of convergence $\mathcal{V}\text{-}\lim:\ \mathcal{V} \to \mathbb{K}$. The sum of a double series $\sum_{k,l} u_{kl}$ with respect to this notion of convergence will be defined by $\mathcal{V}\text{-}\sum_{k,l} u_{kl} := \mathcal{V}\text{-}\lim_{m,n} \sum_{k=1}^{m} \sum_{l=1}^{n} u_{kl}$. Generally $\mathcal{V}$ will be omitted when no confusion may arise.

Let $B = (b_{mnk})$ be a 3-dimensional matrix. The summability method $B$ induced by the summability domain

$$\mathcal{V}_B := \left\{ z \in \omega |\ Bz := \left( \sum_k b_{mnk} z_k \right)_{m,n} \text{ exists and } Bz \in \mathcal{V} \right\}$$

and the limit functional

$$\mathcal{V}\text{-}\lim_B:\ \mathcal{V}_B \to: \mathbb{K},\quad z \mapsto \mathcal{V}\text{-}\lim_{m,n} \sum_k b_{mnk} z_k$$

is called a *$\mathcal{V}$-SM-method* (cf. [1]). Following [1], a sequence of numbers $z = (z_k)$ is said to be *summable by a $\mathcal{V}$-SM-method $B$ to a number $s$* if the limit $\mathcal{V}\text{-}\lim_B z$ exists and is equal to $s$.

In [1] the consistency and the structure of summability domains of $\mathcal{C}_{be}$-SM-methods are examined. Our aim is to give necessary and sufficient conditions for a $\mathcal{C}_e$-SM- ($\mathcal{C}_{be}$-SM-) method $B = (b_{mnk})$ to be *conservative* (i.e. to sum every convergent sequence) or *coercive* (i.e. to sum every bounded sequence).

**Remark 1.1.** The summation in Volkov's sense (cf. [4]) can be considered as a special $\mathcal{C}_e$-SM-method. Given a matrix $A = (a_{nk})$, we put $b_{mnk} := a_{nk}$ for $k = 1, \ldots, m$ and $b_{mnk} := 0$ otherwise $(m, n \in \mathbb{N})$. Then the summability domain $\mathcal{C}_{eB}$ of the $\mathcal{C}_e$-SM-method $B = (b_{mnk})$ coincides with the domain $V_A$ of all sequences, summable by $A$ in Volkov's sense, and $\mathcal{C}_e\text{-}\lim_B x$ equals $V\text{-}\lim_A x$ for all $x \in \mathcal{C}_{eB}$.

## 2. CONSERVATIVE SM-METHODS

In [1], Theorem 2.4, it was proved that $\mathcal{C}_e$ is an LFH-space (i.e. it can be written as a union of countably many FH-spaces, [3]) with $H = \Omega$. More precisely, $\mathcal{C}_e = \bigcup_n \mathcal{C}_e^n$, where

$$\mathcal{C}_e^n := \left\{ x \in \Omega \middle| \sup_{l \geq n} \overline{\lim_k} |x_{kl}| < \infty \text{ and } \exists a \in \mathbb{K} : \left( \overline{\lim_k} |x_{kl} - a| \right)_{l \geq n} \in c_0 \right\}$$

is an FH-space with $H = \Omega$ $(n \in \mathbb{N})$. Note that $\mathcal{C}_e^1 = \mathcal{C}_{be}$. We will verify that for every conservative $\mathcal{C}_e$-SM-method $B$ there exists $N \in \mathbb{N}$ such that $B$ maps $c$ into $\mathcal{C}_e^N$. Here we will make use of the following result.

**Lemma 2.1** (cf. [3], Theorem 4.2.2). *Let $Y$ be an FH-space, $X$ an F-space, and $T : X \to Y$ a linear map. If $T : X \to H$ is continuous, then $T : X \to Y$ is continuous.*

**Lemma 2.2.** *Let $E$ be an FK-space and suppose that $F = \bigcup_n F_n$ is an LFH-space with $H = \Omega$ and $F_n \subset F_{n+1}$ $(n \in \mathbb{N})$. If a 3-dimensional matrix $B = (b_{mnk})$ maps $E$ into $F$, then there exists $N \in \mathbb{N}$ such that $B(E) \subset F_N$.*

*Proof.* By Lemma 2.1 the matrix map $B$ is continuous, hence (cf. [5], 19.5 (4)) there exists $N \in \mathbb{N}$ such that $B(E) \subset F_N$. ▼

**Theorem 2.3.** *A 3-dimensional matrix $B = (b_{mnk})$ maps $c$ into $\mathcal{C}_e$ if and only if each of the following conditions holds*:

(i) *for every $k \in \mathbb{N}$ the limit $b_k := \mathcal{C}_e\text{-}\lim_{m,n} b_{mnk}$ exists,*
(ii) $\sum_k |b_{mnk}| < \infty$ *for all $m, n \in \mathbb{N}$,*
(iii) *the limit $v := \mathcal{C}_e\text{-}\lim_{m,n} \sum_k b_{mnk}$ exists,*
(iv) *there exists $N \in \mathbb{N}$ such that $\sup_{m \in \mathbb{N}} \sum_k |b_{mnk}| < \infty$ for all $n \geq N$, and*
(v) *for every index sequence $(L_n)$ there exists $N \in \mathbb{N}$ such that*

$$M := \sup_{n \geq N} \overline{\lim_m} \sum_{k=1}^{L_n} |b_{mnk}| < \infty.$$

*Under these circumstances, $(b_k) \in \ell$ and*

$$\lim_B x = \sum_k b_k x_k + \Big( v - \sum_k b_k \Big) \lim_k x_k \quad (x \in c).$$

*Proof.*

**Necessity.** The Necessity of (i)–(iii) is evident.

(iv) By Lemma 2.2 there exists $N \in \mathbb{N}$ such that $B(c) \subset \mathcal{C}_e^N$. For every $m, n \in \mathbb{N}$ we consider the operator $B_{mn} : c \to \mathbb{R}$, $B_{mn} : x \mapsto [Bx]_{mn}$. Since the sequence of operators $(B_{mn})_m$ is pointwise bounded for every $n \geq N$, (iv) follows from the Uniform Boundedness Principle.

(v) Since $B$ is a continuous operator from $c$ into $\mathcal{C}_e^N$ (cf. Lemma 2.1), there exists $K \in \mathbb{N}$ such that

$$\sup_{n \geq N} \overline{\lim_m} \Big| \sum_{k=1}^{\infty} b_{mnk} x_k \Big| \leq K \parallel x \parallel_\infty \text{ for every } x \in c.$$

Let $(L_n)$ be an index sequence. By (iv)

$$\sup_m \sum_{k=1}^{L_n} |b_{mnk}| \leq \sup_m \sum_k |b_{mnk}| =: M_n < \infty \quad \text{for } n \geq N.$$

Let $(m_{in})$ be a double sequence satisfying

$$\lim_i \sum_{k=1}^{L_n} |b_{m_{in}nk}| = \overline{\lim_m} \sum_{k=1}^{L_n} |b_{mnk}| \quad (n \geq N).$$

Passing to a subsequence of $(m_{in})_i$ if necessary $(n \in \mathbb{N})$, we may suppose that

$$\operatorname{sgn}\Re(b_{m_{i_1n}nk}) = \operatorname{sgn}\Re(b_{m_{i_2n}nk}) \text{ for } k = 1, \ldots, L_n;\ i_1, i_2 \in \mathbb{N},\ n \geq N.$$

For every fixed $n \geq N$ we put $y_k := \operatorname{sgn}\Re(b_{m_{1n}nk})$ for $1 \leq k \leq L_n$ and $y_k := 0$ otherwise. Then $\| y \|_\infty \leq 1$ and

$$\overline{\lim_m} \sum_{k=1}^{L_n} |\Re(b_{mnk})| = \lim_i \Big| \Re\Big(\sum_k b_{m_{in}nk} y_k\Big)\Big| \leq K.$$

Analogously, $\overline{\lim}_m \sum_{k=1}^{L_n} |\Im(b_{mnk})| \leq K$. So $\sup_{n \geq N} \overline{\lim}_m \sum_{k=1}^{L_n} |b_{mnk}| \leq 2K$.

**Sufficiency.** Note that (i) and (v) imply $\underline{(b_k)} \in \ell$. Really, by (i) for a fixed $s \in \mathbb{N}$ we may find $n \geq \max\{N, s\}$ such that $\overline{\lim}_m \sum_{k=1}^{s} |b_{mnk} - b_k| \leq 1$. Hence by (v) we get $\sum_{k=1}^{s} |b_k| \leq 1 + \sup_{i \geq N} \overline{\lim}_j \sum_{k=1}^{i} |b_{ijk}| < \infty$.

It is sufficient to verify that $B$ maps $c_0$ into $\mathcal{C}_e$, since in this case by (iii) the limit

$$\lim_B x = \lim_i x_i \cdot \mathcal{C}_e\text{-}\lim_{m,n} \sum_k b_{mnk} + \mathcal{C}_e\text{-}\lim_{m,n} \sum_k b_{mnk}(x_k - \lim_i x_i)$$

exists for every $x \in c$.

So let $x \in c_0$ and $\varepsilon > 0$ be arbitrarily fixed. By (iv) we may find $N_1 \in \mathbb{N}$ such that $M_n := \sup_{m \in \mathbb{N}} \sum_k |b_{mnk}| < \infty$ $(n \geq N_1)$. Now we choose an index sequence $(L_n)$ such that $|x_k| \leq \varepsilon/(4M_n)$ for $k \geq L_n$. By (v) there exist $N_2 > N_1$, $M > 0$ and an index sequence $(m_n)$ such that $\sum_{k=1}^{L_n} |b_{mnk}| \leq M$ for all $n \geq N_2$, $m \geq m_n$. Select $K \in \mathbb{N}$ with $\sum_{k=K}^{\infty} |b_k| \leq 1$ and $|x_k| \leq \varepsilon/(4M)$ for $k \geq K$. By (i) we may find $N_3 > N_2$ and an index sequence $(m'_n)$ with $m'_n > m_n$ $(n \geq N_3)$ such that $\sum_{k=1}^{K} |b_{mnk} - b_k||x_k| \leq \varepsilon/4$ for all $n \geq N_3$ and $m \geq m'_n$. Now for every $n \geq N_3$ and $m \geq m'_n$ we get

$$\Big| \sum_k b_{mnk} x_k - \sum_k b_k x_k \Big|$$
$$\leq \sum_{k=1}^{K} |b_{mnk} - b_k||x_k| + \frac{\varepsilon}{4M} \sum_{k=K}^{L_n} |b_{mnk}| + \frac{\varepsilon}{4} \sum_{k=K}^{\infty} |b_k| + \frac{\varepsilon}{4M_n} \sum_{k=L_n}^{\infty} |b_{mnk}| \leq \varepsilon.$$

Hence $\lim_B x = \sum_k b_k x_k$. ▼

Note that condition (ii) is independent of all others. The matrix $B = (b_{mnk})$ with $b_{11k} := (-1)^k/k$ and $b_{mnk} := 0$ $(m, n, k \in \mathbb{N};\ (m, n) \neq (1, 1))$ satisfies all the hypotheses of Theorem 2.3 except (ii). At the same time it is possible to find $x \in c$ such that the series $\sum_k b_{11k} x_k$ diverges.

**Theorem 2.4.** *A* 3*-dimensional matrix* $B = (b_{mnk})$ *maps* $c$ *into* $\mathcal{C}_{be}$ *if and only if each of the following conditions holds*:

(i) *for every* $k \in \mathbb{N}$ *the limit* $b_k := \mathcal{C}_{be}\text{-}\lim_{m,n} b_{mnk}$ *exists,*

(ii) $\sup_{m\in\mathbb{N}} \sum_k |b_{mnk}| < \infty$ *for all* $n \in \mathbb{N}$,

(iii) *the limit* $v := \mathcal{C}_{be}\text{-}\lim_{m,n} \sum_k b_{mnk}$ *exists, and*

(iv) $\sup_n \overline{\lim}_m \sum_{k=1}^{L_n} |b_{mnk}| < \infty$ *for every index sequence* $(L_n)$.

*Under these circumstances,* $(b_k) \in \ell$ *and*

$$\lim_B x = \sum_k b_k x_k + \Big(v - \sum_k b_k\Big) \lim_k x_k \quad (x \in c).$$

*Proof.*

**Necessity.** (i) and (iii) are evident; (ii) and (iv) follow from Theorem 2.3, since for every fixed $n \in \mathbb{N}$ the matrix $(b_{mnk})_{m,k}$ maps $c$ into $\mathfrak{m}$.

**Sufficiency.** By Theorem 2.3 the limit $\mathcal{C}_e\text{-}\lim_{m,n}[Bx]_{mn}$ exists for any fixed $x \in c$. Now (iv) implies that $([Bx]_{mn})_m \in \mathfrak{m}$ for every $n \in \mathbb{N}$. Hence for every $x \in c$ the limit $\mathcal{C}_{be}\text{-}\lim_{m,n}[Bx]_{mn}$ exists. ▼

## 3. COERCIVE SM-METHODS

**Theorem 3.1.** *A* 3*-dimensional matrix* $B = (b_{mnk})$ *maps* $\mathfrak{m}$ *into* $\mathcal{C}_e$ *if and only if each of the following conditions holds*:

(i) *for every* $k \in \mathbb{N}$ *the limit* $b_k := \mathcal{C}_e\text{-}\lim_{m,n} b_{mnk}$ *exists,*

(ii) $\sum_k |b_{mnk}| < \infty$ *for all* $m, n \in \mathbb{N}$,

(iii) *there exists* $N \in \mathbb{N}$ *such that* $\sup_{n\geq N} \overline{\lim}_m \sum_k |b_{mnk}| < \infty$*, and*

(iv) $\lim_n \overline{\lim}_m \sum_k |b_{mnk} - b_k| = 0$.

*Under these circumstances,* $(b_k) \in \ell$ *and*

$$\lim_B x = \sum_k b_k x_k \quad (x \in \mathfrak{m}).$$

In proving this proposition we make use of two nonsummability lemmas involving gliding hump arguments.

Let $\varphi:\ \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection defined inductively by

$$\varphi[(1,1)] = 1, \quad \varphi[(1,2)] = 2, \quad \varphi[(2,1)] = 3;$$
$$\varphi[(1,n)] = \frac{(n-1)n}{2} + 1, \quad \varphi[(2,n-1)] = \frac{(n-1)n}{2} + 2,\ \ldots,$$
$$\varphi[(n,1)] = \frac{n(n+1)}{2}.$$

Let $\pi_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $(a,b) \to a$ and $\pi_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $(a,b) \to b$ be the projection maps. We put $\lambda_i := \pi_i \varphi^{-1}$ $(i = 1,2)$.

We say that a double sequence $(m_{ij})$ in $\mathbb{N}$ is *increasing* if $m_{i,j+1} > m_{ij}$ $(i,j \in \mathbb{N})$.

In proving the lemmas mentioned above we will use the following

**Remark 3.2.** Let a 3-dimensional matrix $B = (b_{mnk})$ and $x \in \omega$ be fixed. If there exists an index sequence $(n_i)$ and an increasing double sequence $(m_{ij})$ in $\mathbb{N}$ such that $x \notin \mathcal{C}_{eD}$, where $D := (b_{m_{ij}n_ik})_{i,j,k}$, then $x \notin \mathcal{C}_{eB}$.

**Lemma 3.3.** *Let $B = (b_{mnk})$ be a 3-dimensional matrix such that*

$$\sup_m \sum_k |b_{mnk}| < \infty \ (n \in \mathbb{N}) \quad \textit{and} \quad \lim_n \lim_s \overline{\lim_m} \sum_{k=s}^{\infty} |b_{mnk}| \neq 0.$$

*Then there exists an $x \in \mathfrak{m} \backslash \mathcal{C}_{eB}$.*

*Proof.* Without loss of generality we may suppose that there exists an index sequence $(n_r)$ such that

$$\lim_s \overline{\lim_m} \sum_{k=s}^{\infty} |\Re(b_{mn_rk})| > 5\gamma \quad (r \in \mathbb{N})$$

for some suitably chosen $\gamma > 0$.

Setting $s_{r1} := 0$ $(r \in \mathbb{N})$, we choose inductively increasing double sequences $(\mu_{rj})$ and $(s_{rj})$ of indexes such that

$$\sum_{k=s_{rj}+1}^{\infty} |\Re(b_{\mu_{rj}n_rk})| > 4\gamma, \qquad \sum_{k=s_{r,j+1}+1}^{\infty} |b_{\mu_{rj}n_rk}| < \gamma \quad (r,j \in \mathbb{N}).$$

So

$$\sum_{k=s_{rj}+1}^{s_{r,j+1}} |\Re(b_{\mu_{rj}n_rk})| > 3\gamma \quad (r,j \in \mathbb{N}).$$

Setting $t_1 := s_{11}$ and putting $t_r := s_{\lambda_1(r)j_r}$, $m_{\lambda_1(r)\lambda_2(r)} := \mu_{\lambda_1(r)j_r}$ for $r > 1$, where $j_r \in \mathbb{N}$ is chosen such that $s_{\lambda_1(r)j_r} > s_{\lambda_1(r-1),j_{r-1}+1}$, we obtain an index sequence $(t_i)$ and an increasing double sequence $(m_{ij})$ such that $(m_{ij})_j$ is a subsequence of $(\mu_{ij})_j$, $\sum_{k=t_i+1}^{t_{i+1}} |b_{m_{\lambda_1(i)\lambda_2(i)}n_{\lambda_1(i)}k}| > 3\gamma$ and $\sum_{k=t_{i+1}+1}^{\infty} |b_{m_{\lambda_1(i)\lambda_2(i)}n_{\lambda_1(i)}k}| < \gamma$ $(i \in \mathbb{N})$.

Fixing $x_k := 0$ for $k \leq t_1$, for $k = t_i + 1, \ldots, t_{i+1}$ we put

$$x_k := \begin{cases} \operatorname{sgn} \Re(b_{m_{\varphi^{-1}(i)} n_{\lambda_1(i)} k}) & \text{if } \lambda_2(i) = 1 \\ & \text{or } \sum_{l=1}^{t_i} \Re(b_{m_{\lambda_1(i),\lambda_2(i)-1} n_{\lambda_1(i)} l} x_l) < \sum_{l=1}^{t_i} \Re(b_{m_{\lambda_1(i)\lambda_2(i)} n_{\lambda_1(i)} l} x_l), \\ -\operatorname{sgn} \Re(b_{m_{\varphi^{-1}(i)} n_{\lambda_1(i)} k}) & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} &\Big| \sum_{k=1}^{t_i} \Re(b_{m_{\lambda_1(i),\lambda_2(i)-1} n_{\lambda_1(i)} k} x_k) - \sum_{k=1}^{t_{i+1}} \Re(b_{m_{\lambda_1(i)\lambda_2(i)} n_{\lambda_1(i)} k} x_k) \Big| \\ &\qquad \geq \sum_{k=t_i+1}^{t_{i+1}} |\Re(b_{m_{\lambda_1(i)\lambda_2(i)} n_{\lambda_1(i)} k})| \; > \; 3\gamma \quad (i \in \mathbb{N} : \; \lambda_2(i) > 1). \end{aligned}$$

Hence

$$\begin{aligned} &\Re\Big( [Bx]_{m_{\lambda_1(i),\lambda_2(i)-1} n_{\lambda_1(i)}} - [Bx]_{m_{\lambda_1(i)\lambda_2(i)} n_{\lambda_1(i)}} \Big) \\ &\quad = \; \Big| \Re\Big( \sum_k b_{m_{\lambda_1(i),\lambda_2(i)-1} n_{\lambda_1(i)} k} x_k \Big) - \Re\Big( \sum_k b_{m_{\lambda_1(i)\lambda_2(i)} n_{\lambda_1(i)} k} x_k \Big) \Big| \\ &\quad \geq \; \sum_{k=t_i+1}^{t_{i+1}} |\Re(b_{m_{\lambda_1(i)\lambda_2(i)} n_{\lambda_1(i)} k})| - \sum_{k=t_i+1}^{\infty} |b_{m_{\lambda_1(i),\lambda_2(i)-1} n_{\lambda_1(i)} k}| \\ &\qquad - \sum_{k=t_{i+1}+1}^{\infty} |b_{m_{\lambda_1(i)\lambda_2(i)} n_{\lambda_1(i)} k}| \\ &\quad \geq \; 3\gamma - \gamma - \gamma \; = \; \gamma \end{aligned}$$

for every $i \in \mathbb{N}$ with $\lambda_2(i) > 1$. Therefore, by Remark 3.2, $x \notin \mathcal{C}_{eB}$. ▼

**Lemma 3.4.** *Let $B = (b_{mnk})$ be a 3-dimensional matrix such that*

$$\mathcal{C}_e\text{-}\lim_{m,n} b_{mnk} = 0 \; (k \in \mathbb{N}) \quad \textit{and} \quad \lim_n \lim_s \overline{\lim_m} \sum_{k=s}^{\infty} |b_{mnk}| = 0.$$

*If $\lim_n \overline{\lim}_m \sum_k |b_{mnk}| \neq 0$, then there exists an $x \in \mathfrak{m} \backslash \mathcal{C}_{eB}$.*

*Proof.* Without loss of generality we may assume that there exist a $\gamma > 0$ and an index sequence $(n^{(i)})$ such that

$$\overline{\lim_m} \sum_k |\Re(b_{mn^{(i)}k})| > 5\gamma \quad (i \in \mathbb{N}).$$

Fixing $k_1 := 1$, we construct inductively two index sequences $(k_i)$ and $(n_i)$ choosing the second sequence as a subsequence of $(n^{(i)})$.

Suppose that $k_1, \ldots, k_r$ and $n_1, \ldots, n_{r-1}$ are fixed. Then we may choose $n_r$ from $(n^{(j)})$ such that $n_r > n_{r-1}$ and

$$\overline{\lim_m} \sum_{k=1}^{k_r} |b_{mn_rk}| < \gamma \quad \text{and} \quad \lim_s \overline{\lim_m} \sum_{k=s}^{\infty} |b_{mn_rk}| < \gamma.$$

Now we take $k_{r+1}$ with $k_{r+1} > k_r$ such that $\overline{\lim}_m \sum_{k=k_{r+1}+1}^{\infty} |b_{mn_rk}| < \gamma$. Hence

$$\overline{\lim_m} \sum_{k=k_r+1}^{k_{r+1}} |\Re(b_{mn_rk})| > 3\gamma \quad (r \in \mathbb{N}).$$

Then we find an increasing double sequence $(m_{rj})$ such that

$$\mathrm{sgn}\Re(b_{m_{ri}n_rk}) = \mathrm{sgn}\Re(b_{m_{rj}n_rk}) \ \text{ for } \ k = k_r+1, \ldots, k_{r+1},$$

$$\sum_{k=k_{r+1}+1}^{\infty} |b_{m_{rj}n_rk}| < \gamma, \quad \sum_{k=1}^{k_r} |b_{m_{rj}n_rk}| < \gamma, \quad \sum_{k=k_r+1}^{k_{r+1}} |\Re(b_{m_{rj}n_rk})| > 3\gamma$$

for all $r, i, j \in \mathbb{N}$ . We put $x_k := 0$ for $k \leq k_1$ and $x_k := (-1)^r \mathrm{sgn}\Re(b_{m_{rj}n_rk})$ for $k_r < k \leq k_{r+1}$ $(r \in \mathbb{N})$. Then $x \in \mathfrak{m}$ and for all $r, i, j \in \mathbb{N}$ we get

$$\begin{aligned}
&\Re\big([Bx]_{m_{rj}n_r} - [Bx]_{m_{r+1,i}n_{r+1}}\big) \\
&\quad = \Big|\sum_k \Re(b_{m_{rj}n_rk}x_k) - \sum_k \Re(b_{m_{r+1,i}n_{r+1}k}x_k)\Big| \\
&\quad \geq \sum_{k=k_r+1}^{k_{r+1}} |\Re(b_{m_{rj}n_rk})| + \sum_{k=k_{r+1}+1}^{k_{r+2}} |\Re(b_{m_{r+1,i}n_{r+1}k})| - \sum_{k=1}^{k_r} |b_{m_{rj}n_rk}| \\
&\qquad - \sum_{k=1}^{k_{r+1}} |b_{m_{r+1,i}n_{r+1}k}| - \sum_{k=k_{r+1}+1}^{\infty} |b_{m_{rj}n_rk}| - \sum_{k=k_{r+2}+1}^{\infty} |b_{m_{r+1,i}n_{r+1}k}| \\
&\quad > 3\gamma + 3\gamma - 4\gamma = 2\gamma.
\end{aligned}$$

Hence, by Remark 3.2, $x \notin \mathcal{C}_{eB}$. ▼

*Proof of Theorem 3.1*.

**Necessity.** (i) and (ii) are evident.

(iii) By Lemma 2.2 there exists $N \in \mathbb{N}$ such that $B(\mathfrak{m}) \subset \mathcal{C}_e^N$. Applying Lemma 3.3 to the matrix $(b_{m,n+N,k})$, we get $\lim_n \lim_s \overline{\lim}_m \sum_{k=s}^{\infty} |b_{mnk}| = 0$. Hence there exists an index sequence $(L_n)$ such that

$$\sup_{n \geq N} \lim_s \overline{\lim}_m \sum_{k=L_n+1}^{\infty} |b_{mnk}| < \infty.$$

By Theorem 2.3 (v) $\sup_{n\geq N} \overline{\lim}_m \sum_{k=1}^{L_n} |b_{mnk}| < \infty$. Hence (iii) follows.

(iv) By (i) and (iii) we get $(b_k) \in \ell$. We may assume that $b_k = 0$ $(k \in \mathbb{N})$. So (iv) follows by Lemma 3.4.

**Sufficiency.** From (i) and (iii) it follows that the series $\sum_k |b_k x_k|$ converges for every $x \in \mathfrak{m}$. Let $\gamma_{mn} := \sum_k |b_{mnk} - b_k|$ $(m, n \in \mathbb{N})$. By (iv) $\lim_n \overline{\lim}_m |\gamma_{mn}| = 0$. For every $x \in \mathfrak{m}$ we get

$$\Big|\sum_k b_{mnk} x_k - \sum_k b_k x_k\Big| \leq \gamma_{mn} \|x\|_\infty \quad (m, n \in \mathbb{N}).$$

Hence $\mathcal{C}_e$-$\lim_{m,n} \sum_k b_{mnk} x_k = \sum_k b_k x_k$. So $\mathfrak{m} \subset \mathcal{C}_{eB}$. ▼

**Theorem 3.5.** *A* 3*-dimensional matrix* $B = (b_{mnk})$ *maps* $\mathfrak{m}$ *into* $\mathcal{C}_{be}$ *if and only if* $B$ *satisfies* (iv) *of Theorem* 3.1 *and*

(i') *for every* $k \in \mathbb{N}$ *the limit* $b_k := \mathcal{C}_{be}$-$\lim_{m,n} b_{mnk}$ *exists,*

(ii') $\sup_n \overline{\lim}_m \sum_k |b_{mnk}| < \infty$.

*Under these circumstances,* $(b_k) \in \ell$ *and*

$$\lim_B x = \sum_k b_k x_k \quad (x \in \mathfrak{m}).$$

*Proof.* It may be obtained in the same way as the proof of Theorem 3.1. ▼


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## KOONDUVUST SÄILITAVAD JA TEKITAVAD SM-MENETLUSED

Maria ZELTSER

On vaadeldud topeltjadade ruumi

$$C_e := \{x = (x_{kl}) | \exists a \in \mathbb{K} : \lim_{l\to\infty} \overline{\lim_{k\to\infty}} |x_{kl} - a| = 0\}.$$

Libiseva küüru meetodi abil on leitud tarvilikud ja piisavad tingimused selleks, et kolmemõõtmeline maatriks (ehk SM-menetlus) teisendaks iga koonduva või tõkestatud jada $(x_k)$ ruumi $\mathcal{C}_e$ või tema alamruumi

$$\mathcal{C}_{be} := \left\{x \in \mathcal{C}_e \,|\, \forall l \in \mathbb{N} : \ (x_{kl})_k \in \mathfrak{m}\right\}.$$