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# AN INVERSE PROBLEM ARISING IN COMPRESSION OF PORO-VISCOELASTIC MEDIUM

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**Abstract.** An inverse problem for determining a time- and space-dependent memory kernel in a model of compression of poro-viscoelastic medium is considered. The kernel is represented in a form of a finite sum of products of known functions of the local coordinates and of unknown time-dependent functions. An existence and uniqueness theorem is proved.

**Key words:** integrodifferential equation, inverse problem.

## 1. INTRODUCTION

In recent years a number of papers have appeared on inverse problems for identification of time-dependent memory kernels (see, for instance, [1–3] and the references therein). Particularly, the work [3] deals with the determination of four independent relaxation kernels of poro-viscoelastic materials described by coupled systems of wave and diffusion equations.

In the present paper an attempt is made to identify the memory kernels of poro-viscoelastic media which depend both on time and local coordinates. To simplify this rather complicated problem we consider the case of consolidation [4] in which only a single kernel appears. Our main idea is to represent (or approximate) this kernel by a finite sum of products of known functions of the local coordinates and of unknown time-dependent functions. In this way we arrive at a certain parabolic inverse problem containing finitely many time-dependent unknown kernels. Furthermore, as in [5], for this problem an existence and uniqueness theorem in infinite time interval is stated using the Laplace transform method.

Identification problems for other classes of memory kernels in diffusion and viscoelasticity depending on time and some local coordinates are treated in recent papers [6–8].

## 2. FORMULATION OF THE PROBLEM

In the theory of consolidation [4,9] the compression of the water saturated porous medium is governed by the parabolic equation

$$[\lambda(x)u_x(x,t)]_x = \beta(x)u_t(x,t) + \left[\int_0^t m(x,t-\tau)u(x,\tau)d\tau\right]_t + \gamma(x,t), \quad x \in D = (0,1),\ t \in (0,\infty), \tag{1}$$

where $u$ is the effective stress, $\beta > 0$ is the instantaneous elastic compressibility, $m$ is the corresponding memory kernel, and $\gamma$ is the source density. Besides Eq. (1), the function $u$ satisfies the initial and boundary conditions

$$u(x,0) = \varphi(x) \qquad \text{on } D, \tag{2}$$

$$u(0,t) = u(1,t) = 0, \quad t \in (0,\infty) \tag{3}$$

with the given continuous function $\varphi$.

In the sequel we take the kernel $m(x,t)$ in the form

$$m(x,t) = \sum_{k=1}^{N} \mu_k(x)m_k(t), \tag{4}$$

where $\mu_k,\ k = 1, ..., N$, are given functions of $x \in D$ and $m_k,\ k = 1, ..., N$, are unknown memory kernels depending on $t \in (0,\infty)$.

In the *inverse problem* the unknowns $m_k,\ k = 1, ..., N$, are to be determined by $N$ additional conditions of the form

$$u(x_k,t) = h_k(t), \quad t \in (0,\infty), \ k = 1, \ldots, N, \tag{5}$$

where $x_1, \ldots, x_N$ are some points in the interior of the interval [0,1] and $h_1, \ldots, h_N$ are given observations.

## 3. APPLICATION OF THE LAPLACE TRANSFORM

Let us apply the Laplace transform [10] to Eq. (1) with (2) and (3). Then, for the image of $u$,

$$U(x,p) = \mathcal{L}_{t\to p}u = \int_0^\infty e^{-pt}u(x,t)dt, \quad \mathrm{Re}\, p > \sigma$$

with some real number $\sigma$, there holds the equation

$$[\lambda(x)U_x(x,p)]_x - \beta(x)pU(x,p) = p\sum_{k=1}^{N} \mu_k(x)M_k(p)U(x,p) + \Gamma(x,p) - \beta(x)\varphi(x) \tag{6}$$

in $D$, where

$$M_k = \mathcal{L}_{t\to p} m_k, \;\; \Gamma = \mathcal{L}_{t\to p}\gamma \,.$$

The boundary conditions (3) are transformed to

$$U(0,p) = U(1,p) = 0. \tag{7}$$

Denoting Green's function of the left-hand differential operator in (6) with boundary condition (7) by $G(x,y,p)$, we get the solution of Eq. (6) with (7) as

$$U(x,p) = \sum_{k=1}^{N} M_k(p)\int_0^1 p\,G(x,y,p)\mu_k(y)U(y,p)dy + F(x,p), \quad 0 < x < 1, \tag{8}$$

where

$$F(x,p) = \int_0^1 G(x,y,p)[\Gamma(y,p) - \beta(y)\varphi(y)]dy, \;\; 0 < x < 1\,. \tag{9}$$

Further, the additional conditions (5) take the form

$$U(x_k,p) = H_k(p), \;\; k = 1,\ldots,N, \tag{10}$$

where $H_k(p) = \mathcal{L}_{t\to p} h_k$. Inserting (8) into (10), we have the system

$$\sum_{k=1}^{N} M_k(p)\int_0^1 pG(x_i,y,p)\mu_k(y)U(y,p)dy = H_i(p) - F(x_i,p)\,, \quad i = 1,\ldots,N. \tag{11}$$

Defining the coefficients

$$\gamma_{i,k} := -\frac{1}{\beta(x_i)}\mu_k(x_i)\varphi(x_i), \tag{12}$$

we get Eqs. (11) equivalent to the following *system of N equations* for the functions $M_k(p)$, $k = 1, ..., N$:

$$\sum_{k=1}^{N} \gamma_{i,k} M_k(p)$$

$$+\sum_{k=1}^{N} M_k(p)\left[\frac{1}{\beta(x_i)}\mu_k(x_i)\varphi(x_i) + \int_0^1 p\, G(x_i, y, p)\mu_k(y)\varphi(y)dy\right] \tag{13}$$

$$+\sum_{k=1}^{N} M_k(p) \int_0^1 p\, G(x_i, y, p)\mu_k(y)[pU(y,p) - \varphi(y)]dy = \Phi_i(p),$$

$$i = 1, ..., N,$$

where

$$\Phi_i(p) = p[H_i(p) - F(x_i, p)], \; i = 1, \ldots, N. \tag{14}$$

In Eqs. (13) the first sum is the main part of the left-hand side for Re $p \to +\infty$ since

$$-\int_0^1 pG(x_i, y, p)\mu_k(y)\varphi(y)dy \to \frac{1}{\beta(x_i)}\mu_k(x_i)\varphi(x_i) \tag{15}$$

and

$$pU(y,p) \to \varphi(y), \; 0 < y < 1, \tag{16}$$

for Re $p \to +\infty$.

The asymptotic relation (15) follows from the assertion (19) of Lemma 1 below and the limit (16) is a consequence of the known relation $p\, G(p) \to g(0)$ for the Laplace transform $G$ of a function $g$ (cf. [10]).

The inverse problem (1)–(3), (5) is now reduced to the system of equations (13), where $U(x, p)$ is the solution of the integral equation (8), and in the main case the regularity assumption

$$\det(\gamma_{ik}) \neq 0 \qquad (i, k = 1, ..., N) \tag{17}$$

holds.

## 4. PREPARATIONS

Before stating and proving the existence theorem, we shall make some preparations. At first we prove a lemma concerning Green's function $G(x, y, p)$.

**Lemma 1.** *If $\lambda, \beta \in C^2[0,1]$ and $\lambda, \beta > 0$ in $[0,1]$, then the following estimates hold*

$$\|G\| := \sup_{\substack{0\le x\le 1\\ \operatorname{Re} p>0}} \int_0^1 |pG(x,y,p)|dy < \infty, \tag{18}$$

*and for any $\nu \in C^1[0,1]$*

$$\sup_{\substack{0\le x\le 1\\ \operatorname{Re} p>0}} \left| \sqrt{p}\left[\int_0^1 pG(x,y,p)\nu(y)dy + \frac{\nu(x)}{\beta(x)}\right]\right| \le C_1\|\nu\|_{C^1[0,1]}, \tag{19}$$

*where*

$$\|\nu(x)\|_{C^1[0,1]} = \|\nu(x)\|_{C[0,1]} + \|\nu'(x)\|_{C[0,1]}$$

*and the constants $C_1$ and $\|G\|$ depend on $\lambda, \beta$, only.*

*Proof.* It was shown in [5] that under the assumptions $\lambda, \beta \in C^2[0,1]$ and $\lambda, \beta > 0$ in $[0,1]$ Green's function $G(x,y,p)$ admits the representation

$$G(x,y,p) = \frac{(\lambda\beta)^{-1/4}(x)(\lambda\beta)^{-1/4}(y)}{C_0 s^2} \times \begin{cases} sh\, sz \cdot sh\, s(w-1) + O_1, & x \le y \\ sh\, sw \cdot sh\, s(z-1) + O_2, & y \le x, \end{cases} \tag{20}$$

where $z, w$ are given by

$$z = \frac{1}{l}\int_0^x \sqrt{\frac{\beta(\eta)}{\lambda(\eta)}}d\eta, \quad w = \frac{1}{l}\int_0^y \sqrt{\frac{\beta(\eta)}{\lambda(\eta)}}d\eta, \tag{21}$$

respectively,

$$l = \int_0^1 \sqrt{\frac{\beta(\eta)}{\lambda(\eta)}}d\eta, \quad s^2 = \frac{p}{l^2}, \ \operatorname{Re} s > 0, \tag{22}$$

the quantities $O_1, O_2$ possess the following asymptotics

$$O_1 = O\left[\frac{e^{s(1-w+z)}}{s}\right], \ O_2 = O\left[\frac{e^{s(w+1-z)}}{s}\right] \quad \text{as } \operatorname{Re} p \to \infty \tag{23}$$

uniformly in $x, y$, and

$$C_0 = \frac{l}{s} sh\, s + O\left(\frac{e^s}{s^2}\right) \quad \text{as } \operatorname{Re} p \to \infty, \ C_0 \ne 0. \tag{24}$$

Here and in the sequel $sh$ and $ch$ stand for the hyperbolic sine and cosine, respectively.

Using the representation (20), the asymptotic relations (23), and the obvious inequalities

$$|e^{st}| = e^{\mathrm{Re}\, s \cdot t}, \ |sh\, st| \le ch(\mathrm{Re}\ s \cdot t), \ |sh\, st| \ge sh(\mathrm{Re}\ s \cdot t), \tag{25}$$

for $t > 0$ we deduce that

$$\int_0^1 |pG(x,y,p)|dy \le C_2 \frac{|s|}{\mathrm{Re}\, s} \frac{sh(\mathrm{Re}\, s) + \dfrac{e^{\mathrm{Re}\, s}}{|s|}}{|sC_0|} \tag{26}$$
$$\text{for } 0 \le x \le 1, \quad \mathrm{Re}\, p > 0$$

with some positive constant $C_2$. Observe that for $\mathrm{Re}\, p > 0$ there holds

$$\mathrm{Re}\, s = \frac{1}{l} \mathrm{Re}\, \sqrt{p} > \frac{\sqrt{2}}{2l}\sqrt{|p|} = \frac{\sqrt{2}}{2l}|s|.$$

Thus, the first factor on the right-hand side of (26) is bounded for $\mathrm{Re}\ p > 0$. Due to (24) the second factor on the right-hand side of (26) is bounded, too. Therefore, the assertion (18) holds.

In order to prove (19), we first define the following integrated Green's function:

$$H(x,y,p) = \begin{cases} \displaystyle\int_1^y G(x,\tau,p)d\tau, & x \le y \\ \displaystyle\int_0^y G(x,\tau,p)d\tau, & y \le x. \end{cases} \tag{27}$$

Let us substitute $G(x,y,p)$ by (20) in (27) and integrate by parts the products $(\lambda\beta)^{-1/4}(y) sh\, s(w-1)$ and $(\lambda\beta)^{-1/4}(y) sh\, sw$. Observing (22) and (23) after some computations, we obtain the following asymptotic relation for $H(x,y,p)$:

$$\begin{aligned} H(x,y,p) \ &= \ \frac{l(\lambda\beta)^{-1/4}(x)\lambda^{1/4}(y)}{C_0 s^3 \beta^{3/4}(y)} \\ &\quad \times \begin{cases} sh\, sz[ch\, s(w-1) - 1] + O_3, & x < y \\ (ch\, sw - 1) sh\, s(z-1) + O_4, & y < x, \end{cases} \end{aligned} \tag{28}$$

where

$$O_3 = O\left[\frac{e^{s(1-w+z)}}{s}\right], \ O_4 = O\left[\frac{e^{s(w+1-z)}}{s}\right] \quad \text{as } \mathrm{Re}\, p \to \infty \tag{29}$$

uniformly in $x, y$.

Further, let us split the integral $\int_0^1 G(x,y,p)\nu(y)dy$ into the integral from 0 to $x$ and the one from $x$ to 1. Integrating by parts in view of (24), (28), (29) and the relation $s^2 = \frac{p}{l^2}$, we have

$$\sqrt{p}\left[\int_0^1 pG(x,y,p)\nu(y)dy + \frac{\nu(x)}{\beta(x)}\right] = -p\sqrt{p}\int_0^1 H(x,y,p)\nu'(y)dy$$

$$+\sqrt{p}\left[\frac{\nu(x)}{\beta(x)} + pH(x,y,p)\nu(y)|_{y=0}^{y=x-0} + pH(x,y,p)\nu(y)|_{y=x+0}^{y=1}\right]$$

$$= -p\sqrt{p}\int_0^1 H(x,y,p)\nu'(y)dy + \sqrt{p}O\left(\frac{1}{s}\right)[\nu(0)+\nu(1)]$$

$$+\frac{\nu(x)}{\beta(x)}\sqrt{p}\left[1 - \frac{sh\,s + sh\,s(z-1) - sh\,z + O\left(\frac{e^s}{s}\right)}{sh\,s + O\left(\frac{e^s}{s}\right)}\right], \tag{30}$$

where the $O$-terms are uniform with respest to $x, y$. Observing the formula (28), we can estimate the term $p\sqrt{p}\int_0^1 H(x,y,p)\nu'(y)dy$ in (30) as in the equality (18) for $\int_0^1 |pG(x,y,p)|dy$. The other terms in (30) are obviously bounded for $0 \le x, y \le 1$ and Re $p > 0$. This proves the assertion (19). Lemma 1 is proved. $\square$

In the following we need an estimation of the function

$$B(x,p) = B[M_1, ..., M_N](x,p) = p\,U(x,p) - \varphi(x)\,, \tag{31}$$

which is the Laplace transform of the derivative $u_t$ of the solution $u$ to the direct problem (1)–(3). By Eq. (8), for $U$ the function $B$ is the solution of the integral equation

$$\begin{aligned} B(x,p) \;\; &= +\sum_{k=1}^{N} M_k(p)\int_0^1 pG(x,y,p)\mu_k(y)B(y,p)dy \\ &\quad +\sum_{k=1}^{N} M_k(p)\int_0^1 pG(x,y,p)\mu_k(y)\varphi(y)dy \\ &\quad +F_1(x,p), \end{aligned} \tag{32}$$

where

$$F_1(x,p) = pF(x,p) - \varphi(x) \tag{33}$$

with $F$ given by (9).

Our intention is to estimate the function $B$ in terms of the kernels $M_1, \ldots, M_N$. To this end, we first introduce the following functional spaces for complex-valued scalar and vector functions:

$$\mathcal{A}_{\gamma,\sigma} = \{V : V(p) \quad \text{is holomorphic on} \quad \mathrm{Re}\, p > \sigma$$

and

$$\|V\|_{\gamma,\sigma} < \infty\}, \ \gamma > 0, \ \sigma > 0,$$

where

$$\|V\|_{\gamma,\sigma} := \sup_{\operatorname{Re} p > \sigma} |p|^{\gamma} |f(p)|$$

and

$$(\mathcal{A}_{\gamma,\sigma})^N = \{V = (V_1, ..., V_n) : V_i(p) \in \mathcal{A}_{\gamma,\sigma}, \ i = 1, ..., N\}$$

with the norm

$$\|V\|_{\gamma,\sigma} := \sum_{i=1}^{N} \|V_i\|_{\gamma,\sigma} \qquad \text{for} \quad V \in (\mathcal{A}_{\gamma,\sigma})^N.$$

We note that

$$\mathcal{A}_{\gamma,\sigma} \subset \mathcal{A}_{\gamma,\sigma'}, \ (\mathcal{A}_{\gamma,\sigma})^N \subset (\mathcal{A}_{\gamma,\sigma'})^N$$

with $\|\cdot\|_{\gamma,\sigma'} \leq \|\cdot\|_{\gamma,\sigma}$ if $\sigma' > \sigma$.

Let $\gamma$ and $\alpha$ be two real numbers such that

$$1/2 < \gamma \leq 1 \qquad \text{and} \qquad 1 < \alpha < 1/2 + \gamma. \tag{34}$$

Moreover, let $w = (w_1, ..., w_N)(t)$ be a real-valued vector function with Laplace transform $W = (W_1, ..., W_N)(p) = \mathcal{L}_{t \to p} w$ satisfying the condition

$$W \in (\mathcal{A}_{\gamma,\sigma_0})^N \qquad \text{with some} \quad \sigma_0 > 0\,. \tag{35}$$

We introduce the space for the kernels $M = (M_1, ..., M_N)(p)$

$$\mathcal{M}_{W,\sigma} = \{M : M(p) = W(p) + Z(p), \ \ Z \in (\mathcal{A}_{\alpha,\sigma})^N\}, \ \sigma \geq \sigma_0$$

and the space for the scalar functions $B(x, p)$

$$\begin{aligned} \mathcal{B}_{\sigma} = \{B : B(x,p) \in \mathcal{A}_{\gamma,\sigma} \ \text{ for a.e. } \ x \in (0,1), \\ B(\cdot, p) \in L^{\infty}(0,1) \ \text{ for } \ \operatorname{Re} p > \sigma\}, \ \sigma \geq \sigma_0\,, \end{aligned}$$

with the norm

$$\|B\|_{\sigma} = \operatorname*{essup}_{x \in (0,1)} \|B(x,\cdot)\|_{\gamma,\sigma} = \operatorname*{essup}_{x \in (0,1)} \ \sup_{\operatorname{Re} p > \sigma} |p|^{\gamma} |B(x,p)|.$$

Let us prove the following lemma.

**Lemma 2.** *Let beside* (34) *and* (35) *the following assumptions be fulfilled*:

$$\begin{aligned}&\lambda,\beta\in C^2[0,1] \text{ and } \lambda,\beta>0 \text{ in } [0,1]\,,\\ &\mu_k\in C[0,1],\ k=1,...,N,\ \varphi\in C[0,1]\,,\\ &F_1\in\mathcal{B}_{\sigma_0} \text{ with } \sigma_0 \text{ from (35).}\end{aligned}\tag{36}$$

*Then, for each* $\sigma\geq\sigma_0$ *and* $M=W+Z\in\mathcal{M}_{W,\sigma}$ *such that*

$$\eta(M,\sigma):=\|G\|\|\mu\|\left(\frac{\|W\|_{\gamma,\sigma}}{\sigma^\gamma}+\frac{\|Z\|_{\alpha,\sigma}}{\sigma^\alpha}\right)<1\,,\tag{37}$$

*where* $\|G\|$ *as in Lemma* 1 *and* $\|\mu\|=\max_{1\leq k\leq N}\|\mu_k\|_{C[0,1]}$, *Eq.* (32) *has a unique solution* $B=B[M]\in\mathcal{B}_\sigma$. *This solution satisfies the estimate*

$$\|B\|_\sigma\leq\frac{1}{1-\eta(M,\sigma)}\left[\|F_1\|_\sigma+\|G\|\|\mu\|\,\|\varphi\|\left(\|W\|_{\gamma,\sigma}+\frac{\|Z\|_{\alpha,\sigma}}{\sigma^{\alpha-\gamma}}\right)\right],\tag{38}$$

*where* $\|\varphi\|=\|\varphi\|_{C[0,1]}$. *Moreover, for each* $\sigma\geq\sigma_0$ *and* $M=W+Z\in\mathcal{M}_{W,\sigma}$, $\tilde M=W+\tilde Z\in\mathcal{M}_{W,\sigma}$ *such that* $\eta(M,\sigma)<1$ *and* $\eta(\tilde M,\sigma)<1$, *the difference* $B[M]-B[\tilde M]$ *can be estimated by*

$$\begin{aligned}\|B[M]-B[\tilde M]\|_\sigma\leq\frac{1}{1-\eta(\tilde M,\sigma)}\|G\|\|\mu\|&\left[\frac{\|\varphi\|}{\sigma^{\alpha-\gamma}}\right.\\ +\frac{1}{\sigma^\alpha}\frac{1}{1-\eta(M,\sigma)}&\left.\left\{\|F_1\|_\sigma+\|G\|\|\mu\|\,\|\varphi\|\left(\|W\|_{\gamma,\sigma}+\frac{\|Z\|_{\alpha,\sigma}}{\sigma^{\alpha-\gamma}}\right)\right\}\right]\\ &\times\|Z-\tilde Z\|_{\alpha,\sigma}\,.\end{aligned}\tag{39}$$

*Proof.* The integral equation (32) for $B=B[M]$ can be written in the operator form

$$B\;=\;b\;+\;A_0B\;\text{ in }\;\mathcal{B}_\sigma\,,\tag{40}$$

where

$$b=b(x,p)=\sum_{k=1}^N M_k(p)\int_0^1 pG(x,y,p)\mu_k(y)\varphi(y)dy+F_1(x,p)\tag{41}$$

and the linear operator $A_0\,:\,\mathcal{B}_\sigma\to\mathcal{B}_\sigma$ is defined by

$$(A_0B)(x,p)=\sum_{k=1}^N M_k(p)\int_0^1 pG(x,y,p)\mu_k(y)B(y,p)dy\,.$$

Provided $M \in \mathcal{M}_{W,\sigma}$, the assertion (18) of Lemma 1 with (36) implies $b \in \mathcal{B}_\sigma$. Analogously, by (18) we have $A_0 B \in \mathcal{B}_\sigma$ if $B \in \mathcal{B}_\sigma$. Moreover, the norm of $A_0$ in $L(\mathcal{B}_\sigma)$ can be estimated by

$$\|A_0\| \le \|G\| \|\mu\| \left( \frac{\|W\|_{\gamma,\sigma}}{\sigma^\gamma} + \frac{\|Z\|_{\alpha,\sigma}}{\sigma^\alpha} \right) = \eta(M,\sigma) . \tag{42}$$

Consequently, in the case $\eta(M,\sigma) < 1$, by the contraction principle, Eq. (40) [or, equivalently, (32)] has a unique solution $B \in \mathcal{B}_\sigma$.

To prove (38) we first estimate $b$ from (41). We have

$$\|b\|_\sigma \le \|F_1\|_\sigma + \|G\| \|\mu\| \|\varphi\| \left( \|W\|_{\gamma,\sigma} + \frac{\|Z\|_{\alpha,\sigma}}{\sigma^{\alpha-\gamma}} \right) .$$

Using this inequality and (42) in (40), we obtain (38).

It remains to derive the estimate (39) for the difference $B[M] - B[\tilde{M}]$. From (32) we have

$$\begin{aligned}
&(B[M] - B[\tilde{M}])(x,p) \\
&= \sum_{k=1}^{N} \tilde{M}_k(p) \int_0^1 pG(x,y,p)\mu_k(y)(B[M] - B[\tilde{M}])(y,p)dy \\
&+ \sum_{k=1}^{N} \left[ Z_k(p) - \tilde{Z}_k(p) \right] \int_0^1 pG(x,y,p)\mu_k(y)[B[M](y,p) + \varphi(y)]dy .
\end{aligned}$$

Using here (18) and the definition (37), we obtain

$$\begin{aligned}
\|B[M] \quad - \quad & B[\tilde{M}]\|_\sigma \\
& \le \frac{1}{1-\eta(\tilde{M},\sigma)} \|G\| \|\mu\| \left[ \frac{1}{\sigma^\alpha} \|B[M]\|_\sigma + \frac{\|\varphi\|}{\sigma^{\alpha-\gamma}} \right] \|Z - \tilde{Z}\|_{\alpha,\sigma} .
\end{aligned}$$

Substituting $\|B[M]\|_\sigma$ by (38), we derive (39). Lemma 2 is proved. □

## 5. EXISTENCE THEOREM

We can now state our main result.

**Theorem.** *Let* (34), (35), *and* (36) *be fulfilled, where we assume* $\mu_k$ *and* $\varphi$ *to satisfy stronger smoothness conditions*

$$\mu_k \in C^1[0,1], \; k = 1, ..., N, \; \varphi \in C^1[0,1] . \tag{43}$$

*Moreover, let the regularity condition* (17) *be satisfied for the matrix* $\Gamma = (\gamma_{ik})$ *and* $\Phi = (\Phi_1, \ldots, \Phi_N)$, *defined in* (14), *have the representation*

$$\Phi = \Gamma W + \Psi \in \mathcal{M}_{AW,\sigma_0} \tag{44}$$

*with some* $\Psi = (\Psi_1, \ldots, \Psi_N) \in (\mathcal{A}_{\alpha,\sigma_0})^N$. *Then there exists* $\sigma_1 \geq \sigma_0$ *such that the system of equations* (13) *has a unique solution* $M = (M_1, \ldots, M_N)(p)$ *of the form* $M = W + Z \in \mathcal{M}_{W,\sigma_1}$.

*Proof.* The system of equations (13) is equivalent to the operator equation

$$Z = AZ \quad \text{in } (\mathcal{A}_{\alpha,\sigma})^N \tag{45}$$

with the operator $A = \Gamma^{-1}A_1 \; : \; (\mathcal{A}_{\alpha,\sigma})^N \to (\mathcal{A}_{\alpha,\sigma})^N$, where $\Gamma^{-1}$ is the inverse of the matrix $\Gamma$ and the operator $A_1 \; : \; (\mathcal{A}_{\alpha,\sigma})^N \to (\mathcal{A}_{\alpha,\sigma})^N$ is defined by

$$\begin{aligned}(A_1 Z)_i(p) \;&=\; \Psi_i - \sum_{k=1}^{N} [W_k(p) + Z_k(p)] g_{ik}(p) \\ &\quad - \sum_{k=1}^{N} M_k(p) \int_0^1 pG(x_i, y, p)\mu_k(y) B[W+Z](y,p)\,dy, \end{aligned} \tag{46}$$

where

$$g_{ik}(p) \;=\; \frac{1}{\beta(x_i)}\mu_k(x_i)\varphi(x_i) + \int_0^1 pG(x_i, y, p)\mu_k(y)\varphi(y)\,dy$$

and $B$ is given by (31).

For the proof of the existence of a unique solution to Eq. (45) we introduce the balls

$$D_{\alpha,\sigma}(\rho) \;=\; \{Z \in (\mathcal{A}_{\alpha,\sigma})^N \; : \; \|Z\|_{\alpha,\sigma} \leq \rho\}$$

and show that $A$ is a contraction in such a ball.

First note that by the assertion (19) of Lemma 1 we have

$$\|g_{ik}\|_{1/2,\sigma} \;\leq\; C_3, \;\; 1 \leq i, k \leq N\,, \tag{47}$$

with some constant $C_3$ depending on $\lambda, \beta, \mu = (\mu_1, \ldots, \mu_N)$ and $\varphi$.

Let us fix some $\sigma \geq \sigma_0$. Using the estimates (18) of Lemma 1, (38) of Lemma 2, and (47) in (46), we obtain

$$\begin{aligned}
\|AZ\|_{\alpha,\sigma} &\leq |\Gamma^{-1}| \|A_1 Z\|_{\alpha,\sigma} \\
&\leq |\Gamma^{-1}| \left\{ \|\Psi\|_{\alpha,\sigma_0} + C_3 \left( \frac{\|W\|_{\gamma,\sigma}}{\sigma^{1/2+\gamma-\alpha}} + \frac{\|Z\|_{\alpha,\sigma}}{\sigma^{1/2}} \right) \right. \\
&\quad \left. + \|G\| \|\mu\| \left( \frac{\|W\|_{\gamma,\sigma}}{\sigma^{2\gamma-\alpha}} + \frac{\|Z\|_{\alpha,\sigma}}{\sigma^{\gamma}} \right) \|B[W+Z]\|_{\sigma} \right\} \\
&\leq C_4 \left[ \|\Psi\|_{\alpha,\sigma_0} + \frac{\|W\|_{\gamma,\sigma_0}}{\sigma^{1/2+\gamma-\alpha}} + \frac{\|Z\|_{\alpha,\sigma}}{\sigma^{1/2}} \right. \\
&\quad \left. + \frac{1}{1-\eta(W+Z,\sigma)} \left( \frac{\|W\|_{\gamma,\sigma_0}}{\sigma^{2\gamma-\alpha}} + \frac{\|Z\|_{\alpha,\sigma}}{\sigma^{\gamma}} \right) \left( 1 + \|W\|_{\gamma,\sigma_0} + \frac{\|Z\|_{\alpha,\sigma}}{\sigma^{\alpha-\gamma}} \right) \right]
\end{aligned} \tag{48}$$

provided the quantity $\eta(W+Z,\sigma)$ given by (37) is less than one. Here the norm of $\Gamma^{-1}$ is defined by

$$|\Gamma^{-1}| = \max_{1 \leq i,k \leq N} |(\Gamma^{-1})_{ik}|$$

and $C_4$ is a positive constant depending on $\lambda, \beta, \mu, \varphi$, and $F_1$.

Let now $Z \in D_{\alpha,\sigma}(\rho)$. Then, from (48) we get

$$\begin{aligned}
\|AZ\|_{\alpha,\sigma} \leq C_4 &\left[ \|\Psi\|_{\alpha,\sigma_0} + \frac{\|W\|_{\gamma,\sigma_0}}{\sigma^{1/2+\gamma-\alpha}} + \frac{\rho}{\sigma^{1/2}} \right. \\
&\left. + \frac{\left( \frac{\|W\|_{\gamma,\sigma_0}}{\sigma^{2\gamma-\alpha}} + \frac{\rho}{\sigma^{\gamma}} \right) \left( 1 + \|W\|_{\gamma,\sigma_0} + \frac{\rho}{\sigma^{\alpha-\gamma}} \right)}{1 - \|G\| \|\mu\| \left( \frac{\|W\|_{\gamma,\sigma_0}}{\sigma^{\gamma}} + \frac{\rho}{\sigma^{\alpha}} \right)} \right].
\end{aligned}$$

From this estimate, in view of the assumptions (34) about $\gamma$ and $\alpha$, we see that for every $\rho > \rho_0 = C_4 \|\Psi\|_{\alpha,\sigma_0}$ there exists $\sigma_2 = \sigma_2(\rho)$ such that

$$\|AZ\|_{\alpha,\sigma} \leq \rho \quad \text{for} \quad \sigma \geq \sigma_2(\rho)\,. \tag{49}$$

Further, by Lemma 2 we have $B = B[W+Z] \in \mathcal{B}_{\sigma_2(\rho)}$, which implies that the function $B = B(y,p)$ is holomorphic with respect to $p$ on Re $p > \sigma_2(\rho)$. This, together with the holomorphy of $Z$ in $D_{\alpha,\sigma}(\rho)$ and the other terms in (46), shows the holomorphy of $AZ = \Gamma^{-1} A_1 Z$ on the half-plane Re $p > \sigma_2(\rho)$. Hence, by (49) it follows that

$$A : D_{\alpha,\sigma}(\rho) \to D_{\alpha,\sigma}(\rho) \quad \text{for} \ \sigma \geq \sigma_2(\rho)\,, \rho > \rho_0\,. \tag{50}$$

Next, let $Z^1, Z^2 \in D_{\alpha,\sigma}(\rho)$ with some $\sigma \geq \sigma_0$. Then, by virtue of the inequalities (18), (19) of Lemma 1 and (38), (39) of Lemma 2, again we obtain the estimate

$$\|AZ^1 - AZ^2\|_{\alpha,\sigma} \leq |\Gamma^{-1}| \Big\{ \frac{C_3}{\sigma^{1/2}} \|Z^1 - Z^2\|_{\alpha,\sigma}$$

$$+\|G\|\|\mu\| \left( \frac{\|W\|_{\gamma,\sigma}}{\sigma^{2\gamma-\alpha}} + \frac{\|Z^1\|_{\alpha,\sigma}}{\sigma^\gamma} \right) \|B[W+Z^1] - B[W+Z^2]\|_\sigma$$

$$+\|G\|\|\mu\| \frac{1}{\sigma^\gamma} \|B[W+Z^2]\|_\sigma \|Z^1 - Z^2\|_{\alpha,\sigma} \Big\}$$

$$\leq C_5 \Bigg\{ \frac{1}{\sigma^{1/2}} + \frac{\dfrac{\|W\|_{\gamma,\sigma_0}}{\sigma^{2\gamma-\alpha}} + \dfrac{\rho}{\sigma^\gamma}}{1 - \eta_0(\rho,\sigma)}$$

$$\times \left[ \frac{1}{\sigma^{\alpha-\gamma}} + \left( \frac{1}{\sigma^\alpha} + \frac{1}{\sigma^\gamma} \right) \frac{1 + \|W\|_{\gamma,\sigma_0} + \dfrac{\rho}{\sigma^{\alpha-\gamma}}}{1 - \eta_0(\rho,\sigma)} \right] \Bigg\}$$

$$\times \|Z^1 - Z^2\|_{\alpha,\sigma}, \tag{51}$$

where

$$\eta_0(\rho,\sigma) = \|G\|\|\mu\| \left( \frac{\|W\|_{\gamma,\sigma_0}}{\sigma^\gamma} + \frac{\rho}{\sigma^\alpha} \right)$$

and the constant $C_5$ depends on $\lambda, \beta, \mu, \varphi$, and $F_1$.

From the estimate (51), in view of the assumptions (34) on $\gamma$ and $\alpha$, again we see that there exists $\sigma_3 = \sigma_3(\rho)$ such that the operator $A$ is a contraction for $\sigma \geq \sigma_3(\rho)$. This, together with (50), implies that Eq. (45) has a unique solution in every ball $D_{\alpha,\sigma}(\rho)$ when $\sigma \geq \sigma_4(\rho) = \max\{\sigma_2(\rho), \sigma_3(\rho)\}$ and $\rho > \rho_0 = C_4 \|\Psi\|_{\alpha,\sigma_0}$. Therefore, the existence assertion of Theorem is proved for $\sigma_1 = \sigma_4(\rho_1)$ with some $\rho_1 > \rho_0$.

Since any two solutions $Z^1, Z^2$ of Eq. (45) from $(\mathcal{A}_{\alpha,\sigma_1})^N$ are lying in some common ball $D_{\alpha,\sigma}(\rho)$ with $\sigma \geq \sigma_4(\rho)$, $\rho > \max\{\|Z^1\|_{\alpha,\sigma_1}, \|Z^2\|_{\alpha,\sigma_1}, \rho_1\}$, the uniqueness of the solution $Z$ in the space $(\mathcal{A}_{\alpha,\sigma_1})^N$ follows from the proven uniqueness in these balls. Theorem is proved. □

**Corollary.** *Under the assumptions of Theorem the inverse problem* (1)–(3), (5) *has a unique solution* $m$ *of the form*

$$m(t) = w(t) + \frac{1}{2\pi i} \int_{\xi - i\infty}^{\xi + i\infty} e^{tp} Z(p) dp \quad (\xi > \sigma_1)$$

*with* $Z \in (\mathcal{A}_{\alpha,\sigma_1})^N$.

Since $\alpha > 1$ by (34), this follows from a known inversion formula for the Laplace transform (see [10], Th. 21.3). A vector function $w$, often occurring in applications, contains components with power-type singularities at $t = 0$, i.e.

$$w_i(t) = \sum_{j=1}^{n_i} c_{ij} t^{-\delta_{ij}} e^{-s_{ij}t}, \quad i = 1, \ldots, N,$$

where $c_{ij}$, $s_{ij} \geq 0$ and $0 \leq \delta_{ij} \leq 1 - \gamma$. Then

$$W_i(p) = \sum_{j=1}^{n_i} c_{ij} \frac{\Gamma(1 - \delta_{ij})}{(p + s_{ij})^{1-\delta_{ij}}}$$

with $\Gamma$ standing for Euler's gamma function.

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# POORSE VISKOELASTSE KESKKONNA KOKKUSURUMISEL TEKKIVAST PÖÖRDÜLESANDEST

Jaan JANNO

On vaadeldud pöördülesannet aja- ja ruumimuutujast sõltuva tuumafunktsiooni määramiseks mudelist, mis kirjeldab poorse viskoelastse keskkonna kokkusurumist. Tuum on esitatud etteantud ruumimuutujast sõltuvate ja tundmatute ajast sõltuvate funktsioonide korrutiste lõpliku summana. On tõestatud käsitletava pöördülesande lahendi olemasolu ja ühesus.