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ON THE APPROXIMATE CALCULATION OF DOUBLE INTEGRALS WITH WEIGHT FUNCTION

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Abstract. Cubature formulas with the weight function $x^{\mu}\,(\mu > -1)$ which are optimal on given sets of functions are obtained for the approximate calculation of a double integral. One type of these formulas consists of the line integrals which may be estimated by optimal quadrature formulas. The optimal cubature formulas include both knots and coefficients of the corresponding values of optimal quadrature formulas. The application of a cubature formula with weight function to the calculation of double integrals over the domain of integration limited by a parabolic curve is shown.

Key words: multivariate singular integral, optimal cubature formula, weight function, approximate formula.

1. INTRODUCTION: NOTATIONS AND DEFINITIONS

For the calculation of multivariate singular integrals it is important to have efficient approximate formulas with weight function. Below, we consider the problem of obtaining optimal cubature formulas with the weight function $x^{\mu}\,(\mu > -1)$. The approach is based on the idea set forth in [1,2].

Let $p^{-1} + q^{-1} = 1$ and $D = [0,1] \times [0,1]$. Let us introduce the notation of derivatives

$$f^{(i,j)}(x, y) = \frac{\partial^{i+j}}{\partial x^i \partial y^j} f(x, y), \quad i, j = 0,1,\ldots,$$

of norms

$$\|f(\cdot)\|_p = \begin{cases} \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p}, & p < \infty \\ \operatorname*{supvrai}_{0 \le x \le 1} |f(x)|, & p = \infty \end{cases}$$

$$\|f(\cdot,\cdot)\|_p = \begin{cases} \left(\iint_D |f(x,y)|^p \, dxdy \right)^{1/p}, & p < \infty \\ \operatorname*{supvrai}_{x,y \in D} |f(x,y)|, & p = \infty \end{cases}$$

and the sets of functions

$$W_p^k = \left\{ f(x): \quad f^{(k)}(x) \quad \text{piecewise continuous on } [0,1], \quad \left\| f^{(k)}(\cdot) \right\|_p \le 1 \right\},$$

$$V_p^{r,s} =$$

$$\left\{ f(x,y): f^{(i,j)}(x,y)(i \le r, \, j \le s) \text{ piecewise continuous on } D, \left\| f^{(r,s)}(\cdot,\cdot) \right\|_p \le p \right\},$$

$$\overline{W}_p^{r,s} = \left\{ f(x,y): \quad f \in V^{r,s}, \quad \left\| \int_0^1 f^{(r,0)}(\cdot,y) \, dy \right\|_p \le M, \left\| \int_0^1 x^{\mu} f^{(0,s)}(x,\cdot) \, dx \right\|_p \le N \right\}.$$

Let Q be a set of linear functionals defined on a set H of functions f, and let Z be a given linear functional defined on the set H. Consider approximate formulas of the form

$$Zf = \sum_{k=1}^{n} L_k f + r(f), \quad L_k \in Q, \quad k = 1, \ldots, n, \tag{1}$$

where $r(f)$ is the error of the formula.

Formula (1) is called an optimal formula on the set H if the functionals $L_k \in Q$ are chosen so that the quantity

$$r[H] = \sup_{f \in H} |r(f)| \tag{2}$$

has the minimal value.

Designate by $A_k^{(j)}$, x_k^* $(k=1,...,n;\ \ j=0,...,\rho_1;\ \ 0\le\rho_1\le r-1)$ the coefficients and the knots of the optimal (best) quadrature formula with the weight function $x^\mu\,(\mu>-1)$ ([3,4])

$$\int_0^1 x^\mu f(x)\,dx=\sum_{k=1}^{n}\sum_{j=0}^{\rho_1}A_k^{(j)}f^{(j)}(x_k)+R_n(f), \tag{3}$$

where $f\in W_p^r$ and $0\le x_1<x_2<...<x_n\le 1$, and by $B_l^{(i)}$, y_l^* $(l=1,...,m;$ $i=0,...,\rho_2;\ \ 0\le\rho_2\le s-1)$ respectively the coefficients and knots of the optimal formula ([5,6])

$$\int_0^1 f(y)\,dy=\sum_{l=1}^{m}\sum_{i=0}^{\rho_2}B_l^{(i)}f^{i}(y_l)+R_m(f), \tag{4}$$

where $f\in W_p^s$ and $0\le y_1<...<y_m\le 1$.

Denote by δ_1 and δ_2 the exact bounds for remainders of the formulas (3) and (4), which are assumed to be finite, i.e.,

$$\begin{cases}\delta_1=\inf\limits_{\{A_n^j,x_n\}}\ \sup\limits_{f\in W_p^r}\left|R_n(f)\right|,\\ \delta_2=\inf\limits_{\{B_l^i,y_l\}}\ \sup\limits_{f\in W_p^s}\left|R_m(f)\right|.\end{cases} \tag{5}$$

2. CUBATURE FORMULAS BASED ON OPTIMAL QUADRATURE FORMULAS

It is shown in [6,7] that the optimal formula

$$\iint\limits_D f(x,y)\,dxdy=\sum_{k=1}^{l}\sum_{j=1}^{n}C_{kj}f(x_k,y_j)+R(f),$$

$$0\le x_1<...<x_l\le 1,\quad 0\le y_1<...<y_n\le 1,$$

on the set $W_p^{r,s}$ has the coefficients $C_{kj}=A_k^{(l)}\ B_j^{(n)}$ and the knots

$$(x_k^*,y_j^*)=(x_k^{(l)},y_j^{(n)}).$$

Now we shall show that some optimal cubature formulas with the weight function $x^\mu\,(\mu>-1)$ may be constructed as a product of the optimal formulas (3) and (4).

First, we shall find an optimal formula of the form

$$\iint\limits_D x^{\mu} f(x,y)\,dxdy = \sum_{k=1}^{n} \alpha_k \int_0^1 f(x_k,y)\,dy + \sum_{l=1}^{m} \beta_l \int_0^1 x^{\mu} f(x,y_l)\,dx$$

$$+\sum_{k=1}^{n}\sum_{l=1}^{m} \gamma_{kl}\, f(x_k,y_l) + E(f)\,, \tag{6}$$

$$0 \le x_1 < \ldots < x_n \le 1; \quad 0 \le y_1 < \ldots < y_m \le 1\,,$$

on the set $V_p^{r,s}$. In other words, we shall find the formula (6) with the minimal value of

$$E[V_p^{r,s}] = \sup_{f \in V_p^{r,s}} |E(f)|. \tag{7}$$

For $\mu = 0$ this formula was considered in [1].

Theorem 1. *The coefficients and knots*

$$\begin{gathered} \alpha_k = A_k^0\,,\ \beta_l = B_l^0\,,\ \gamma_{kl} = -A_k^0\, B_l^0\,, \\ x_k = x_k^*\,,\ y_l = y_l^*,\ k = 1,\ldots,n\,;\ l = 1,\ldots,m, \end{gathered} \tag{8}$$

and the exact bound of the remainder

$$E\,[V_p^{r,s}] = P\,\delta_1\,\delta_2 \tag{9}$$

are respectively the coefficients, knots, and the exact bound of the remainder of the optimal formula (6) *on the set* $V_p^{r,s}$.

Proof. It is known (see [6]) that the optimal formula (6) must statisfy the condition

$$\begin{gathered} E[\varphi(x)\, y^{\upsilon}] = E[\varphi(y)\, x^{\lambda}] = 0, \quad \forall \varphi(x)\, y^{\upsilon}, \quad \varphi(y)\, x^{\lambda} \in V_p^{r,s}, \\ \upsilon = 0,\ldots, s-1; \quad \lambda = 0,\ldots, r-1. \end{gathered} \tag{10}$$

Taking in (6) $f(x,y) = \varphi(x)\, y^{\upsilon}$, $f(x,y) = \varphi(y)\, x^{\lambda}$, where $\varphi(x)$ is an arbitrary function, we obtain by (10)

$$\sum_{l=1}^{m} \beta_l\, y_l^{\upsilon} = \frac{1}{\upsilon+1}, \quad \sum_{k=1}^{n} \alpha_k\, x_k^{\lambda} = \frac{1}{\mu+\lambda+1}\,,$$

$$\frac{\alpha_k}{\upsilon+1} = -\sum_{l=1}^{m} \upsilon_{kl}\, y_l^{\upsilon}, \quad \frac{\beta_l}{\mu+\lambda+1} = -\sum_{k=1}^{n} \upsilon_{kl}\, x_k^{\lambda}, \tag{11}$$

$$k = 1,\ldots,n; \quad l = 1,\ldots,m; \quad \upsilon = 0,\ldots, s-1; \quad \lambda = 0,\ldots, r-1.$$

Now let $f(x,y)\in V_p^{r,s}$, $\quad u_+^k=\begin{cases} u^k, & u>0 \\ 0, & u\le 0. \end{cases}$

By Taylor's formula

$$f(x,y)=\sum_{i=0}^{r-1}\psi_i(y)\,x^i+\sum_{j=0}^{s-1}\varphi_j(x)\,y^j$$

$$+\iint_D f^{(r,s)}(t,u)\,\frac{(x-t)_+^{r-1}}{(r-1)!}\,\frac{(y-u)_+^{s-1}}{(s-1)!}\,dtdu,$$

where

$$\psi_i(y)=\frac{1}{i!}f^{(i,0)}(0,y),$$

$$\varphi_j(x)=\frac{1}{j!}\int_0^1 f^{r,s}(t,0)\,\frac{(x-t)_+^{r-1}}{(r-1)!}\,dt.$$

Hence, by (10)

$$E(f)=\iint_D f^{r,s}(t,u)\,E\left(\frac{(x-t)_+^{r-1}\quad(y-u)_+^{s-1}}{(r-1)!\,(s-1)!}\right)dtdu$$

$$=\iint_D f^{(r,s)}(t,u)\,K(t,u)\,dtdu, \tag{12}$$

where $K(t,u)$ is a spline function,

$$K(t,u)=\frac{1}{(r-1)!}\frac{1}{(s-1)!}\Bigg[z(u)\,w(t)-z(u)\sum_{k=1}^{n}\alpha_k\,(x_k-t)_+^{r-1}$$

$$-w(t)\sum_{l=1}^{m}\beta_l\,(y_l\;u)_+^{s-1}-\sum_{k=1}^{n}\sum_{l=1}^{m}\gamma_{kl}\,(x_k-t)_+^{r-1}\,(y_l-u)_+^{s-1}\Bigg],$$

where $z(u)=\dfrac{(1-u)^s}{s}$ and $w(t)=\displaystyle\int_0^1 x^{\mu}(x-t)_+^{r-1}\,dx$.

The function $K(t,u)$ satisfies the condition

$$K^{(\lambda,0)}(0,y)\equiv K^{(0,\upsilon)}(x,0)\equiv 0,\quad \lambda=0,\ldots,r-1;\quad \upsilon=0,\ldots,s-1, \tag{13}$$

by the equalities (11).

By Hölder's inequality we obtain from (12) that

$$|E(f)|\le P\|K(\cdot,\cdot)\|_q. \tag{14}$$

For the function

$$f_0(x,t)=\frac{P}{\|K(\cdot,\cdot)\|_q^{q/p}}\iint\limits_D |K(t,u)|^{q-1}\ \operatorname{sgn} K(t,u)\frac{(x-t)_+^{r-1}(y-u)_+^{s-1}}{(r-1)!\,(s-1)!}\,dtdu$$

belonging to $V_p^{r,s}$, it follows from (12) that

$$E(f_0)=P\,\|K(\cdot,\cdot)\|_q.$$

Then we have from (14) the equality

$$E\,[V_p^{r,s}]=P\,\|K(\cdot,\cdot)\|_q. \tag{15}$$

Now we have to minimize this value. Let us consider only the function $K(t,u)$ satisfying the condition (13) and the splines

$$K_1(t)=\frac{1}{(r-1)!}\left[\int_0^1 x^{\mu}(x-t)_+^{r-1}\,dx-\sum_{k=1}^{n}\alpha_k(x_k-t)_+^{r-1}\right], \tag{16}$$

$$K_1^{(i)}(0)=0,\quad i=0,\dots,r-1,$$

$$K_2(t)=\frac{(1-u)^s}{s\,!}-\frac{1}{(s-1)!}\sum_{l=1}^{m}\beta_l(y_l-u)_+^{s-1} \tag{17}$$

$$K_2^{(j)}(0)=0,\quad j=0,\dots,s-1.$$

Let

$$\inf_{\{\alpha_k,\beta_k,\gamma_{kl},x_k,y_l\}}\|K(\cdot,\cdot)\|_q=\|K^*(\cdot,\cdot)\|_q,$$

$$\inf_{\{\alpha_k,x_k\}}\|K_1(\cdot)\|_q=\|K_1^*(\cdot)\|_q,$$

$$\inf_{\{\beta_l,y_l\}}\|K_2(\cdot)\|_q=\|K_2^*(\cdot)\|_q.$$

It is shown in [3–6] that

$$K_1^*(t)=\frac{1}{(r-1)!}\left[\int_0^1 x^{\mu}(x-t)^{r-1}\,dx-\sum_{k=1}^{n}A_k^0(x_k^*-t)_+^{r-1}\right],$$

$$K_2^*(t)=\frac{1}{(s-1)!}\left[\frac{(1-u)^s}{s}-\sum_{l=1}^{m}B_l^0(y_l^*-u)_+^{s-1}\right], \tag{18}$$

$$\|K_1^*(\cdot)\|_q=\delta_1,\quad \|K_2^*(\cdot)\|_q=\delta_2.$$

By the proof method used in [1,7] we have the equality

$$K^{*}(t,u)=K_1^{*}(t)\,K_2^{*}(t). \tag{19}$$

This, together with (18), gives us the values (8) and (9). As the formula (6) with coefficients and knots (8) satisfies the condition (10), the theorem is proved.

Much in the same manner as Theorem 4.2.3 in [6] the following theorem can be proved.

Theorem 2. *Among the set of cubature formulas*

$$\iint_D x^{\mu} f(x,y)\,dxdy=\sum_{k=1}^{n}\sum_{l=1}^{m}\ \sum_{i=0}^{\rho_1}\sum_{j=0}^{\rho_2} C_{kl}^{ij} f^{(i,j)}(x_k,y_l)+R_{mn}(f), \tag{20}$$

where

$$\mu>-1,\quad 0\le\rho_1\le r-1,\quad 0\le\rho_2\le s-1,\quad 0\le x_1<\ldots<x_n\le 1,$$
$$0\le y_1<\ldots<y_m\le 1.$$

The optimal formula for the set $\overline{W}_p^{\,r,s}$ *has the knots*

$$x_k=x_k^{*},\quad y_l=y_l^{*} \tag{21}$$

and the coefficients

$$\begin{gathered} C_{kl}^{ij}=A_k^i\,B_l^j, \\ k=1,\ldots,n;\quad l=1,\ldots,m;\quad i=0,\ldots,\rho_1;\quad j=0,\ldots,\rho_2. \end{gathered} \tag{22}$$

The exact bound for the remainder is

$$R_{\text{opt}}[\overline{W}_p^{\,r,s}]=\inf_{\{x_k,y_l,C_{kl}^{ij}\}}\ \sup_{f\in\overline{W}_p^{\,r,s}}\left|R_{mn}(f)\right|=M\delta_1+N\delta_2+P\delta_1\delta_2. \tag{23}$$

Finally let us find an efficient cubature formula for the calculation of double integrals over the domain of integration

$$D=\{(x,y):\ 0\le x\le 1;\ \ 0\le y\le x^{\mu},\ \ \mu>-1\}.$$

Let us assume that $f(x,x^{\mu}y)=F(x,y)\in\overline{W}_p^{\,r,s}$. Then, by changing the variables $y=x^{\mu}z,\ \ x=x,$ we obtain

$$\iint_D f(x,y)\,dxdy$$

$$= \int_0^1 \int_0^{x^\mu} f(x,y)\,dxdy = \int_0^1 \int_0^1 x^\mu f(x, x^\mu z)\,dxdz = \int_0^1 \int_0^1 x^\mu F(x,y)\,dxdy$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{m} \sum_{i=0}^{\rho_1} \sum_{j=0}^{\rho_2} C_{kl}^{ij} F^{(i,j)}(x_k, y_l) + R_{nm}(F). \tag{24}$$

By Theorem 2 the optimal cubature formula of the form (24) has the knots (21), the coefficients (22), and the exact bound of the remainder (23).

REFERENCES

1. Levin, M. On the approximate calculation of double integrals. *Math. Comput.*, 1983, **40**, 273–282.
2. Gismalla, D. A. Quadrature formula with weight function $x^{1/2}$. *Int. J. Comput. Math.*, 1989, **26**, 57–67.
3. Arro, V. Best quadrature formula with weight function x^α. *ENSV TA Toim. Füüs. Mat.*, 1975, **24**, 387–392 (in Russian).
4. Arro, V. The best quadrature formula with weight function x^α for set of functions W^2L_2. *Tallinna Polütehnil. Inst. Toim.*, 1976, 393, 3–14 (in Russian).
5. Nikolsky, S. M. *Quadrature Formulas*. Nauka, Moscow, 1980 (in Russian).
6. Levin, M. and Girshovich, J. *Optimal Quadrature Formulas*. Teubner, Leipzig, 1979.
7. Levin, M. and Girshovich, J. Extremal problems for cubature formulas. *Soviet Math. Dokl.*, 1977, **18**, 1355–1358.

KAALUFUNKTSIOONIGA KAHEKORDSE INTEGRAALI LIGIKAUDNE ARVUTAMINE

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On leitud antud funktsioonide hulgal optimaalsed kubatuurvalemid kaalufunktsiooniga x^μ $(\mu > -1)$ kahekordse integraali ligikaudseks arvutamiseks. Üks kubatuurvalemite tüüp sisaldab joonintegraale, mis on kergesti hinnatavad optimaalsete kvadratuurvalemite abil. Optimaalsete kubatuurvalemite sõlmed ja kordajad sisaldavad optimaalsete kvadratuurvalemite vastavaid väärtusi. On näidatud, kuidas saab kaalufunktsiooniga kubatuurvalemit kasutada astmefunktsiooniga piiratud integreerimispiirkonnaga kahekordse integraali ligikaudseks arvutamiseks.