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TEMPERATURE DISTRIBUTION IN A FINITE ATMOSPHERE SUBJECTED TO COSINE VARYING COLLIMATED FLUX

Tõnu VIIK

Tartu Observatory, 61602 Tõravere, Tartumaa, Estonia; viik@jupiter.aai.ee

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Abstract. The radiation field is calculated in an optically finite, two-dimensional, plane-parallel, absorbing–emitting but nonscattering grey atmosphere subjected to collimated cosine-varying incident boundary radiation. As in the optically semi-infinite case, we approximate the kernel of the integral equation for the emissive power by a sum of exponents. After this approximation the integral equation can again be solved exactly. The solution may be written in x and y functions which were introduced for a one-dimensional atmosphere and which are generalizations of the Chandrasekhar–Ambarzumian X and Y functions. An algorithm to find these functions is given.

This approximation allowed us to find the accurate values for the temperature distribution and the radiative flux at arbitrary optical depths in the atmosphere.

Key words: two-dimensional radiative transfer, X and Y functions, emissive power, radiative flux.

1. INTRODUCTION

In a previous paper [1] the radiative transfer in a two-dimensional, optically semi-infinite atmosphere subjected to collimated cosine radiation was studied. By applying the kernel approximation method to a simplified integral equation for the temperature distribution (or for the emissive power), it was possible to find the radiative field at any point in the atmosphere. In the present paper these results are generalized for an optically finite atmosphere, allowing for collimated cosine radiation incident on one of its boundaries. Breig and Crosbie, who have found the external radiation field both for optically semi-infinite and finite atmospheres [2,3], have already stressed that the cosine boundary conditions are not physically realistic. Nevertheless, these are useful since the solutions for other, more realistic problems can be expressed in terms of the cosine solutions.

As the main relations of a similar problem were described in [1], below the respective equations are referred to as Eq. (1,n), where n is the number of an equation used in that paper.

2. SOLUTION OF THE RADIATIVE TRANSFER EQUATION

The temperature distribution (or the emissive power) and the radiative flux are looked for in an optically finite, homogeneous, nonscattering but absorbing and emitting plane-parallel, two-dimensional, grey atmosphere which is in local thermodynamic equilibrium. The radiative transfer in such an atmosphere is described by Eq. (1,1).

By applying integrating factor techniques to Eq. (1,1), the formal relation for the intensity of radiation moving downwards is defined as Eq. (1,2) and the intensity of radiation moving upwards as

$$I^-(\tau_y, \tau_z, \mu, \tau_0) = \frac{1}{\pi} \int_{\tau_z}^{\tau_0} \overline{\sigma} T^4(\tau_y', \tau_z', \tau_0) \exp(-(\tau_z' - \tau_z)/\mu) \mathrm{d}\tau_z'/\mu, \tag{1}$$

where $\tau_y^+ = \tau_y - \tau_z \tan\theta \sin\phi$, $\tau_y' = \tau_y + (\tau_z' - \tau_z) \tan\theta \sin\phi$, $\mu = \cos\theta$, I_0^+ is the intensity incident on the boundary of the atmosphere, and τ_0 is the optical thickness of the atmosphere.

It is required again that the atmosphere be in radiative equilibrium, i.e., the relation expressed by Eq. (1,4) is effective.

The equation for the emissive power coincides with the respective equation for the optically semi-infinite atmosphere, Eq. (1,5), with the only difference that we have to integrate with respect to τ_z from 0 to τ_0. In the following it is assumed that no radiation is incident on the lower boundary of the optically finite atmosphere and the upper boundary is subjected to collimated cosine radiation

$$I_0^+(\tau_y^+) = I_0 \left[1 + \epsilon \exp(i\beta\tau_y^+)\right] \delta(\mu - \mu_0)\delta(\phi), \tag{2}$$

where I_0 is a constant, $(\mu_0 = \cos\theta_0, \phi)$ defines the direction of the incident collimated radiation, ϵ is the amplitude of the cosine wave, and δ is the Dirac delta function. Boundary condition (2) means that the top of the atmosphere is illuminated stripwise by a parallel beam at an angle θ_0, while the strips are parallel to the x-axis and their widths are defined by the spatial frequency β as π/β in units of optical length τ_y^+. The illumination in the direction parallel to the y-axis varies according to the cosine law.

By applying the concept of separation of variables to Eq. (1,1), it can be obtained that

$$\overline{\sigma} T^4(\tau_y, \tau_z, \tau_0) = \frac{1}{4} I_0 \left[B_{\beta=0}(\tau, \mu_0, \tau_0) + \epsilon B_\beta(\tau, \mu_0, \tau_0) \exp(i\beta\tau_y)\right], \tag{3}$$

where B_β is the dimensionless emissive power and $\tau = \tau_z$. Using Eq. (1,7) in Eq. (1,5) and taking into account the finiteness of the atmosphere, a simple integral equation for B_β is obtained in the form

$$B_\beta(\tau, \mu_0, \tau_0) = \exp(-\tau/\mu_0) + \frac{1}{2}\int_0^{\tau_0} \mathcal{E}_1(\tau - \tau')B_\beta(\tau', \mu_0, \tau_0)\mathrm{d}\tau', \tag{4}$$

where the generalized exponential integral $\mathcal{E}_1$ is defined as in Eq. (1,9). By manipulating Eq. (4) in the same manner as in [2], it can be shown that the integral equation for the resolvent function Φ_β has the form

$$\Phi_\beta(\tau, \tau_0) = \frac{1}{2}\mathcal{E}_1(\tau, \beta) + \frac{1}{2}\int_0^{\tau_0} \mathcal{E}_1(\tau - \tau')\Phi_\beta(\tau', \tau_0)\mathrm{d}\tau', \tag{5}$$

where

$$\Phi_\beta(\tau, \tau_0) = \frac{1}{2}\int_1^\infty \frac{B_\beta\left(\tau, \sqrt{t^2+\beta^2}, \tau_0\right)\mathrm{d}t}{\sqrt{t^2+\beta^2}}. \tag{6}$$

Next we introduce two functions $x_\beta(\tau, \mu_0, \tau_0)$ and $y_\beta(\tau, \mu_0, \tau_0)$ as follows [4]:

$$x_\beta(\tau, \mu_0, \tau_0) = 1 + \int_\tau^{\tau_0} \Phi_\beta(t, \tau_0)\exp(-(t-\tau)/\mu_0)\mathrm{d}t \tag{7}$$

and

$$y_\beta(\tau, \mu_0, \tau_0) = \exp(-\tau/\mu_0) + \int_0^\tau \Phi_\beta(t, \tau_0)\exp(-(\tau - t)/\mu_0)\mathrm{d}t. \tag{8}$$

The well-known Ambarzumian–Chandrasekhar X and Y functions are limiting values of these two functions,

$$X_\beta(\mu_0, \tau_0) = x_\beta(0, \mu_0, \tau_0), \tag{9}$$

and

$$Y_\beta(\mu_0, \tau_0) = y_\beta(\tau_0, \mu_0, \tau_0). \tag{10}$$

According to Sobolev [5], the solution of Eq. (4) may be written in the form

$$\begin{aligned} B_\beta(\tau, \mu_0, \tau_0) = X_\beta(\mu_0, \tau_0)&\left[\exp(-\tau/\mu_0) + \int_0^\tau \Phi_\beta(t, \tau_0)\exp(-(\tau - t)/\mu_0)\mathrm{d}t\right] \\ -Y_\beta(\mu_0, \tau_0)&\left[\int_0^\tau \Phi_\beta(\tau_0 - t, \tau_0)\exp(-(\tau - t)/\mu_0)\mathrm{d}t\right], \end{aligned} \tag{11}$$

or, in our notation,

$$B_\beta(\tau,\mu_0,\tau_0) = X_\beta(\mu_0,\tau_0)y(\tau,\mu_0,\tau_0) - Y_\beta(\mu_0,\tau_0)\left[x_\beta(\tau_0-\tau,\mu_0,\tau_0)-1\right]. \tag{12}$$

This is the formal solution of determining the emissive power (and the temperature distribution) in an optically finite atmosphere subjected to collimated cosine varying radiation.

Next we find the functions $x_\beta(\tau,\mu_0,\tau_0)$ and $y_\beta(\tau,\mu_0,\tau_0)$. We use the same technique as in [4]: first we approximate the kernel of Eq. (5) by a sum of exponents and after that the integral equation can be solved exactly. The solution of Eq. (5) is

$$\Phi_\beta(\tau,\tau_0) = \sum_{i=1}^{N}\left[a_i\exp(-s_i\tau) + b_i\exp(-s_i(\tau_0-\tau))\right]. \tag{13}$$

In order to determine the coefficients a_i, b_i, and s_i in Eq. (13), we use Eq. (13) in Eq. (5) and by equating the similar exponents we obtain the characteristic equation

$$1-2\sum_{i=1}^{N}\frac{w_i\Psi(\mu_i,\beta)}{1-\mu_i^2 s^2} = 0, \tag{14}$$

and a linear algebraic system for coefficients a_i and b_i

$$\begin{aligned}\sum_{k=1}^{N}\frac{a_k}{1-\mu_i s_k} + \sum_{k=1}^{N}\frac{b_k\exp(-s_k\tau_0)}{1+\mu_i s_k} - \mu_i^{-1} &= 0,\\ \sum_{k=1}^{N}\frac{a_k\exp(-s_k\tau_0)}{1+\mu_i s_k} + \sum_{k=1}^{N}\frac{b_k}{1-\mu_i s_k} &= 0, \qquad i=1,...,N.\end{aligned} \tag{15}$$

When analysing Eq. (14) we encounter an interesting feature. While our problem is completely similar to the respective one-dimensional case with pure scattering, i.e., no photon perishes in the act of scattering, there is still a remarkable difference. In the one-dimensional case Eq. (14) has a double zero $s_1 = 0$. This is not so in our case, and that is the reason why there are no polynomial terms in Eq. (13).

We have described in [4] how to solve Eqs. (14) and (15). For Eq. (15) we apply a simple change of unknowns

$$\bar{a}_k = a_k + b_k, \qquad \bar{b}_k = a_k - b_k, \qquad k=1,...,N, \tag{16}$$

which helps us to substitute a $2N$ by $2N$ system by two N by N systems

$$\begin{aligned}\sum_{k=1}^{N}\bar{a}_k\left[\frac{1}{1-\mu_i s_k} + \frac{\exp(-s_k\tau_0)}{1+\mu_i s_k}\right] - \mu_i^{-1} &= 0,\\ \sum_{k=1}^{N}\bar{b}_k\left[\frac{1}{1-\mu_i s_k} - \frac{\exp(-s_k\tau_0)}{1+\mu_i s_k}\right] - \mu_i^{-1} &= 0.\end{aligned} \tag{17}$$

These two systems may easily be solved by any of the well-known methods. It appears that for all $\tau_0 < \infty$ coefficients b_k are negative.

Taking into account Eqs. (7), (8), and (13), we find the approximations for the x_β and y_β functions

$$x_\beta(\tau, \mu_0, \tau_{0)} = \mu_0 \sum_{k=2}^{N} \frac{a_k \left[\exp(-s_k\tau) - \exp\left(-s_k\tau_0 - (\tau_0 - \tau)/\mu_0\right)\right]}{1 + s_k\mu_0}$$

$$+\mu_0 \sum_{k=2}^{N} \frac{b_k \left[\exp(-s_k(\tau_0 - \tau)) - \exp\left(-(\tau_0 - \tau)/\mu_0\right)\right]}{1 - s_k\mu_0} \qquad (18)$$

and

$$y_\beta(\tau, \mu_0, \tau_{0)} = \mu_0 \sum_{k=2}^{N} \frac{a_k \left[\exp(-s_k\tau) - \exp(-\tau/\mu_0)\right]}{1 - s_k\mu_0}$$

$$+\mu_0 \sum_{k=2}^{N} \frac{b_k \left[\exp(-s_k(\tau_0 - \tau)) - \exp(-s_k\tau_0 - \tau/\mu_0)\right]}{1 + s_k\mu_0}. \qquad (19)$$

Respectively, from Eqs. (10), (11), and (14) we obtain

$$X_\beta(\mu_0, \tau_{0)} = \mu_0 \sum_{k=2}^{N} \frac{a_k \left[1 - \exp(-\tau_0(s_k + 1/\mu_0))\right]}{1 + s_k\mu_0}$$

$$+\mu_0 \sum_{k=2}^{N} \frac{b_k \left[\exp(-s_k\tau_0) - \exp-(\tau_0/\mu_0)\right]}{1 - s_k\mu_0} \qquad (20)$$

and

$$Y_\beta(\mu_0, \tau_{0)} = \mu_0 \sum_{k=2}^{N} \frac{a_k \left[\exp(-s_k\tau_0) - \exp(-\tau_0/\mu_0)\right]}{1 - s_k\mu_0}$$

$$+\mu_0 \sum_{k=2}^{N} \frac{b_k \left[1 - \exp(-\tau_0(s_k + 1/\mu_0))\right]}{1 + s_k\mu_0}. \qquad (21)$$

It is easy to see that the singularities in Eqs. (18)–(21) at $\mu_0 = 1/s_k$ can be eliminated by substituting the respective term by $(\tau_0 - \tau)\exp\left(-s_k(\tau_0 - \tau)\right)/\mu_0$ in Eq. (18) or by $\tau \exp\left(-s_k\tau\right)/\mu_0$ in Eq. (19) or by $\tau_0 \exp\left(-s_k\tau_0\right)/\mu_0$ in Eq. (21).

This concludes the solution of the radiative transfer equation for the case considered.

3. AUXILIARY EQUATIONS

In deriving the formula for the radiative flux, we need some auxiliary equations connecting the $x_\beta(\tau,\mu_0,\tau_0)$ and $y_\beta(\tau,\mu_0,\tau_0)$ functions. The first set may be obtained by differentiating Eqs. (11) and (12) with respect to τ. As a result, we obtain the following equations

$$-\mu_0 \frac{\partial x_\beta(\tau,\mu_0,\tau_0)}{\partial \tau} + x(\tau,\mu_0,\tau_0) = \mu_0 \Phi_\beta(\tau,\tau_0) + 1, \tag{22}$$

$$\mu_0 \frac{\partial y_\beta(\tau,\mu_0,\tau_0)}{\partial \tau} + y(\tau,\mu_0,\tau_0) = \mu_0 \Phi_\beta(\tau,\tau_0). \tag{23}$$

We shall need also the derivatives of the $x_\beta(\tau,\mu_0,\tau_0)$ and $y_\beta(\tau,\mu_0,\tau_0)$ functions with respect to optical thickness. By differentiating Eqs. (11) and (12) with respect to τ_0, we obtain

$$\frac{\partial x_\beta(\tau,\mu_0,\tau_0)}{\partial \tau_0} = \Phi_\beta(\tau_0,\tau_0) y_\beta(\tau_0-\tau,\mu_0,\tau_0), \tag{24}$$

$$\frac{\partial y_\beta(\tau,\mu_0,\tau_0)}{\partial \tau_0} = \Phi_\beta(\tau_0,\tau_0)\left[x_\beta(\tau_0-\tau,\mu_0,\tau_0) - 1\right] \tag{25}$$

and then, writing Eqs. (11) and (12) for the argument $\tau_0 - \tau$ and again differentiating with respect to τ_0, we obtain

$$\frac{\partial x_\beta(\tau_0-\tau,\mu_0,\tau_0)}{\partial \tau_0} = \frac{1}{\mu_0}\left[x_\beta(\tau_0-\tau,\mu_0,\tau_0) - 1\right] + \Phi_\beta(\tau_0,\tau_0) y_\beta(\tau,\mu_0,\tau_0) - \Phi_\beta(\tau_0-\tau,\tau_0), \tag{26}$$

$$\frac{\partial y_\beta(\tau_0-\tau,\mu_0,\tau_0)}{\partial \tau_0} = -\frac{1}{\mu_0} y_\beta(\tau_0-\tau,\mu_0,\tau_0) + \Phi_\beta(\tau_0,\tau_0)\left[x_\beta(\tau,\mu_0,\tau_0) - 1\right] + \Phi_\beta(\tau_0-\tau,\tau_0). \tag{27}$$

Setting τ equal to zero in Eq. (24) and in Eq. (27), we obtain the well-known set for the X_β and Y_β functions

$$\frac{\partial X_\beta(\mu_0,\tau_0)}{\partial \tau_0} = \Phi_\beta(\tau_0,\tau_0) Y_\beta(\mu_0,\tau_0), \tag{28}$$

$$\frac{\partial Y_\beta(\mu_0,\tau_0)}{\partial \tau_0} = -\frac{1}{\mu_0} Y_\beta(\mu_0,\tau_0) + \Phi_\beta(\tau_0,\tau_0) X_\beta(\tau,\mu_0,\tau_0). \tag{29}$$

The value of the resolvent function at $\tau = \tau_0$ can be found from Eq. (11) of [2] and Eq. (12) in the form

$$\Phi_\beta(\tau_0, \tau_0) = \frac{1}{2}\int_0^p Y_\beta(\mu, \tau_0)\mathrm{d}\mu/\mu. \tag{30}$$

Equations (28) and (29) may be used to determine the X_β and Y_β functions. If we approximate the integral in Eq. (30) by a Gaussian sum, we obtain a set of ordinary differential equations together with boundary conditions which we get from Eqs. (7) and (8)

$$\begin{aligned} X_\beta(\mu_0, 0) &= 1, \\ Y_\beta(\mu_0, 0) &= 1. \end{aligned} \tag{31}$$

When we solve Eqs. (29)–(30) subject to boundary conditions (31), we obtain the values of the X_β and Y_β functions at the points of the Gauss quadrature only. If we are interested in values of these functions at other than Gaussian points, we may use the approach described by Breig and Crosbie [3]. The essence of this approach is that we solve the same equations at the values of μ_0 we are interested in, together with Eqs. (28) and (29). This means that if we need to know the values of $X_\beta(\mu_0, \tau_0)$ and $Y_\beta(\mu_0, \tau_0)$ functions at, say 10 values of μ_0, we have to solve the set of $2N + 10$ ordinary differential equations.

4. RADIATIVE FLUX

In this section we consider the formulation of the equations for the z-component of radiative flux in the atmosphere and respective calculations. According to [2], the z-component of radiative flux can be shown to satisfy the relationship

$$q_z(\tau_y, \tau, \tau_0) = I_0 Q_{\beta=0}(\tau, \mu_0, \tau_0) + \epsilon I_0 Q_\beta(\tau, \mu_0, \tau_0)\exp(i\beta\tau_y), \tag{32}$$

where the dimensionless radiative flux is given by

$$\begin{aligned} Q_\beta(\tau, \mu_0, \tau_0) = \mu_0 \exp(-\tau/\mu_0) &+ \tfrac{1}{2}\int_0^\tau \mathcal{E}_2(\tau - \tau', \beta)\, B_\beta(\tau', \mu_0, \tau_0)\mathrm{d}\tau' \\ &- \tfrac{1}{2}\int_\tau^{\tau_0} \mathcal{E}_2(\tau' - \tau, \beta)\, B_\beta(\tau', \mu_0, \tau_0)\mathrm{d}\tau'. \end{aligned} \tag{33}$$

In Eq. (33) the generalized second exponential integral is given by Eq. (1,56). Substituting Eq. (1,56) into Eq. (33), changing the order of integration, and taking into account Eqs. (1,2), (1), (22), and (23), we obtain

$$Q_\beta(\tau,\mu_0,\tau_0) = \mu_0 \exp(-\tau/\mu_0)$$

$$+\mu_0 X_\beta(\mu_0,\tau_0)\int_0^p \frac{u\psi_1(u,\beta)\mathrm{d}u}{\mu_0 - u}\left[y_\beta(\tau,\mu_0,\tau_0) - y_\beta(\tau,u,\tau_0)\right]$$

$$-\mu_0 Y_\beta(\mu_0,\tau_0)\int_0^p \frac{u\psi_1(u,\beta)\mathrm{d}u}{\mu_0 - u}\left[x_\beta(\tau_0-\tau,\mu_0,\tau_0) - x_\beta(\tau_0-\tau,u,\tau_0)\right]$$

$$-\mu_0 X_\beta(\mu_0,\tau_0)\int_0^p \frac{u\psi_1(u,\beta)\mathrm{d}u}{\mu_0 + u}\left[x_\beta(\tau,u,\tau_0) + y_\beta(\tau,\mu_0,\tau_0) - 1\right]$$

$$+\mu_0 Y_\beta(\mu_0,\tau_0)\int_0^p \frac{u\psi_1(u,\beta)\mathrm{d}u}{\mu_0 + u}\left[x_\beta(\tau_0-\tau,\mu_0,\tau_0) + y_\beta(\tau_0-\tau,u,\tau_0) - 1\right], \tag{34}$$

where

$$\psi_1(u,\beta) = \frac{1}{2\left(1-\beta^2u^2\right)^{3/2}} \tag{35}$$

and

$$p = \frac{1}{\sqrt{1+\beta^2}}. \tag{36}$$

The radiative flux at the boundaries of an atmosphere is thus

$$Q_\beta(0,\mu_0,\tau_0)$$

$$= \mu_0 - \mu_0 \int_0^p \frac{u\psi_1(u,\beta)\mathrm{d}u}{\mu_0+u}\left[X_\beta(u,\tau_0)X_\beta(\mu_0,\tau_0) - Y_\beta(u,\tau_0)Y_\beta(\mu_0,\tau_0)\right] \tag{37}$$

and

$$Q_\beta(\tau_0,\mu_0,\tau_0) = \mu_0 \exp(-\tau_0/\mu_0)$$

$$-\mu_0 \int_0^p \frac{u\psi_1(u,\beta)\mathrm{d}u}{\mu_0-u}\left[Y_\beta(u,\tau_0)\,X_\beta(\mu_0,\tau_0) - X_\beta(u,\tau_0)\,Y_\beta(\mu_0,\tau_0)\right]. \tag{38}$$

The apparent singularities in Eqs. (34) and (38) can be eliminated by means of the L'Hospital rule, since the derivatives of the x_β and y_β functions (or X_β and Y_β functions) with respect to μ_0 may easily be found. However, the expressions of these derivatives are rather unwieldy [4].

5. NUMERICAL RESULTS

In the numerical experiments we have used the same quadrature scheme as in [1], i.e., we divided the integration range $(0, p)$ into four subintervals $(0, 0.1p)$, $(0.1p, 0.9p)$, $(0.9p, 0.99p)$, and $(0.99p, p)$ and in each subinterval we used the Gaussian rule with $N/4$ points. To ensure the accuracy of at least five significant figures in the region $0 \leq \beta \leq 10\,000$ we set $N = 84$.

Contrary to Breig and Crosbie [2] who used the integration range $(0, 1)$, we chose the integration range $(0, p)$, because this approach, though a little bit more time-consuming in calculations, is free of awkward angle variables of the type $\sqrt{1+\beta^2}/\mu$.

Using the method described above, the temperature distribution and the radiative flux were found according to Eqs. (3), (12), and (35) for different sets of input parameters.

Figure 1 shows the temperature distribution as a function of the optical depth τ and the spatial frequency β, while the radiation is incident perpendicularly on the atmosphere of optical thickness $\tau_0 = 1$. It is clearly seen that for small values of β there exists a maximum of the temperature distribution at quite small optical depths which disappears when $\beta > 10$. In the case of optically thicker atmospheres, this behaviour remains the same, as shown in Fig. 2. The asymptotic behaviour of the temperature distribution surface indicates that for $\beta \to 0$ the solution is approaching the one-dimensional solution [6] as it should.

At the grazing incidences only the outermost layers define the radiation field for optically thicker atmospheres, since the temperature distribution decreases rapidly towards zero when the optical depth increases (Fig. 2).

The clearly defined maximum at normal incidence disappears for grazing incidence. Figures 3 and 4 show evidence of maxima at $\mu_0 = 0.3$ neither for $\tau_0 = 1$ nor for $\tau_0 = 10$. For optically thicker atmospheres $(\tau_0 = 10)$ the rapid fall-off of the temperature distribution is very conspicuous.

If the collimated radiation is incident perpendicularly on the atmosphere, then the dimensionless radiative flux $Q_\beta(\tau, \mu_0, \tau_0)$ shows monotonous behaviour in optically not so thick atmospheres $(\tau_0 \leq 1)$. There is a strange feature to that behaviour, though. For small optical depths the radiative flux increases with the spatial frequency β and reaches an asymptotic value at large values of β, but for large optical depths this distribution is the opposite: the radiative flux decreases with β and reaches again an (different) asymptotic value. At the same time, the radiative flux at the values of the spatial frequency $\log\, \beta < -0.5$ is constant throughout the atmosphere (Fig. 5).

For oblique incidence $(\arccos\, \mu_0 = 72^\circ.54)$ the dimensionless radiative flux shows a maximum when considering the flux as a function of the spatial frequency β. For an atmosphere of optical thickness of $\tau_0 = 10$ the maximum for the emerging at $\tau = \tau_0$ flux occurs at $\log\, \beta = -0.2$ (Fig. 6).

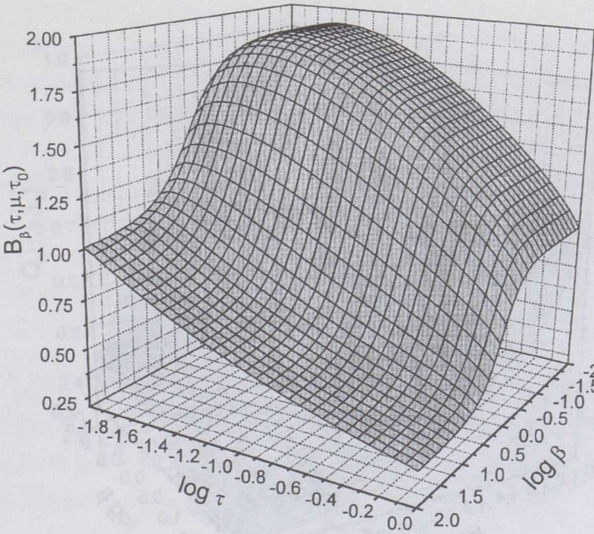

Fig. 1. The dimensionless temperature distribution $B_\beta(\tau, \mu, \tau_0)$ as a function of the optical depth τ and the spatial frequency β for the atmosphere with the optical thickness of $\tau_0 = 1$. The collimated radiation is incident on the atmosphere at the angle of 0°. In Figs. (1)–(6) the index 0 of the angular variable μ is dropped for simplicity.

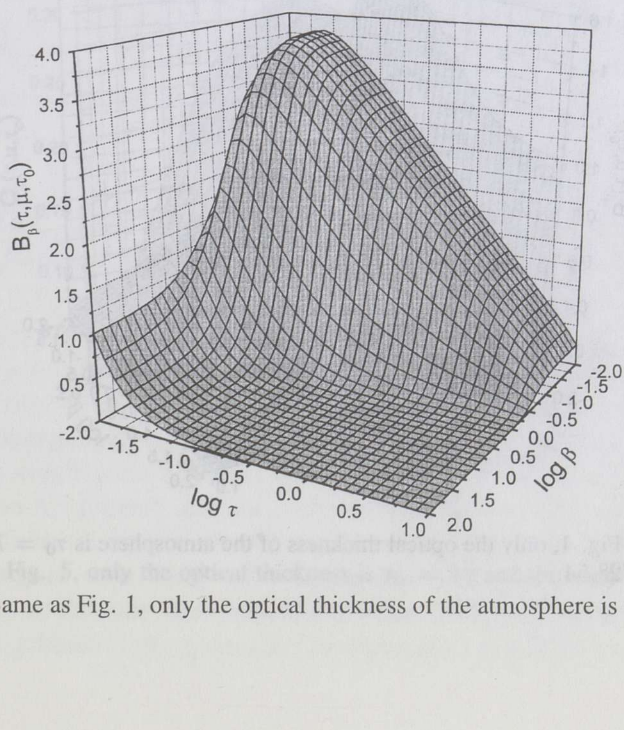

Fig. 2. Same as Fig. 1, only the optical thickness of the atmosphere is $\tau_0 = 10$.

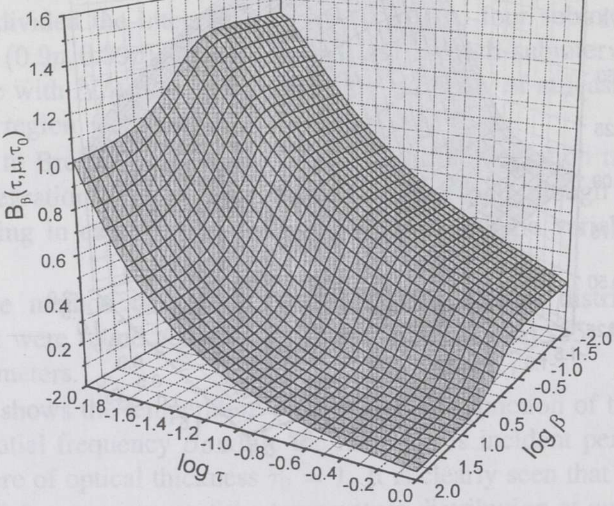

Fig. 3. Same as Fig. 1, only the angle of incidence is 72° .54.

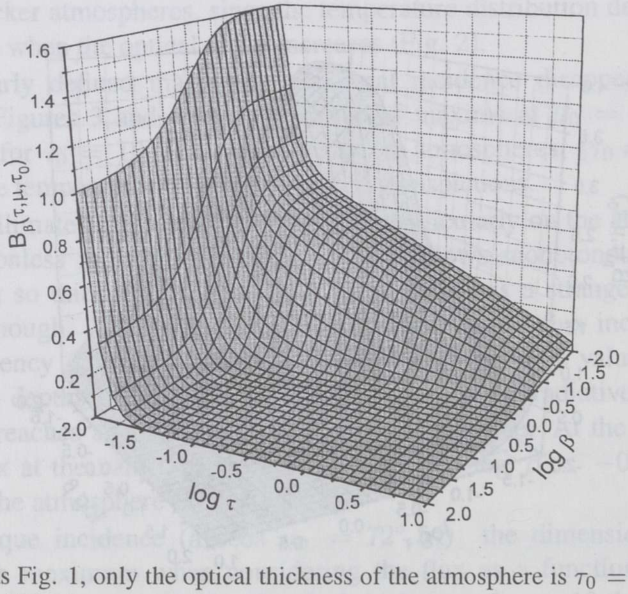

Fig. 4. Same as Fig. 1, only the optical thickness of the atmosphere is $\tau_0 = 10$ and the angle of incidence is 72° .54.

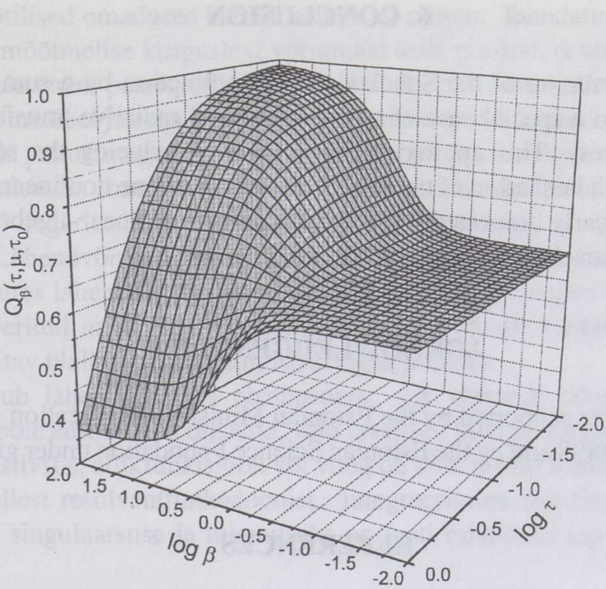

Fig. 5. The dimensionless radiative flux $Q_\beta(\tau, \mu, \tau_0)$ as a function of the optical depth τ and the spatial frequency β for the atmosphere with the optical thickness of $\tau_0 = 1$. The collimated radiation is incident on the atmosphere at the angle of $0°$.

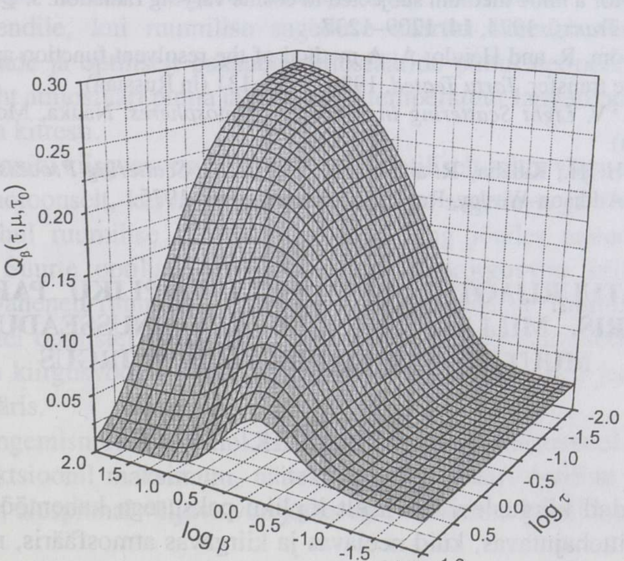

Fig. 6. Same as Fig. 5, only the optical thickness is $\tau_0 = 10$ and the angle of incidence is $72°.54$.

6. CONCLUSION

The approximation of the Sobolev resolvent function by a sum of exponents can be applied to a special case of two-dimensional radiative transfer in optically finite atmospheres. This approximation allows of reducing the solution of the integro-differential equation of transfer to a solution of one nonlinear characteristic equation with clearly bracketed roots and to two sets of linear algebraic equations. This simple approach gives accurate and reliable results.

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TEMPERATUURIJAOTUS OPTILISELT LÕPLIKU PAKSUSEGA ATMOSFÄÄRIS, MILLELE LANGEB KOOSINUSSEADUSE JÄRGI MUUTUV KOLLIMEERITUD KIIRGUS

Tõnu VIIK

On vaadeldud kiirguslevi optiliselt lõpliku paksusega kahemõõtmelises tasaparalleelses mittehajutavas, kuid neelavas ja kiirgavas atmosfääris, millele langeb koosinusseaduse järgi muutuv kollimeeritud kiirgus. Samuti kui eelmises artiklis (Eesti TA Toim. Füüs. Matem., 2000, **49**, 40–57) on oletatud, et atmosfäär on hall ja ta on kiirguslikus tasakaalus. Sel juhul saab kiirguslevi võrrandi taandada integraalvõrrandiks, mille omakorda saab muutujate eraldamise teel taandada suhteliselt lihtsaks ühemõõtmeliseks integraalvõrrandiks, kui oletada, et

atmosfääri optilised omadused x-telje suunas ei muutu. Taandatud võrrand erineb tavalisest ühemõõtmelise kiirguslevi võrrandist selle poolest, et tema karakteristlik funktsioon pole enam paarispolünoom, kuid siiski paarsuse säilitanud funktsioon. Nagu ühemõõtmelisel juhulgi, saab selle integraalvõrrandi lahendamiseks kasutada meetodit, mille puhul integraalvõrrandi tuum lähendatakse eksponentide reaga. See lähend on kahtlemata ebatäpne, sest üldistatud eksponentfunktsioonil on argumendi väärtusel null logaritmiline singulaarsus, kuid eksponentide rida pole kusagil singulaarne. Lähendvõrrandil on siiski üks suurepärane omadus, nimelt ta lahendub täpselt, kusjuures lahendiks on samuti eksponentide rida. Seejärel on funktsioonid x ja y defineeritud nagu ühemõõtmelisel juhulgi. Nende funktsioonide abil on kiirgusväli leitav ülalkirjeldatud atmosfääri igas punktis.

Mis puutub lähendvõrrandi ebatäpsusse, siis võrrandi lahend – resolventfunktsioon – pole eesmärgiks omaette, vaid tavalised huvipakkuvad suurused, nagu kiirguse intensiivsus, allikfunktsioon või voog on kõik teatud kaalufunktsioonidega integraalid sellest resolventfunktsioonist. Integreerimine aga silub koordinaatide alguses oleva singulaarsuse ja tulemuseks on igati rahuldava täpsusega vajalikud suurused.

Numbrilised eksperimendid parameetrite erinevate väärtuste puhul näitasid, et temperatuuri jaotusfunktsioon võib üsna väikestel optilistel sügavustel saavutada maksimumi, kui kiirgus langeb atmosfääri pinnale risti või peaaegu risti ja langeva kiirguse ruumiline sagedus on väiksem kui 10. Suuremate ruumiliste sageduste ja suuremate langemisnurkade puhul maksimum kaob. Temperatuuri jaotusfunktsioon näitab selgesti asümptootset lähenemist ühemõõtmelisele ülesande lahendile, kui ruumilise sageduse väärtus läheneb nullile. Suurte langemisnurkade ja optiliselt paksude atmosfääride puhul defineerib kiirgusvälja vaid õhuke kiht atmosfääri pinna lähedal, sest temperatuur langeb optilise sügavuse kasvades väga kiiresti.

Dimensioonitu kiirgusvoog käitub sellistes optiliselt mitte väga paksudes atmosfäärides monotoonselt, kasvades suurte ruumiliste sageduste ja väikeste optiliste sügavuste puhul ruumilise sageduse kasvuga ning jõudes teatud asümptootse väärtuseni. Suurte optiliste paksuste puhul on kiirgusvoo käik vastupidine: kiirgusvoog väheneb ruumilise sageduse kasvuga väikeste ruumiliste sageduste juures väikestel optilistel sügavustel, jõudes jälle (teise) asümptootse väärtuseni. Samal ajal on kiirgusvoog ruumiliste sageduste $\log \beta < -0,5$ jaoks konstantne kogu atmosfääris.

Suurte langemisnurkade puhul on dimensioonitul kiirgusvool kui ruumilise sageduse funktsioonil maksimum, näiteks kui atmosfääri optiline paksus on 10, siis atmosfääri aluspinnal väljuva voo jaoks on maksimum kohal $\log \beta = -0,2$.