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ONE-STEP MEMORY CLOSED-LOOP CONTROL OF LINEAR INTERCONNECTED SUBSYSTEMS

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Abstract. An example of one-step memory control of the linear-quadratic grouped system is presented. It is shown that in the case of unidirectional interconnections between cost functions, one subsystem can indirectly control the behaviour of the whole system and can establish the trajectory it desires only through a properly chosen equivalent representation of its control law in cost functions of other subsystems.

Key words: incentive control, equivalent representation.

1. INTRODUCTION

The control actions performed by layered controllers depend on the information available. The open-loop information structure means that only the initial state is known. To distinguish between the actions depending on the nonzero and the zero memory information, the former are produced by closed-loop control laws (may use all past state vectors) and the latter by a feedback control law (uses only the current state). The information structures with nonzero memory are rarely used in deterministic control strategies.

In this paper a linear-quadratic example of one-step memory intervention in the local cost functions of the grouped system is presented. It is shown that in the case of unidirectional interconnections between cost functions, one subsystem can establish the trajectory it desires for the whole system only through properly chosen equivalent representation of its (unchanged) control law. The selected representation of the control law may be implemented in the cost function of other subsystems, in the state equation, or in both of them. In this paper only the first possibility is considered. The algorithm is based on a short communication [1]. A similar problem has been discussed in [2] and [3] in the context of closed-loop Stackelberg solution of two-person nonzero-sum dynamic games. In [2,3] conditions are obtained under which a one-step memory strategy of the leader (implemented in the state equation and in the cost function) forces the follower to a team-optimal solution.

In the present communication the group of subsystems with no decision-making priorities is studied. The problem is posed in the context of an arbitrary desired trajectory of the whole system. The situation shows some similarity to the sponsoring system (where the grant-holder's behaviour is also controlled indirectly by the attached prescriptions) and to intelligent controllers (multi-layered architecture like [4] may cause problems with the coordination of goals).

2. DESCRIPTION OF THE PROBLEM

Consider a linear discrete-time object governed by m controllers

$$x_{k+1} = Ax_k + B_1 u_{1,k} + \sum_{i=2}^{m} B_i u_{i,k}. \tag{2.1}$$

The controls $u_{i,k}$ are unconstrained and minimize the local cost function

$$J_i = x_n' Q_i x_n + \sum_{k=0}^{n-1} (x_k' Q_i x_k + u_{1,k}' R_{i1} u_{1,k} + u_{i,k}' R_i u_{i,k}), \quad i = 2,\ldots,m,$$

$$J_1 = x_n' Q_1 x_n + \sum_{k=0}^{n-1} (x_k' Q_1 x_k + u_{1,k}' R_1 u_{1,k}), \tag{2.2}$$

at a given initial state x_0. The matrices $A, B_i,\ Q_i \geq 0,\ R_i > 0,\ R_{i1} > 0$ are of appropriate dimensions; they are prescribed to all controllers by the higher level of decision-making.

Suppose the controllers are currently involved in the determination of the optimal trajectory under the restriction that the minimization of cost functions must be accomplished with no priorities in the order of decision-making. As is known from the basic literature, the optimal controls for this open-loop problem are linear functions of the current state. In other words, the open-loop optimal controls are represented in the feedback form:

$$u_{i,k} = -K_{i,k} x_k, \quad k = 0,1,\ldots,n-1. \tag{2.3}$$

Also, the implementation of (2.3) establishes the Nash-type equilibrium in the grouped system.

The main goal of this paper is to expose the additional possibilities of the first subsystem to control the trajectory x_k, $k = 0,1,\ldots,n-1$, by changing the representation (2.3) of $u_{1,k}$ in local cost functions (2.2). Since $R_{i1} > 0$ for

$i = 2,\ldots,m$ and $R_{11} = 0$, the first subsystem actually possesses the formal ability for such kind of intervention.

Suppose the controller of the first subsystem (as well as of all others) can model the behaviour of the grouped system (2.1) subject to the changes of the parameters of local goals (2.2) and can create feasible trajectories of the system. Suppose now the first subsystem decides to establish in the system one of these trajectories, say x_k°, $k = 0,1,2,\ldots,n$, which is obtained by the feedback control law

$$u_{i,k}^\circ = -K_{i,k}^\circ x_k. \tag{2.4}$$

In other words, it decides to establish in the whole system the trajectory

$$x_k^\circ = A_{k-1} x_{k-1}^\circ, \quad x_0^\circ = x_0,$$

where

$$A_{k-1} = A - \sum_{i=1}^{m} B_i K_{i,k-1}^\circ.$$

It is assumed henceforth that each set of selected controls admits a unique desired trajectory and a unique final value of each J_i.

As was stated in (2.2), the first subsystem must expose its control law in the local cost functions of other subsystems. It can use different ways, for example, pick up some equivalent nonzero-memory form. We adopt the one-step memory representation

$$u_{1,k}^{i\circ} = -K_{1,k}^\circ x_k + P_{i,k}(x_k - A_{k-1} x_{k-1}). \tag{2.5}$$

This equation is also used as one-step-ahead expectation in systems with filters [5]. In the current nonstochastic case it is simply an identity. The index i points to the subsystem, for the cost function of which the particular representation (2.5) is prepared, and $P_{i,k}$ is an arbitrary nonzero weight matrix.

Now each subsystem $i = 2,\ldots,m$, must (independently of other subsystems) find the optimal controls $u_{i,k}$, $k = 0,1,2,\ldots,n-1$, such that the local cost function

$$J_i = x_n' Q_i x_n + \sum_{k=1}^{n-1} \left(x_k' Q_i x_k + (u_{1,k}^{i\circ})' R_{i,1} (u_{1,k}^{i\circ}) + u_{i,k}' R_i u_{i,k} \right)$$
$$+ x_0' Q_i x_0 + x_0' K_{i,0}'^\circ R_{i1} K_{i,0}^\circ x_0 + u_{i,0}' R_i u_{i,0} \tag{2.6}$$

is minimal subject to

$$x_{k+1} = A x_k - \sum_{j=1,\, j\neq i}^{m} B_j K_{j,k}^\circ x_k + B_i u_{i,k}, \quad x_0 = x_0^\circ,\ k = 1,2,\ldots,n-1, \tag{2.7}$$

where

$$u_{1,k}^{i\circ} = -K_{1,k}^{\circ} x_k + P_{i,k}^{*}(x_k - A_{k-1}x_{k-1}),$$

and

$$A_{k-1} = A - \sum_{j=1}^{m} B_j K_{j,k-1}^{\circ}.$$

All design parameters, the initial state and controls but $u_{i,k}$ are known and fixed.

The weight matrices $P_{i,k}^{*}$ must cause significant loss in the final value of the local cost function if any deviation from the desired trajectory takes place, i.e.,

$$J_i(u_{i,0}^{\circ},\ldots,u_{i,N-1}^{\circ}) \le J_i(u_{i,0}^{\circ},\ldots,u_{i,k},u_{i,k+1}^{\circ},\ldots,u_{i,N-1}^{\circ}), \quad i = 2,\ldots,m .$$

If the first subsystem can find such weight matrices, then it is able to force other subsystems to choose exactly the prescribed control laws and implement the desired trajectory.

3. DETERMINATION OF OPTIMAL WEIGHT MATRICES

We assume that for every subsystem there exists a sequence of optimal weight matrices $P_{i,k}^{*}$ under which the final value of the local cost function J_i, calculated along the desired trajectory, is minimal. In this case at every stage k of the n-stage process the value of the vector of partial derivatives of $J_{i,k}$ with respect to the control at the desired trajectory is equal to zero:

$$\partial J_{i,k} / \partial u_{i,k} = 0 \text{ at } u_{i,k} = K_{i,k}^{\circ} x_k^{\circ}, \; k = n-2, n-3, \ldots, 1, 0, \; i = 2, 3, \ldots, m.$$

Suppose for a moment that all optimal weight matrices $P_{i,n-2}^{*}, P_{i,n-3}^{*}, \ldots, P_{i,3}^{*}, P_{i,2}^{*}$ are already found except for the last one, $P_{i,1}^{*}$.

The local cost function (2.6) is exposed as a function of the free control variable $u_{i,0}$:

$$J_i = x_n' Q_i x_n + \sum_{k=1}^{n-1} \left(x_k' Q_i x_k + (u_{1,k}^{i\circ})' R_{i1} (u_{1,k}^{i\circ}) + u_{i,k}'^{\circ} R_i u_{i,k}^{\circ} \right)$$
$$+ x_0' Q_i x_0 + (u_{1,0}'^{i\circ})' R_{i1} (u_{1,0}^{i\circ}) + u_{i,0}' R_i u_{i,0}, \; i = 2, \ldots, m, \qquad (3.1)$$

where

$$u^{i\circ}_{1,0} = -K^{\circ}_{1,0} x^{\circ}_0,$$

$$u^{i\circ}_{1,k} = (P^*_{i,k} - K^{\circ}_{1,k}) x_k - P^*_{i,k} A_{k-1} x_{k-1},$$

$$u^{\circ}_{i,k} = -K^{\circ}_{i,k} x_k,$$

and

$$x_1 = Ax_0 - \sum_{j=1, j\neq i}^{m} B_j K^{\circ}_{j,0} x_0 + B_i u_{i,0}, \; x_0 = x^{\circ}_0, \tag{3.2}$$

$$x_{k+1} = A_k x_k, \; k = n-2, n\ 3, \dots, 2, 1,$$

where

$$A_k = A - \sum_{j=1}^{m} B_j K^{\circ}_{j,k}.$$

The cost function J_i is a quadratic function of the initial state x_0 and the control variable $u_{i,0}$. Its partial derivative $\partial J_i / \partial u_{i,0} = W_{i,0}(P^*_{i,1}) x^{\circ}_0$ must be zero at any initial state $x^{\circ}_0 \neq 0$, so $W_{i,0}(P^*_{i,1}) = 0$. This equation determines $P^*_{i,1}$ if all other weight matrices are known and $W_{i,0}(P^*_{i,1}) = 0$ has a solution.

As the desired trajectory is optimal (with respect to some linear-quadratic control problem), then its remaining part is also optimal. So the weight matrices could be found stage by stage from the end of the process by using the described approach. At the last, nth stage, the cost function does not depend on the controls, so the corresponding weight matrices are missing. At the $(n-1)$th stage the weight matrices cannot influence the final value of the cost function and are also absent.

At the next stage we can find the expression $W_{i,n-2}(P_{i,n-1})$ we are searching for. The problem (3.1), (3.2) at the last stages is

$$J_{i,n-2} = x'_n Q_i x_n + x'_{n-1} Q_i x_{n-1}$$
$$+ \left((P_{i,n-1} - K^{\circ}_{1,n-1}) x_{n-1} - P_{i,n-1} A_{n-2} x_{n-2} \right)' R_{i1} \left((P_{i,n-1} - K^{\circ}_{1,n-1}) x_{n-1} - P_{i,n-1} A_{n-2} x_{n-2} \right)$$
$$+ x'_{n-1} K'^{\circ}_{i,n-1} R_i K^{\circ}_{i,n-1} x_{n-1} + x'_{n-2} (Q_i + K'^{\circ}_{1,n-2} R_{i1} K^{\circ}_{1,n-2}) x_{n-2} + u'_{i,n-2} R_i u_{i,n-2}.$$

The restrictions in hand are

$$x_n = A_{n-1}(A_{n-2} + B_i K^{\circ}_{i,n-2}) x_{n-2} + A_{n-1} B_i u_{i,n-2},$$
$$x_{n-1} = (A_{n-2} + B_i K^{\circ}_{i,n-2}) x_{n-2} + B_i u_{i,n-2},$$
$$x_{n-2} = x^{\circ}_{n-2}.$$

After using $-K^{\circ}_{i,n-2}x_{n-2}$ instead of optimal $u_{i,n-2}$ in $\partial J_{i,n-2}/\partial u_{i,n-2}$ and substantial simplifications, we get the condition $W_{i,n-2}(P_{i,n-1})$ in the following form:

$$B_i'P_{i,n-1}'(B_1'Q_iA_{n-1}-R_{i1}K^{\circ}_{1,n-1})A_{n-2}=$$

$$R_iK^{\circ}_{i,n-2}-B_i'(Q_i+A_{n-1}'Q_iA_{n-1}+K'^{\circ}_{1,n-1}R_{i1}K^{\circ}_{1,n-1}+K'^{\circ}_{i,n-1}R_iK^{\circ}_{i,n-1})A_{n-2}.$$

Repeating the same steps at other stages, we will obtain the relatively simple matrix equation

$$B_i'(S_{i,k}-P_{i,k}'R_{i1}K^{\circ}_{1,k})A_{k-1}=R_iK^{\circ}_{i,k-1}, \tag{3.3}$$

which determines the optimal $P^*_{i,k}$. The matrix $S_{i,k}$ is calculated recursively from the end of the process:

$$S_{i,k}=Q_i+A_k'S_{i,k+1}A_k+K'^{\circ}_{1,k}R_{i1}K^{\circ}_{1,k}+K'^{\circ}_{i,k}R_iK^{\circ}_{i,k},\quad S_{i,n}=Q_i,$$
$$k=n-1,\ldots,1,\ \ i=2,\ldots,m. \tag{3.4}$$

We can conclude that if $R_{i1}>0$, $K^{\circ}_{1,k}>0$, and $B_i'S_{i,k}A_{k-1}\neq R_iK^{\circ}_{i,k-1}$, then there exists a sequence of optimal weight matrices $P^*_{i,k}$, given by (3.4) and (3.3) such that $\partial J_{i,k}/\partial u_{i,k}=0$ at $u_{i,k}=-K^{\circ}_{i,k}x_k$.

As expected, P^*_i is a constant matrix if n is sufficiently large.

4. ILLUSTRATIVE EXAMPLE

Let us take a scalar system consisting of three subsystems:

$$x_{k+1}=ax_k-K^{\circ}_1x_k+u_{2,k}+u_{3,k},$$

$$J_i=\sum_{k=0}^{n-1}\left(2x_k^2+2(u^{i0}_{1,k})^2+u^2_{i,k}\right),\ i=2,3,$$

where

$$u^{i0}_{1,k}=-K^{\circ}_1x_k+P^*_i\left(x_k-(a-K^{\circ}_1-K^{\circ}_2-K^{\circ}_3)x_{k-1}\right),$$

and

$$x_0=30,\quad a=0.9,\quad n=16.$$

Table 1 gives the final values of $J_3(\alpha K^{\circ}_3)-2x_0^2$ (calculated at $\alpha=\{0.8;0.9;1.0;1.1;1.2\}$) for six different desired trajectories (determined by the values of K_1°, K_2°, K_3°, and x_0). Optimal weights P_i^* satisfy Eqs. (3.3) and (3.4).

Table 1. Results of calculations

K_1°	K_2°	P_2^*	K_3°	P_3^*	$J_3(\alpha K_3^\circ) - 2x_0^2$				
					$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$	$\alpha = 1.1$	$\alpha = 1.2$
0.2	0.05	7.24	0.1	7.04	949	910	897	909	946
0.2	0.05	6.69	0.15	6.26	810	744	723	743	803
0.2	0.05	6.25	0.2	5.53	707	620	592	619	697
0.2	0.15	5.76	0.1	6.00	591	567	558	567	590
0.2	0.15	5.32	0.15	5.32	508	466	453	466	505
0.2	0.15	4.92	0.2	4.61	446	391	374	391	442

The first subsystem can actually manipulate the trajectory of the whole system and the behaviour of other subsystems in a wide range only by carefully selected weights in the equivalent representation of the same control law.

5. CONCLUDING REMARKS

It was shown that in the case of unidirectional interconnections between the cost functions of one subsystem and of the other subsystems, the former subsystem can cause significant loss in the final values of the local cost functions of other subsystems. That subsystem is able to force other subsystems to choose the control laws which establish the trajectory it desires for a grouped system. Moreover, the desired trajectory can be enforced only through the properly chosen equivalent representation of the (same) closed-loop control law of that subsystem. It was also shown that in the case of one-step memory representations the sequence of optimal weight matrices for any desired trajectory is determined step by step from the end of the process by one matrix equation.

The equivalent representation of a control law can be inserted also into the state equation. In this case the optimal weights are determined by the system of recursive matrix equations [1]. Implementation of equivalent representations in the state equation means, in fact, direct intervention of the first subsystem if any (unintentional) deviation from the desired trajectory takes place.

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LINEAARSETE ALAMSÜSTEEMIDE KOGUMI JUHTIMINE ÜHESAMMULISE MÄLUGA MÕJUTUSTE ABIL

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On käsitletud lineaarsete seostatud alamsüsteemide kogumi juhtimist, kus ühe (näiteks esimese) alamsüsteemi juhttoime sõltub kaudselt kõigi teiste alamsüsteemide juhttoimetest. Kuna see alamsüsteem on teadlik ülejäänute reageeringutest, siis võib ta oma juhttoime valikuga mõjutada kogu süsteemi käitumist soovitud suunas. On vaadeldud kaudse juhtimise sellist varianti, kus esimene alamsüsteem võib valida liitsüsteemi suvalise trajektoori ning sundida teisi sellest kinni pidama, muutes vaid oma juhtimisseaduse esitust ülejäänute sihifunktsioonides. On leitud algoritm juhtimisseaduse ekvivalentse esituse optimaalse variandi määramiseks.