Proc. Estonian Acad. Sci. Phys. Math., 1999, **48**, 2, 90–100
https://doi.org/10.3176/phys.math.1999.2.03

THE NUMERICAL SOLUTION OF WEAKLY SINGULAR FIRST-KIND VOLTERRA INTEGRAL EQUATIONS WITH DELAY ARGUMENTS

Hermann BRUNNER

Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada A1C 5S7; hermann@math.mun.ca

Received 6 October 1998

Abstract. We analyze the numerical solution of a class of weakly singular neutral functional integro-differential equations and related Volterra integral equations with delay arguments by collocation methods in spaces of piecewise polynomials. It is shown that many problems (choice of collocation points, local superconvergence) remain open.

Key words: Volterra integral equations of the first kind, weakly singular kernel, delays, collocation methods.

1. INTRODUCTION

It is well known that "classical" numerical methods like linear multistep methods and collocation methods on uniform meshes exhibit a drastic reduction of their orders when applied to Volterra integral or integro-differential equations whose kernels contain a weak (integrable) singularity (see, for example, [[1–5]]). This order reduction occurs also in second-kind Fredholm integral equations with weakly singular kernels ([[6]]; compare also references in [[4]]). This is due to the fact that solutions corresponding to (nontrivial) smooth data have low regularity at the endpoint(s) of the interval of integration.

During the last 15 years various ways of designing high-order methods for the discretization of weakly singular functional equations of Volterra type have been explored. Here, we mention the fractional linear multistep methods on uniform meshes ([[3]]), the piecewise polynomial collocation methods on graded meshes ([[1,2,4,5,7]] and their references), and the nonpolynomial spline collocation methods on uniform meshes ([8,9]). The theory of fractional linear multistep methods for weakly singular Volterra integral equations is now well understood. While this is true also for piecewise polynomial collocation methods applied to second-kind Volterra integral equations and Volterra integro-differential equations with weakly singular kernels ([4,5] and their references), the situation is far less satisfactory for first-kind Volterra integral equations and certain Volterra functional integro-differential equations ([7,10–12]) where many questions remain to be answered.

In this survey paper we shall study the numerical solution of a class of neutral functional integro-differential equations (NFIDEs),

$$\frac{d}{dt}\left(a_0 x(t) + \int_{-r}^{0} a_1(s)x(t+s)\,ds\right) = f(t), \quad 0 < t \le T, \tag{1.1}$$

and the related delay Volterra integral equations,

$$a_0 x(t) + \int_{t-r}^{t} a_1(s-t)x(s)\,ds = F(t), \quad 0 < t \le T, \tag{1.2}$$

where typically $a_1(s) = (-s)^{-\alpha}$, $0 < \alpha < 1$. The right-hand side of the integrated form (1.2) of (1.1) is

$$F(t) = \int_0^t f(s)\,ds + a_0 x(0) + \int_{-r}^{0} a_1(s)\phi(s)\,ds =: g(t) + a_0 x(0) + D_\alpha \phi. \tag{1.3}$$

On $[-r, 0]$ the solution is subject to the initial condition $x(s) = \phi(s)$ where ϕ is a given (continuous) function.

Functional integro-differential equations of the form (1.1) arise, for example, in the mathematical modelling of the elastic motions of an airfoil section with flap in a 2-dimensional incompressible flow (see [13–16] for the underlying mathematical theory and references on applications; compare also [17]). When solving NFIDEs of the form (1.1), the greatest challenge arises in the case where $a_0 = 0$ on which we will focus in the following.

When designing high-order methods for the numerical solution of (1.1) and (1.2), one has to deal with the following problems:

(i) The weakly singular nature of the kernel in the integral operator leads to solutions with low regularity at the point $t = 0$: for smooth f and ϕ the analytical solution x behaves like $Ct^{1-\alpha}$ (if $a_0 \neq 0$), or Ct^{α} (if $a_0 = 0$) at $t = 0^+$.

(ii) The constant (finite) delay $r > 0$ leads in general to low regularity at the points $\xi_\mu := \mu r$ $(\mu = 1, 2, \ldots)$: if $a_0 = 0$, then, typically, one has $|x(t)| \le C(t - \xi_\mu)^\alpha$ at $t = \xi_\mu^+$.

(iii) If $a_0 = 0$, Eq. (1.2) is a Volterra integral equation of the first kind with constant delay r. The collocation solution approximating x will converge uniformly to x only for certain choices of the collocation points.

In Section 2 we shall describe the framework for the piecewise collocation methods for the delay integral equation (1.2). Section 3 begins with a brief review of convergence results in the case where $a_0 \neq 0$; here, the theory is now essentially complete. We then turn to the more difficult case $a_0 = 0$: while there are convergence results for collocation by piecewise constant functions, the general theory remains to be established. The final Section 4 deals with additional questions and future work: for example, should one solve (1.1) directly or indirectly, using (1.2)?

2. COLLOCATION METHODS FOR DELAY VOLTERRA INTEGRAL EQUATIONS

We begin by describing the framework for the piecewise polynomial collocation method which underlies the discretization for the delay integral equation (1.2). Set $\xi_\mu := \mu r$ ($\mu \in \mathrm{N}_0$) and assume without loss of generality that $T = \xi_{M+1}$ for some $M \geq 1$. In analogy to delay differential equations (see [18]), we shall refer to the points $\{\xi_\mu\}$ as *primary discontinuities* of the solution to (1.2) (or (1.1)). Let $I_\mu := (\xi_\mu, \xi_{\mu+1}]$ ($\mu = 0, 1, \ldots, M$), and denote by $\Pi_N^{(\mu)}$ the mesh for the closed interval $\bar{I}_\mu$ given by the points

$$t_n^{(\mu)} := \xi_\mu + (n/N)^q r \qquad (n = 0, 1, \ldots, N), \tag{2.1}$$

where the grading exponent q satisfies $q \geq 1$; it will depend on α and be governed by the degree of regularity of the solution at $t = \xi_\mu^+$.

An approximation u to the solution of the Volterra integral equation (1.2) on the interval I_μ will be sought in the linear space $S_{m-1}^{(-1)}(\Pi_N^{(\mu)})$ of discontinuous (real) piecewise polynomials of degree not exceeding $m - 1 \geq 0$,

$$S_{m-1}^{(-1)}(\Pi_N^{(\mu)}) := \{u = u^{(\mu)} \,:\, u\,|_{I_\mu} \in \pi_{m-1}\ (n = 0, 1, \ldots, N-1)\},$$

whose dimension is Nm. This *collocation solution* $u \in S_{m-1}^{(-1)}(\Pi_N^{(\mu)})$ is to satisfy the integral equation on a suitable set $X_N^{(\mu)} \subset I_\mu$ of *collocation points*, namely,

$$X_N^{(\mu)} := \{t_n^{(\mu)} + c_i h_n^{(\mu)} \,:\, 0 < c_1 < \ldots < c_m \leq 1\ (n = 0, 1, \ldots, N-1)\}; \tag{2.2}$$

here, $h_n^{(\mu)} := t_{n+1}^{(\mu)} - t_n^{(\mu)}$. This set is completely determined by the given mesh $\Pi_N^{(\mu)}$ and the prescribed *collocation parameters* $\{c_i\}$.

A commonly used *local representation* of the collocation solution $u \in S_{m-1}^{(-1)}(\Pi_N^{(\mu)})$ is given by

$$u(t_n^{(\mu)} + s h_n^{(\mu)}) = \sum_{j=1}^{m} L_j(s) U_{n,j}^{(\mu)}, \quad s \in (0, 1], \tag{2.3}$$

where $U_{n,j}^{(\mu)} := u(t_n^{(\mu)} + c_j h_n^{(\mu)})$; the $L_j(s)$ denote the Lagrange fundamental polynomials with respect to the set $\{c_j\}$.

Detailed background information on the discretization of various types of differential and integral equations by spline collocation methods, as well as on the use of graded meshes for weakly singular Volterra equations, may be found, for example, in [1,2,7].

On I_μ the collocation solution $u = u_\mu \in S_{m-1}^{(-1)}(\Pi_N^{(\mu)})$ is determined from the collocation equation

$$a_0 u(t) + \int_{t-r}^{t} a_1(s-t)u(s)\,ds = F(t), \quad t \in X_N^{(\mu)}, \tag{2.4}$$

with $F(t)$ given by (1.3) and with values $u(s) = \phi(s)$ if $s \in [-r, 0]$. This equation may be written as

$$a_0 u(t) + \int_{\xi_\mu}^{t} a_1(s-t)u(s)\,ds = G_\mu(t;u), \quad t \in X_N^{(\mu)}, \tag{2.5}$$

where

$$G_\mu(t;u) := F(t) - \int_{t-r}^{\xi_\mu} a_1(s-t)u(s)\,ds. \tag{2.6}$$

For later reference (compare Section 3.2) we write down explicitly the collocation equation for the case $a_0 = 0$ in (1.2); it reads

$$\int_{\xi_\mu}^{t} a_1(s-t)u(s)\,ds = G_\mu(t;u), \quad t \in X_N^{(\mu)}, \tag{2.7}$$

with

$$G_\mu(t;u) := g(t) + D_\alpha \phi - \int_{t-r}^{\xi_\mu} a_1(s-t)u(s)\,ds. \tag{2.8}$$

Using the local representation (2.3) for the collocation solution $u \in S_{m-1}^{(-1)}(\Pi_N^{(\mu)})$, the collocation method for (1.2) is described by the equations (2.9) and (2.10) below: for given $\mu = 0, 1, \ldots, M$, determine the solution $U_n^{(\mu)} := (U_{n,1}^{(\mu)}, \ldots, U_{n,m}^{(\mu)})^T \in \mathbf{R}^m$ of the linear system

$$\begin{aligned} a_0 U_{n,i}^{(\mu)} + h_n^{(\mu)} \sum_{j=1}^{m} \left(\int_0^{c_i} a_1(h_n^{(\mu)}(s-c_i)) L_j(s)\,ds \right) U_{n,j}^{(\mu)} \\ = G_\mu(t_n^{(\mu)} + c_i h_n^{(\mu)}; u) - \Phi_{n,i}^{(\mu)} \qquad (i = 1, \ldots, m), \end{aligned} \tag{2.9}$$

where

$$\Phi_{n,i}^{(\mu)} := \sum_{\ell=0}^{n-1} \int_{t_\ell^{(\mu)}}^{t_{\ell+1}^{(\mu)}} a_1(s - t_n^{(\mu)} - c_i h_n^{(\mu)}) u(s)\,ds\,.$$

Then,

$$u(t_n^{(\mu)} + s h_n^{(\mu)}) = \sum_{i=1}^{m} L_i(s) U_{n,i}^{(\mu)}, \quad s \in (0,1] \quad (n = 0, 1, \ldots, N-1). \tag{2.10}$$

If $a_0 \neq 0$ (we will assume, for simplicity, that $a_0 = 1$), then the so-called *iterated collocation solution* u_{it} corresponding to the collocation solution u on I_μ is also of interest; it is given by

$$u_{it}(t) := G_\mu(t; u) - \int_{\xi_\mu}^{t} a_1(s-t)u(s)\, ds, \quad t \in I_\mu \tag{2.11}$$

(see also [2,19]). We shall return to its importance in Section 3.1.

It is easy to verify that each of the linear systems (2.9) has a unique solution $U_n^{(\mu)} \in \mathbf{R}^m$: if $a_0 = 1$, this holds (by the contraction mapping principle) whenever $h_n^{(\mu)}$ is sufficiently small; for $a_0 = 0$ the statement is true for any $h_n^{(\mu)} > 0$ (recall that $a_s = (-s)^{-\alpha}$).

3. CONVERGENCE RESULTS FOR WEAKLY SINGULAR DELAY VOLTERRA INTEGRAL EQUATIONS

3.1. The case $a_0 = 1$

For each $\mu = 0, 1, \ldots, M$, Eq. (2.5) represents a collocation equation for the second-kind Volterra integral equation

$$x(t) + \int_{\xi_\mu}^{t} a_1(s-t)x(s)\, ds = G_\mu(t; x), \quad t \in I_\mu, \tag{3.1}$$

where, according to (2.6) and (1.3), G_μ is given by

$$G_\mu(t; x) := g(t) + x(0^+) + D_\alpha \phi - \int_{t-r}^{\xi_\mu} a_1(s-t)x(s)\, ds\,. \tag{3.2}$$

If $f \in C^{d-1}[0, T]$ (implying $F \in C^d[0, T]$ in (1.2)), the regularity of G_μ in (3.1) on I_μ is described by

$$|G_\mu^{(k)}(t; x)| \leq C(t - \xi_\mu)^{1-\alpha-k} \quad (1 \leq k \leq d).$$

Hence, for $\mu = 0, 1, \ldots, M$,

$$|x^{(k)}(t)| \leq C(t - \xi_\mu)^{1-\alpha-k} \quad \text{on } I_\mu$$

(see, for example, [2] or [4]). Thus, we obtain results on the (optimal) order of (uniform) convergence of the collocation solution $u \in S_{m-1}^{(-1)}(\Pi_N^{(\mu)})$ on each subinterval I_μ ($\mu = 0, 1, \ldots, M$) by adapting the arguments used in [4] and [20]. We summarize these results in the following theorem but leave the details of its proof to the reader. Recall that the graded meshes $\Pi_N^{(\mu)}$ characterized by the grading exponent $q \geq 1$ were introduced in (2.1).

Theorem 3.1. *Let* $a_0 = 1$ *in* (1.2), *and assume that the given data satisfy* $\phi \in C^d[-r, 0]$, $f \in C^{d-1}[0, T]$ $(d \geq 1)$, *and* $a_1(s) = (-s)^{-\alpha}$ $(0 < \alpha < 1)$. *If* $u \in S_{m-1}^{(-1)}(\Pi_N^{(\mu)})$ $(m \geq 1;\ 0 \leq \mu \leq M)$ *denotes the collocation solution to* (1.2) (*cf.* (2.9), (2.10)), *then, for any set* $\{c_i\}$ *with* $0 \leq c_1 < c_2 < \ldots < c_m \leq 1$ *and* $\mu = 0, 1, \ldots, M$:

(i)

$$\sup_{t \in I_\mu} |x(t) - u(t)| \leq C \begin{cases} h^{1-\alpha} & if \quad q = 1, \\ h^{q(1-\alpha)} & if \quad q \in (1, m/(1-\alpha)], \\ h^m & if \quad q \geq m/(1-\alpha), \end{cases}$$

provided that $d \geq m$. *Here, we have set* $h := r/N$.

(ii) *If the* $\{c_i\}$ *are the* m *Gauss points in* $(0, 1)$ *and if* $d \geq m + 1$, *we obtain*

$$\sup_{t \in I_\mu} |x(t) - u_{it}(t)| \leq C h^{m+1-\alpha} \quad if \quad q \geq m/(1-\alpha),$$

where the iterated collocation solution u_{it} *corresponding to* u *is defined in* (2.11). *This holds whenever* $d \geq m + 1$.

Remark. Using the approach of [4], it is also possible to derive convergence results for L^p norms. Moreover, convergence results for (1.2) with $a_0 \neq 0$ and kernel $a_1 = \log(-s)$ can be obtained, too, again by suitably adapting the analysis in [4] and [20]. Thus, the theory of piecewise polynomial collocation for (1.2) with $a_0 \neq 0$ is now essentially complete (as for classical second-kind Volterra integral equations, the analysis of numerical stability is still largely open). This situation is in stark contrast to that for (1.2) with $a_0 = 0$ which will be discussed in the following section.

3.2. The case $a_0 = 0$

Recall that the first-kind Volterra integral equation corresponding to (1.2) with $a_0 = 0$ is

$$\int_{\xi_\mu}^{t} a_1(s-t)x(s)\, ds = G_\mu(t; x), \quad t \in I_\mu \quad (0 \leq \mu \leq M), \tag{3.3}$$

with

$$G_\mu(t; x) = g(t) + D_\alpha \phi - \int_{t-r}^{\xi_\mu} a_1(s-t)x(s)\, ds. \tag{3.4}$$

Since, by (1.3),

$$G_0(0;x) = G_0(0;\phi) = g(0) + D_\alpha\phi - D_\alpha\phi = 0,$$

(3.3) has a (unique) continuous solution x on $[\xi_0, \xi_1] = [0, r]$, whenever ϕ and f are continuous on $[-r, 0]$ and $[0, r]$, respectively.

Using arguments from [21], it can be shown that

$$G_\mu(\xi_\mu; x) = g(\xi_\mu) + D_\alpha\phi - D_\alpha^{(\mu)}x = 0 \quad (\mu = 1, \ldots M),$$

where we have set

$$D_\alpha^{(\mu)}x := \int_{\xi_{\mu-1}}^{\xi_\mu} a_1(s - \xi_\mu)x(s)\, ds.$$

Hence, for $1 \leq \mu \leq M$, the corresponding weakly singular first-kind Volterra integral equation (3.3) has a continuous solution on $\bar{I}_\mu := [\xi_\mu, \xi_{\mu+1}]$.

According to the inversion formula for (3.3),

$$\begin{aligned} x(t) &= \frac{1}{\gamma_\alpha}\frac{d}{dt}\left(\int_{\xi_\mu}^t (t-s)^{\alpha-1}G_\mu(s;x)\, ds\right) \\ &= \frac{1}{\gamma_\alpha}\left((t-\xi_\mu)^{\alpha-1}G_\mu(\xi_\mu;x) + \int_{\xi_\mu}^t (t-s)^{\alpha-1}G'_\mu(s;x)\, ds\right), \end{aligned} \tag{3.5}$$

where $G_\mu(\xi_\mu; x) = 0$ and $\gamma_\alpha := \Gamma(\alpha)\Gamma(1-\alpha) = \pi/\sin(\alpha\pi)$, the regularity of x at the primary discontinuity points ξ_μ depends on the regularity of $G_\mu(t;x)$ on I_μ (compare also [3,22]). It is easy to verify (recall (3.4)) that for $t \in I_\mu$,

$$|x^{(k)}(t)| \leq C(t-\xi_\mu)^{\alpha-k}, \quad k \geq 1 \ \ (\mu = 0, 1, \ldots, M),$$

whenever ϕ and f are sufficiently smooth.

It is well known that collocation solutions to *regular* first-kind Volterra integral equations in (discontinuous or continuous) piecewise polynomial spaces do not converge uniformly to the exact solution of the equation for every choice of the $\{c_i\}$ (see [2,7] and, especially, [23]): for example, in $S_{m-1}^{(-1)}(\Pi_N)$ uniform convergence holds for any $m \geq 1$ if and only if the condition

$$\prod_{i=1}^m (1-c_i)/c_i \leq 1 \tag{3.6}$$

is satisfied. A similar result exists for spaces of continuous piecewise polynomials ([23]).

For first-kind Volterra integral equations with weakly singular kernels (and hence for (1.2) with $a_0 = 0$) a result of the above type is not yet known, except when $m = 1$ (see [2,10,11,17,24]). If (1.2) ($a_0 = 0$) is solved in $S_{m-1}^{(-1)}(\Pi_N^{(\mu)})$ with

$m = 1$ and if $c_1 = 1$, then uniform convergence of u to x on $\bar{I}_\mu$ can be proved by combining the analysis of [25] (and [2], Section 6.4). More generally, uniform convergence is obtained for all $c_1 \leq 1$ satisfying

$$c_1 \geq c_1^*(\alpha) := \frac{1}{2}\left(\alpha(1-\alpha)\gamma_\alpha\right)^{1/(1-\alpha)}$$

(see [17]). Note that $c_1^*(0) = \frac{1}{2}$ and $c_1(\alpha) < \frac{1}{2}$ for $0 < \alpha < 1$, with $c_1^*(\alpha)$ strictly monotone decreasing. A necessary and sufficient condition analogous to (3.6) (which, for $m = 1$, reduces to $c_1 \geq \frac{1}{2}$) remains to be found.

Instead of explicitly stating these results when $m = 1$ we present a more general theorem indicating that for *feasible* choices of the collocation parameters $\{c_i\}$ with $0 < c_1 < \ldots < c_m \leq 1$, order results analogous to those of Theorem 3.1 hold.

Theorem 3.2. *Let $a_0 = 0$ in (1.2), and assume that the given functions satisfy $\phi \in C^m[-r, 0]$, $f \in C^m[0, T]$, and $a_1(s) = (-s)^{-\alpha}$ $(0 < \alpha < 1)$. Suppose that the collocation parameters $\{c_i\}$ are such that the corresponding collocation solution $u \in S_{m-1}^{(-1)}(\Pi_N^{(\mu)})$ (cf. (2.9), (2.10)) converges uniformly on $\bar{I}_\mu := [\xi_\mu, \xi_{\mu+1}]$ to the solution x of (3.3). Then, for $\mu = 0, 1, \ldots, M$,*

$$\sup_{t \in I_\mu} |x(t) - u(t)| \leq C \begin{cases} h^\alpha & \text{if } q = 1, \\ h^{q\alpha} & \text{if } q \in (1, m/\alpha], \\ h^m & \text{if } q \geq m/\alpha, \end{cases}$$

where we have set $h := r/N$.

Proof. It is sufficient to establish the order results on the initial interval $\bar{I}_0$: the collocation error $e = e_\mu := x - u$ on $\bar{I}_\mu$ $(\mu \geq 1)$ satisfies

$$\int_{\xi_\mu}^{t} a_1(s-t)e_\mu(s)\,ds = E(t; e_{\mu-1}), \quad t \in X_N^{(\mu)},$$

where, by (3.4), we have set

$$E(t; e_{\mu-1}) := -\int_{t-r}^{\xi_\mu} a_1(s-t)e_{\mu-1}(s)\,ds.$$

If, on $I_{\mu-1}$, we have

$$||e_{\mu-1}||_\infty := \sup_{t \in I_{\mu-1}} |e(t)| \leq Ch^p \quad (p > 0),$$

then

$$|E(t; e_{\mu-1})| \leq ||e_{\mu-1}||_\infty \cdot \int_{t-r}^{\xi_\mu} a_1(s-t)ds \leq C(\alpha)h^p, \quad t \in \bar{I}_{\mu-1}.$$

The proof of the attainable order on $\bar{I}_0$ follows by a straightforward adaptation of the proof for a result in [17] (Theorem 3.2).

4. DIRECT VERSUS INDIRECT COLLOCATION

We conclude by pointing to some possible future work in the numerical analysis and computational solution of the weakly singular NFIDE (1.1) and the equivalent delay Volterra equation (1.2). While the piecewise polynomial collocation method described in this paper is a very powerful and accurate tool for solving such equations, especially when used in an adaptive way, it is quite expensive (reminiscent of Runge–Kutta methods for ordinary differential equations). A very promising but yet to be studied alternative is given by the *fractional linear multistep methods* of Lubich [3]: these methods take into account, through their coefficients and a special starting procedure (which may be interpreted as collocation in a certain nonpolynomial spline space; see [8]), the weakly singular nature of the kernel a_1 and thus yield high-order approximations on *uniform* meshes.

So far, all the methods for computing the solutions of weakly singular NFIDEs of the form (1.1) were based on some equivalent reformulation of the given problem. For example, in the recent paper [12] the semigroup framework ([13–15]; also [21]) underlying the functional equation (1.1) with $a_0 = 0$ was used to rewrite the problem as a linear hyperbolic partial differential equation with nonlocal boundary conditions. However, the low regularity of the analytical solution at the points $\{\xi_\mu : \mu \geq 0\}$ leads to low-order numerical approximations if the discretization of the partial differential equation is based on uniform meshes. The analysis and computational implementation of the method for this equivalent initial-boundary-value problem on graded meshes remain to be explored.

We also mention the work in [9]: here, the given weakly singular NFIDE is rewritten as a weakly singular second-kind Volterra integral equation which is then solved by collocation in certain nonpolynomial spline spaces (see also [8]). Here, it would be of interest to study the question of possible (local) superconvergence of iterated collocation solutions in such spaces.

Are there "direct" (collocation) methods for numerically solving (1.1) which possess the local superconvergence property at the mesh points $\Pi_N^{(\mu)}$? In other words, are there collocation spaces and corresponding sets of collocation points such that the collocation solution u to (1.1) satisfies

$$\sup_{t\in I_\mu} |x(t) - u(t)| \leq Ch^{p^*} \quad (0 \leq \mu \leq M),$$

with $p^* > p$, where p denotes the attainable order of uniform convergence on I_μ of u to x? The answer to this question is of particular interest in the case $a_0 = 0$, since ([2,23]) local superconvergence of piecewise polynomial solutions to first-kind Volterra integral equations is not possible at the mesh points.

ACKNOWLEDGEMENTS

This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC Research Grant No. OGP0009406).

Part of the research was carried out in May 1998 at the Mittag-Leffler Institute in Djursholm, Sweden (during the special year on Computational Methods for Differential Equations). The author gratefully acknowledges the hospitality extended to him by the Institute and the director of the program, Professor Vidar Thomeé, during his visit.

The author is also grateful to the Department of Applied Mathematics, in particular to Professor Arvet Pedas, of the University of Tartu and to its Conference Centre at Kääriku for their generous hospitality.

REFERENCES

1. Brunner, H. The numerical solution of weakly singular Volterra integral equations by collocation on graded meshes. *Math. Comp.*, 1985, **45**, 417–437.
2. Brunner, H. and van der Houwen, P. J. The numerical solution of Volterra equations. *CWI Monographs*, 1986, **3**.
3. Lubich, Ch. Fractional linear multistep methods for Abel–Volterra integral equations of the first kind. *IMA J. Numer. Anal.*, 1987, **7**, 97–106.
4. Brunner, H., Pedas, A. and Vainikko, G. The piecewise polynomial collocation method for nonlinear weakly singular Volterra equations. *Math. Comp.* (in press).
5. Hu, Q. Geometric meshes and their application to Volterra integro-differential equations with singularities. *IMA J. Numer. Anal.*, 1998, **18**, 151–164.
6. Vainikko, G. and Uba, P. A piecewise polynomial approximation to the solution of an integral equation with weakly singular kernel. *J. Austral. Math. Soc. Ser. B*, 1981, **22**, 431–438.
7. Brunner, H. The use of splines in the numerical solution of differential and Volterra integral equations. In *Spline Functions and the Theory of Wavelets I,II, Montréal 1996* (Dubuc, S., ed.). *CRM Proc. Lecture Notes*, 1998, **17** (in press).
8. Brunner, H. Nonpolynomial spline collocation for Volterra equations with weakly singular kernels. *SIAM J. Numer. Anal.*, 1983, **20**, 1106–1119.
9. Cerezo, G. M. *Solution Representation and Identification for Singular Neutral Functional Differential Equations*. PhD thesis, Virginia Polytechnic Institute & State University, Blacksburg, VA, 1996.
10. Eggermont, P. P. B. Stability and robustness of collocation methods for Abel-type integral equations. *Numer. Math.*, 1984, **45**, 431–445.
11. Eggermont, P. P. B. On Galerkin methods for Abel-type integral equations. *SIAM J. Numer. Anal.*, 1988, **25**, 1093–1117.
12. Ito, K. and Turi, J. Numerical methods for a class of singular integro-differential equations based on semigroup approximation. *SIAM J. Numer. Anal.*, 1991, **28**, 1698–1722.
13. Burns, J. A., Herdman, T. L. and Stech, H. W. Linear functional differential equations as semigroups on product spaces. *SIAM J. Math. Anal.*, 1983, **14**, 98–116.
14. Staffans, O. J. Some well-posed functional equations which generate semigroups. *J. Differential Equations*, 1985, **58**, 157–191.
15. Burns, J. A., Herdman, T. L. and Turi, J. Neutral functional integro-differential equations with weakly singular kernels. *J. Math. Anal. Appl.*, 1990, **145**, 371–401.

16. Junge, T. D. and Tucker, T. *Approximations for Weakly Singular Integral Equations*. Techn. Report, Interdisciplinary Center for Applied Mathematics, Virginia Polytechnic Institute & State University, Blacksburg, VA, 1997.
17. Brunner, H. On systems of Volterra difference equations associated with collocation methods for weakly singular Volterra integral equations. In *Proc. Third International Conference on Difference Equations and Applications, Taipei 1997* (Cheng, S. S., Elaydi, S. and Ladas, G., eds.) (in press).
18. Zennaro, M. Delay differential equations: theory and numerics. In *Theory and Numerics of Ordinary and Partial Differential Equations* (Ainsworth, M. et al., eds.). *Adv. Numer. Anal.*, 1995, **4**, 291–333.
19. Brunner, H. Iterated collocation methods for Volterra integral equations with delay arguments. *Math. Comp.*, 1994, **62**, 581–599.
20. Pedas, A. and Vainikko, G. Superconvergence of piecewise polynomial collocation for nonlinear weakly singular integral equations. *J. Integral Equations Appl.*, 1997, **9**, 4, 379–406.
21. Kappel, F. and Zhang, K. P. On neutral functional differential equations with nonatomic difference operator. *J. Math. Anal. Appl.*, 1986, **113**, 311–343.
22. Atkinson, K. E. An existence theorem for Abel integral equations. *SIAM J. Math. Anal.*, 1974, **5**, 729–736.
23. Kauthen, J.-P. and Brunner, H. Continuous collocation approximations to solutions of first kind Volterra equations. *Math. Comp.*, 1997, **66**, 1441–1459.
24. Brunner, H. Open problems in the discretization of Volterra integral equations. *Numer. Funct. Anal. Optim.*, 1996, **17**, 717–736.
25. Weiss, R. and Anderssen, R. S. A product integration method for a class of singular first kind Volterra equations. *Numer. Math.*, 1972, **18**, 442–456.

HÄLBIVA ARGUMENDIGA ESIMEST LIIKI NÕRGALT SINGULAARSETE VOLTERRA INTEGRAALVÕRRANDITE NUMBRILINE LAHENDAMINE

Hermann BRUNNER

On vaadeldud teatud neutraalsete diferentsiaal-integraalvõrrandite klassi kuuluvate ülesannete numbrilist lahendamist. Tükiti polünomiaalseid kollokatsioonimeetodeid on rakendatud otse võrrandile ja teise võimalusena ekvivalentsele, hälbiva argumendiga Volterra tüüpi integraalvõrrandile. On uuritud meetodi koonduvust ja seesuguste võrrandite ligikaudse lahendamisega seotud probleeme.