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PERIODIZATION OF INTEGRAL EQUATIONS ON OPEN ARCS

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Abstract. We present a periodization method for three different types of integral equations on open arcs. The periodization is based on the cosine transform. Applying the cosine transform, we obtain an equivalent periodic integral equation of certain parity, and the theory of pseudodifferential equations can be applied to this new formulation. Our results cover logarithmic singular integral equations, Cauchy singular integral equations, as well as hypersingular integral equations.

Key words: integral equation, open arc, cosine transform.

1. INTRODUCTION

In many applications the boundary integral method leads to the solution of an integral equation on an open arc (when two-dimensional phenomena are considered). In the basic examples the arising integral equations can be covered by the following types: logarithmic singular integral equations, Cauchy singular integral equations, and hypersingular integral equations. For the parametrized forms of the model equations, see (1)–(3). Equations of these types come from various fields such as fracture mechanics, aerodynamics, electromagnetism, and elasticity, for example. Starting from the parametrized form, we first apply the cosine transform and obtain for the original problem an equivalent formulation as a periodic problem of certain parity. It is well known that the cosine transform has some obvious advantages. In particular, in the case of logarithmic singular and Cauchy singular equations on an open arc Γ, the solution may have a singularity of order $\mathcal{O}(|x-c|^{-1/2})$ at the endpoint c of Γ even when the right-hand term of the equation is smooth. The cosine transform removes this singularity. Moreover, for smooth $b_k(x,y)$ and $g(x)$ (see (1)–(3)), also the coefficients and the right-hand term of the periodized problem are smooth; consequently, so is the solution of the periodized problem. Thus, the periodized forms of problems (1)–(3) are rather convenient for an approximate solution; see [1] for details and literature.

2. PARAMETRIZED EQUATION AND PERIODIZATION

In our analysis we start from an integral equation which is given on the open interval $I=(0,1)$ and apply the cosine transform to obtain a more convenient form of the equation which allows us to apply an approach for fast solution based on [2], for example. For the parametrized equations, we assume one of the following three types on I:

$$(B_L v)(x) := \int_0^1 \big(b_0(x,y)\log|x-y| + b_1(x,y)\big)v(y)\,dy = g(x), \qquad (1)$$

$$(B_C v)(x) :=$$
$$\int_0^1 \left(\frac{b_0(x,y)}{x-y} + b_1(x,y)\log|x-y| + b_2(x,y)\right)v(y)\,dy = g(x), \qquad (2)$$

$$(B_H v)(x) :=$$
$$\int_0^1 \left(\frac{b_0(x,y)}{|x-y|^2} + b_1(x,y)\log|x-y| + b_2(x,y)\right)v(y)\,dy = g(x). \qquad (3)$$

Here we assume that $b_k \in C^\infty(\overline{I}\times\overline{I})$, $k=0,1,2$, and $b_0(x,x)\neq 0$, $x\in\overline{I}$. The first integrals in (2) and (3) are understood in the sense of the principal value and the finite-part of Hadamard, respectively. We introduce the weighted spaces $L^2_\sigma(I)$, $L^2_{1/\sigma}(I)$, and $H^1_\sigma(I)$ of functions having a finite norm $\|v\|_\sigma = \left(\int_0^1 \sigma(y)|v(y)|^2 dy\right)^{1/2}$, $\sigma(y)=y^{1/2}(1-y)^{1/2}$, $\|v\|_{1/\sigma} = \left(\int_0^1 \frac{1}{\sigma(y)}|v(y)|^2 dy\right)^{1/2}$, and $\|v\|_{1,\sigma} = \left(\|v\|^2_\sigma + \|v'\|^2_\sigma\right)^{1/2}$, respectively. We define also $\overset{o}{H}{}^1_\sigma(I) = \{v\in H^1_\sigma(I) \mid v(0)=v(1)=0\}$ with the norm induced from $H^1_\sigma(I)$. The following mapping properties of B_L, B_C, and B_H can be obtained from our considerations:

$$\begin{aligned}
&B_L : L^2_\sigma(I) \to H^1_\sigma(I) && \text{is Fredholm operator of index 0,}\\
&B_C : L^2_\sigma(I) \to L^2_\sigma(I) && \text{is Fredholm operator of index 1,}\\
&B_C : L^2_{1/\sigma}(I) \to L^2_{1/\sigma}(I) && \text{is Fredholm operator of index } -1,\\
&B_H : \overset{o}{H}{}^1_\sigma(I) \to L^2_\sigma(I) && \text{is Fredholm operator of index 0.}
\end{aligned}$$

Example 2.1. Cauchy singular integral equations. The singular integral equation

$$\int_0^1 \left(\frac{b_0(x,y)}{x-y} + b_1(x,y)\log|x-y| + b_2(x,y) \right) v(y)dy = g(x) \qquad \text{(a)}$$

appears in several applications concerning flow problems around airfoils. In particular, with the constant function $b_0(x,y) = b_0 \neq 0$ and $b_1 = b_2 = 0$, we have the basic airfoil equation. However, Eq. (a) is not yet uniquely solvable in these examples, but the uniqueness is assured by imposing an additional condition of the form

$$\Phi_I v := \int_0^1 v(y)dy = \gamma \qquad \text{(b)}$$

which has the interpretation that the circulation around the profile is given. So, instead of (a) we have to consider the system

$$\begin{aligned} (B_C v)(x) &= g(x), \quad x \in I; \\ \Phi_I v &= \gamma. \end{aligned} \qquad \text{(c)}$$

Having an integral equation on $I = (0,1)$, we apply the cosine transform

$$x = x(t) = \tfrac{1}{2}(1 - \cos 2\pi t), \qquad t \in (0, \tfrac{1}{2}). \qquad (4)$$

After this transform we obtain a new integral equation for the unknown function u and the right-hand side f on $(0, \frac{1}{2})$. The new kernel is defined in a natural way already on the symmetric interval $(-\frac{1}{2}, \frac{1}{2})$ and, moreover, has a natural 1-biperiodic extension to $\mathbb{R}^2$. The final form of the equation is obtained by extending the functions u and f as even or odd functions to $\mathbb{R}$.

2.1. Equations with a logarithmic singular kernel

Consider the equations of the general form (1). Applying the transform (4), extending $x(t)$ by the formula (4) for all $t \in \mathbb{R}$, and writing $u(t) = v(x(t))|x'(t)|$, $f(t) = g(x(t))$, $t \in \mathbb{R}$, we find that (1) is equivalent to the equation for $t \in \mathbb{R}$

$$(A_L u)(t) := \int_{-1/2}^{1/2} \big(a_0(t,s)\kappa_0(t-s) + a_1(t,s)\big)u(s)ds = f(t), \qquad (5)$$

where u and f are 1-periodic, even on $\mathbb{R}$; $a_0(t,s) = b_0(x(t), x(s))$ and $a_1(t,s) = \frac{1}{2}b_1(x(t), x(s))$ are smooth 1-biperiodic even functions; and $\kappa_0(t) = \log|\sin \pi t|$.

2.2. Cauchy singular equations

Here we consider equations of the general form (2) and introduce two different periodizations. Let first $u(t)$, $t \in \mathbb{R}$, be the 1-periodic even extension of the function $u(t) = v(x(t))x'(t)$, $t \in (0, \frac{1}{2})$. Then, putting $f(t) = x'(t)g(x(t))$, we see due to parity properties that (2) is equivalent to

$$(A_{1C}u)(t) := \int_{-1/2}^{1/2} \big(a_0(t,s)\kappa_0(t-s) + a_{11}(t,s)\kappa_1(t-s) + a_{12}(t,s)\big)u(s)ds = f(t), \tag{6a}$$

where u is an even and f is an odd 1-periodic function in $\mathbb{R}$. Moreover,

$$\begin{aligned}\kappa_0(t) = \cot \pi t,\ \kappa_1(t) = \log|\sin \pi t|,\ a_0(t,s) = \pi b_0(x(t), x(s)),\\ a_{11}(t,s) = b_1(x(t), x(s))x'(t),\ a_{12}(t,s) = \tfrac{1}{2}b_2(x(t), x(s))x'(t).\end{aligned} \tag{6b}$$

In the applications connected to Example 2.1 we have to take the additional condition (b) into account. By the cosine transform we obtain

$$\Phi_I v = \Phi u := \tfrac{1}{2}\int_{-1/2}^{1/2} u(s)ds = \gamma. \tag{6c}$$

Define the operator $A_{1C}{\times}\Phi$ by $(A_{1C}{\times}\Phi)u = [A_{1C}u, \Phi u]$. Now the system of Eqs. (6a) and (6c) is given by a single equation: for given $[f,\gamma]$ find the function u such that

$$(A_{1C} \times \Phi)u = [f, \gamma]. \tag{7}$$

In the other periodization of (2) the function $u(t)$ is chosen to be the odd 1-periodic extension of $v(x(t))$, $0 < t < \frac{1}{2}$, and $f(t) = g(x(t))$, $t \in \mathbb{R}$, is even. Proceeding in a similar manner as above, we obtain that (2) is equivalent to

$$\begin{aligned}(A_{2C}u)(t) \quad &:= \int_{-1/2}^{1/2} \big(a_0(t,s)\kappa_0(t-s) + a_{21}(t,s)\kappa_1(t-s) + a_{22}(t,s)\big)u(s)ds\\ &= f(t), \quad t \in \mathbb{R}.\end{aligned} \tag{8}$$

Here the functions $a_0(t,s)$, $\kappa_0(t)$, and $\kappa_1(t)$ are the same as for A_{1C} but

$$a_{21}(t,s) = b_1(x(t), x(s))x'(s),\ a_{22}(t,s) = \tfrac{1}{2}b_2(x(t), x(s))x'(s).$$

In the case of the second formulation we do not use any additional condition for the uniqueness, but we insert a new parameter ω in order to obtain a uniquely solvable equation for all right-hand sides f. Thus we shall consider the solution of the equation

$$A_{2C}u + \omega = f. \tag{9}$$

2.3. Hypersingular equations

Consider Eq. (3). Define u as the periodic odd extension of $v(x(t))$ and put $f(t) = x'(t)g(x(t))$. Applying the cosine transform and multiplying the resulting equation by $x'(t)$, one can see that (3) is equivalent to

$$\begin{aligned}(A_H u)(t) \quad &:= \int_{-1/2}^{1/2} \big(a_0(t,s)\kappa_0(t-s) + a_1(t,s)\kappa_1(t-s) + a_2(t,s)\big)u(s)ds \\ &= f(t), \quad t \in \mathbb{R}, \end{aligned} \tag{10}$$

where $\kappa_0(t) = (\sin^2 \pi t)^{-1}$, $\kappa_1(t) = \log|\sin \pi t|$, $a_0(t,s) = \pi^2 b_0(x(t), x(s))$, $a_1(t,s) = b_1(x(t), x(s))x'(t)x'(s)$, $a_2(t,s) = \frac{1}{2} b_2(x(t), x(s))x'(t)x'(s)$.

Remark 2.1. The cosine substitution was introduced by Multhopp in [3] for the airfoil equation of Prandtl. For Symm's equation it was applied by Yan and Sloan [4] and for the basic hypersingular integral equation on an interval by Bühring [5].

3. ANALYSIS OF THE PERIODIC PROBLEM

We have transformed all Eqs. (1)–(3) to the form

$$\int_{-1/2}^{1/2} \big(a_0(t,s)\kappa_0(t-s) + a_1(t,s)\kappa_1(t-s) + a_2(t,s)\big)u(s)ds = f(t), \tag{11}$$

where u and f are 1-periodic and $a_p \in C_{1,1}^\infty(\mathbb{R}^2)$ (the space of all 1-biperiodic smooth functions). Moreover, there holds $a_0(t,t) \neq 0$, $t \in \mathbb{R}$. Now we assume the general form used in [6] and consider the equation

$$\mathcal{A}u = f, \tag{12}$$

where $\mathcal{A} = \sum_{p=0}^{q} A_p$ and

$$(A_p u)(t) = \int_{-1/2}^{1/2} \kappa_p(t-s)a_p(t,s)u(s)ds, \; a_p \in C_{1,1}^\infty(\mathbb{R}^2). \tag{13}$$

Furthermore, we assume that κ_p, $0 \le p \le q$, are 1-periodic distributions on $\mathbb{R}$ such that the Fourier coefficients satisfy for a $\alpha \in \mathbb{R}$,

$$|\Delta^k \hat{\kappa}_p(l)| \le c_k |l|^{\alpha - p - k}, \; 0 \neq l \in \mathbb{Z}, \; k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \; p = 0, 1, \ldots, q. \tag{14}$$

Here Δ is the difference operator, $\Delta\hat{\kappa}_p(l) = \hat{\kappa}_p(l+1) - \hat{\kappa}_p(l)$. Due to (13), (14), $A_p \in Op \sum^{\alpha-p}$, i.e. A_p is a periodic pseudodifferential operator of order $\alpha - p$. On the main part A_0 of the operator $\mathcal{A}$ we impose the following condition for a positive number c_{00}:

$$|\hat{\kappa}_0(l)| \geq c_{00}|l|^\alpha, \; 0 \neq l \in \mathbb{Z}, \tag{15a}$$

$$a_0(t,t) \neq 0, \; t \in \mathbb{R}. \tag{15b}$$

It follows from (13)–(15) that $\mathcal{A}$ is an elliptic periodic pseudodifferential operator of order α (see [7]) and $\mathcal{A} \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha})$ for any $\lambda \in \mathbb{R}$. Here H^λ is the Sobolev space of 1-periodic distributions u with the norm

$$\|u\|_\lambda = \left(\sum_{k\in\mathbb{Z}} [\max(1,|k|)]^{2\lambda} |\hat{u}(k)|^2 \right)^{1/2}, \quad \hat{u}(k) = \langle u, e^{-ik2\pi t} \rangle.$$

Moreover, $\mathcal{A} : H^\lambda \to H^{\lambda-\alpha}$ is Fredholm operator of index zero for any $\lambda \in \mathbb{R}$, and $N(\mathcal{A}) = \{u \in H^\lambda | \, \mathcal{A}u = 0\} \subset C_1^\infty(\mathbb{R})$ is independent of λ. Therefore, if

$$\mathcal{A}u = 0, \; u \in C_1^\infty(\mathbb{R}) \Rightarrow u = 0, \tag{16}$$

then $\mathcal{A} : H^\lambda \to H^{\lambda-\alpha}$, $\lambda \in \mathbb{R}$, is an isomorphism. We introduce the Sobolev spaces H_e^λ and H_o^λ of the even and odd functions

$$H_e^\lambda = \{u \in H^\lambda | \, u(-t) = u(t)\}, \quad H_o^\lambda = \{u \in H^\lambda | \, u(-t) = -u(t)\}.$$

The space H^λ is represented as the direct sum $H^\lambda = H_e^\lambda + H_o^\lambda$. Let $P_e : H^\lambda \to H_e^\lambda$ and $P_o : H^\lambda \to H_o^\lambda$ be the corresponding projections. For $u \in H^\lambda$, we write $u_e = P_e u$, $u_o = P_o u$. We say that the operator $\mathcal{A}$ is an *even operator* if $\mathcal{A}$ does not change the parity of the function, i.e., there holds $P_e\mathcal{A}u_e = \mathcal{A}u_e, \quad P_o\mathcal{A}u_o = \mathcal{A}u_o$. The operator $\mathcal{A}$ is an *odd operator* if $\mathcal{A}$ changes the parity, i.e., we have $P_e\mathcal{A}u_o = \mathcal{A}u_o, \quad P_o\mathcal{A}u_e = \mathcal{A}u_e$. Consider solution of Eq. (12). If $\mathcal{A}$ is even, Eq. (12) is equivalent to the system

$$\mathcal{A}u_e = f_e, \quad \mathcal{A}u_o = f_o. \tag{17}$$

Similarly, if $\mathcal{A}$ is odd, Eq. (12) is equivalent to the system

$$\mathcal{A}u_o = f_e, \quad \mathcal{A}u_e = f_o. \tag{18}$$

In our applications, which arise from applying the cosine transform to an equation on an open arc, we do not use the whole system (17) or (18) but just one or another of the equations appearing in these systems.

Now we characterize the parity properties of an integral operator through its kernel. Consider a general term $A_p = A$ in the representation of $\mathcal{A}$ defined by

$$(Au)(t) = \int_{-1/2}^{1/2} \kappa(t-s)a(t,s)u(s)ds, \tag{19}$$

$$a \in C_{1,1}^{\infty}(\mathbb{R}^2), \quad |\Delta^k \hat{\kappa}(l)| \leq c|l|^{\alpha-k} \, (0 \neq l \in \mathbb{Z},\, k \in \mathbb{N}_0). \tag{20}$$

We introduce the conditions

$$\kappa \text{ is even, i.e. } \hat{\kappa}(-l) = \hat{\kappa}(l),\ l \in \mathbb{Z}, \tag{21}$$

$$\kappa \text{ is odd, i.e. } \hat{\kappa}(-l) = -\hat{\kappa}(l),\ l \in \mathbb{Z}, \tag{22}$$

$$a \text{ is even, i.e. } a(-t,-s) = a(t,s),\ t,s \in \mathbb{R}, \tag{23}$$

$$a \text{ is odd, i.e. } a(-t,-s) = -a(t,s),\ t,s \in \mathbb{R}. \tag{24}$$

Observe that (23) is equivalent to the condition $\hat{a}(-k,-j) = \hat{a}(k,j),\ k,j \in \mathbb{Z}$, and (24) is equivalent to $\hat{a}(-k,-j) = -\hat{a}(k,j),\ k,j \in \mathbb{Z}$.

Lemma 3.1.
(i) *A is even if conditions* $\{(21),(23)\}$ *or* $\{(22),(24)\}$ *are fulfilled.*
(ii) *A is odd if conditions* $\{(21),(24)\}$ *or* $\{(22),(23)\}$ *are fulfilled.*

We introduce a linear functional Φ by

$$\Phi u = \int_{-1/2}^{1/2} u(s)\overline{\phi(s)}ds, \tag{25}$$

where $\phi \in C_1^{\infty}(\mathbb{R})$ is even. Furthermore, we define the operators $A\times\Phi$ and $A \dot{+} \Phi$ such that

$$(A\times\Phi)u = [Au, \Phi u], \quad u \in H_e^{\lambda}, \tag{26}$$

$$(A \dot{+} \Phi)[u,\omega] = Au + \omega\phi, \quad [u,\omega] \in H_o^{\lambda} \times \mathbb{C}. \tag{27}$$

Lemma 3.2. *In addition to* (20), *assume that $a(t,t) \neq 0$, $t \in \mathbb{R}$, and*

$$|\hat{\kappa}(l)| \geq c_0|l|^{\alpha} \quad (0 \neq l \in \mathbb{Z}),\ c_0 > 0.$$

Then, for any $\lambda \in \mathbb{R}$, the following holds true:
(i) *Under conditions* (21), (23), *the operators $A \in \mathcal{L}(H_e^{\lambda}, H_e^{\lambda-\alpha})$ and $A \in \mathcal{L}(H_o^{\lambda}, H_o^{\lambda-\alpha})$ are Fredholm operators of index* 0,

(ii) *Under conditions* (22), (23), *the operators* $A \in \mathcal{L}(H_e^\lambda, H_o^{\lambda-\alpha})$ *and* $A \in \mathcal{L}(H_o^\lambda, H_e^{\lambda-\alpha})$ *are Fredholm operators of index* 1 *and* -1, *respectively. We have* $A \times \Phi \in \mathcal{L}(H_e^\lambda, H_o^{\lambda-\alpha} \times \mathbb{C})$ *and* $A \dot{+} \Phi \in \mathcal{L}(H_o^\lambda \times \mathbb{C}, H_e^{\lambda-\alpha})$, *moreover, these are Fredholm operators of index* 0.

Now we analyse the solvability of the periodic problem for even and odd operators. Let $C_{1e}^\infty(\mathbb{R})$ and $C_{1o}^\infty(\mathbb{R})$ be the space of all even, respectively, odd functions in $C_1^\infty(\mathbb{R})$. We require on the main part A_0 the following properties:

$$\hat{\kappa}_0(-l) = \hat{\kappa}_0(l),\ l \neq 0\,;\ a_0(-t,-s) = a_0(t,s)\,;\ a_0(t,t) \neq 0. \tag{28}$$

Moreover, we impose the conditions

$$u \in C_{1e}^\infty(\mathbb{R}),\ \mathcal{A}u = 0 \Rightarrow u = 0, \tag{29a}$$

$$u \in C_{1o}^\infty(\mathbb{R}),\ \mathcal{A}u = 0\ \Rightarrow u = 0. \tag{29b}$$

Theorem 3.1. *Let* $\lambda \in \mathbb{R}$ *be given. Assume that* $\mathcal{A}$ *is an even operator with the conditions* (13)–(15) *and* (28). *If* (29a) *is valid, then* $\mathcal{A} : H_e^\lambda \to H_e^{\lambda-\alpha}$ *is an isomorphism. Moreover, if* (29b) *is valid, then* $\mathcal{A} : H_o^\lambda \to H_o^{\lambda-\alpha}$ *is an isomorphism.*

Let us consider the solvability of Eqs. (5) and (10). For these we set

$$v \in L_\sigma^2(I),\ B_L v = 0 \Rightarrow v = 0, \tag{30a}$$

$$v \in \overset{o}{H}{}_\sigma^1(I),\ B_H v = 0 \Rightarrow v = 0. \tag{30b}$$

Lemma 3.3. *The following assertions are valid:*

(i) *The mapping* $v \mapsto u$ *with* $u(t) = v(x(t)),\ t \in \mathbb{R}$, *defines a linear isomorphism between* $L_{1/\sigma}^2(I)$ *and* H_e^0, *as well as between* $H_\sigma^1(I)$ *and* H_e^1.

(ii) *The mapping* $v \mapsto u$ *with* $u(t) = v(x(t))\,\mathrm{sign}\,t,\ |t| \leq \frac{1}{2}$, *extended to a* 1*-periodic function, defines a linear isomorphism between* $L_{1/\sigma}^2(I)$ *and* H_o^0, *as well as between* $\overset{o}{H}{}_\sigma^1(I)$ *and* H_o^1.

(iii) *The mapping* $v \mapsto u$ *with* $u(t) = v(x(t))x'(t),\ t \in \mathbb{R}$, *defines a linear isomorphism between* $L_\sigma^2(I)$ *and* H_o^0.

(iv) *The mapping* $v \mapsto u$ *with* $u(t) = v(x(t))|x'(t)|,\ t \in \mathbb{R}$, *defines a linear isomorphism between* $L_\sigma^2(I)$ *and* H_e^0.

Theorem 3.2. *Assume* (30). *Then the operators* $B_L : L^2_\sigma(I) \to \overset{o}{H}{}^1_\sigma(I)$, $B_H : \overset{o}{H}{}^1_\sigma(I) \to L^2_\sigma(I)$ *and* $A_L : H^\lambda_e \to H^{\lambda+1}_e$, $\quad A_H : H^\lambda_o \to H^{\lambda-1}_o$ *are isomorphic for all* $\lambda \in \mathbb{R}$.

Now we consider the case of an odd operator $\mathcal{A}$ together with the operators $\mathcal{A} \times \Phi$ and $\mathcal{A} \dot{+} \Phi$. We require on the main part A_0 the properties

$$\hat{\kappa}_0(-l) = -\hat{\kappa}_0(l),\ 0 \neq l \in \mathbb{Z}, \tag{31a}$$

$$a_0(-t,-s) = a_0(t,s),\ t, s \in \mathbb{R}, \tag{31b}$$

$$a_0(t,t) \neq 0,\ t \in \mathbb{R}. \tag{31c}$$

Moreover, for the linear functional $\Phi : H^\lambda_e \to \mathbb{C}$, we additionally impose

$$\Phi 1 = \int_{-1/2}^{1/2} \overline{\phi(s)}\, ds \neq 0. \tag{32}$$

We consider the solution of the equation

$$u \in H^\lambda_e : Au = f,\ \Phi u = \gamma,\ f \in H^{\lambda-\alpha}_o,\ \gamma \in \mathbb{C} \tag{33}$$

and assume uniqueness for the homogeneous problem in the form

$$u \in C^\infty_{1e}(\mathbb{R}),\ \mathcal{A}u = 0,\ \Phi u = 0 \Rightarrow u = 0. \tag{34}$$

Theorem 3.3. *Assume* (13)–(15), (25), (31), (32), *and* (34). *Then, for any* $\lambda \in \mathbb{R}$, *the operator* $\mathcal{A} \times \Phi : H^\lambda_e \to H^{\lambda-\alpha}_o \times \mathbb{C}$ *is an isomorphism.*

We apply this result to the solution of the Cauchy singular integral equations on the interval I in the case where the periodization is carried out by the first method described in Section 2.2. The corresponding operator A_{1C} is given in (6a). We set the condition

$$v \in L^2_\sigma(I),\ B_C v = 0,\ \Phi_I v = 0 \Rightarrow v = 0. \tag{35}$$

Theorem 3.4. *Assume* (35) *and define* $\Phi u = \frac{1}{2}\int_{-1/2}^{1/2} u(s)ds$. *Then the mappings* $B_C \times \Phi_I : L^2_\sigma(I) \to L^2_\sigma(I) \times \mathbb{C}$ *and* $A_{1C} \times \Phi : H^\lambda_e \to H^\lambda_o \times \mathbb{C}$ *are isomorphic for all* $\lambda \in \mathbb{R}$.

Now we describe how the second periodization in Section 2.2 can be utilized for the solution of Cauchy singular equations on an interval. This leads to a problem of the general form

$$u \in H_o^\lambda,\ \omega \in \mathbb{C} \,:\, \mathcal{A}u + \omega\phi = f, \quad f \in H_e^{\lambda-\alpha}. \tag{36}$$

Problem (36) can be viewed as a "dual" problem of (33). We put the condition

$$\mathcal{A}u + \omega\phi = 0,\ u \in C_{1o}^\infty(\mathbb{R}),\ \omega \in \mathbb{C} \Rightarrow u = 0,\ \omega = 0, \tag{37}$$

and have a solvability result for (36) given by the operator $\mathcal{A} \dot{+} \Phi$ as follows.

Theorem 3.5. *Assume* (13)–(15), (31), (32), *and* (37). *Then, for any* $\lambda \in \mathbb{R}$, *the operator* $\mathcal{A} \dot{+} \Phi \colon H_o^\lambda \times \mathbb{C} \to H_e^{\lambda-\alpha}$ *is an isomorphism.*

For the Cauchy singular operator we impose the condition

$$v \in L^2_{1/\sigma}(I),\ \omega \in \mathbb{C},\ B_C v \dot{+} \omega\Phi_I = 0 \Rightarrow v = 0,\ \omega = 0. \tag{38}$$

Theorem 3.6. *Assume* (38). *Then* $B_C \dot{+} \Phi_I : L^2_{1/\sigma}(I) \times \mathbb{C} \to L^2_{1/\sigma}(I)$ *is an isomorphism and* $A_{2C} \dot{+} \Phi \colon H_o^\lambda \times \mathbb{C} \to H_e^\lambda$ *is an isomorphism for all* $\lambda \in \mathbb{R}$.

For given $\mathcal{A} \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha})$ we have the adjoint $\mathcal{A}^* \in \mathcal{L}(H^{\alpha-\lambda}, H^{-\lambda})$ defined by

$$(\mathcal{A}u, v) = (u, \mathcal{A}^* v)\ , \quad u \in H^\lambda,\ v \in H^{\alpha-\lambda}. \tag{39}$$

We introduce the duality pairing

$$\langle [u,\omega],[v,\mu]\rangle := (u,v) + \omega\overline{\mu}\ , \quad [u,\omega] \in H^\lambda \times \mathbb{C},\ [v,\mu] \in H^{-\lambda} \times \mathbb{C}.$$

Now we have

$$\langle (\mathcal{A}^* \times \Phi)u, [v,\omega]\rangle = (u, (\mathcal{A} \dot{+} \Phi)[v,\omega]),\ u \in H_e^{\alpha-\lambda},\ v \in H_o^\lambda,\ \omega \in \mathbb{C},$$

and therefore $\mathcal{A} \dot{+} \Phi = (\mathcal{A}^* \times \Phi)^*$. By the general results for Fredholm operators we have:

Theorem 3.7. *Assume that* (13)–(15) *and* (28) *are valid. Then the operator* $\mathcal{A} \dot{+} \Phi$: $H_o^\lambda \times \mathbb{C} \to H_e^{\lambda-\alpha}$ *is an isomorphism for all* $\lambda \in \mathbb{R}$ *if and only if* $\mathcal{A}^* \times \Phi : H_e^\lambda \to H_o^{\lambda-\alpha} \times \mathbb{C}$ *is an isomorphism for all* $\lambda \in \mathbb{R}$.

As an application of Theorem 3.7 we recall the second periodization method described for Cauchy singular equations in Section 2.2. We consider the equation

$$u \in H_o^\lambda,\ \omega \in \mathbb{C} : A_{2C}u + \tfrac{1}{2}\omega = f, \quad f \in H_e^\lambda. \tag{40}$$

Define

$$(B_C^* v)(x) := \int_0^1 \left(\frac{\overline{b_0}(y,x)}{y-x} + \overline{b_1}(y,x)\log|x-y| + \overline{b_2}(y,x) \right) v(y)dy.$$

We impose the condition

$$B_C^* v = 0,\ \Phi_I v = 0, \quad v \in L_\sigma^2(I) \Rightarrow v = 0. \tag{41}$$

Theorem 3.8. *Assume* (41). *Then* $A_{2C} + \frac{1}{2} : H_o^\lambda \times \mathbb{C} \to H_e^\lambda$ *is an isomorphism for all* $\lambda \in \mathbb{R}$.

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LAHTISEL KAAREL ANTUD INTEGRAALVÕRRANDITE PERIODISEERIMINE

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On esitatud kolme tüüpi integraalvõrrandite periodiseerimine koosinus-transformatsiooni abil. Periodiseeritud võrrandi tuum ja vabaliige on 1-perioodised siledad funktsioonid. Seesuguste võrrandite käsitlusel on võimalik kasutada perioodiliste pseudodiferentsiaalvõrrandite teooriat.