Proc. Estonian Acad. Sci. Phys. Math., 1999, **48**, 2, 69–78
https://doi.org/10.3176/phys.math.1999.2.01

SPLINE COLLOCATION METHOD FOR WEAKLY SINGULAR INTEGRAL EQUATIONS

Arvet PEDAS[a] and Gennadi VAINIKKO[b]

[a] Institute of Applied Mathematics, University of Tartu, Liivi 2, 50409 Tartu, Estonia; arvet.pedas@ut.ee
[b] Institute of Mathematics, Helsinki University of Technology, Otakaari 1, 02150 Espoo, Finland; gennadi.vainikko@hut.fi

Received 25 September 1998

Abstract. Second-kind Fredholm integral equations with weakly singular kernels typically have solutions which are nonsmooth near the boundary of integration. In this paper, on the basis of certain regularity properties of the exact solution, the piecewise polynomial collocation method on graded grids is discussed to solve nonlinear multidimensional weakly singular integral equations. Using special collocation points, error estimates at the collocation points are derived showing a more rapid convergence than the global uniform convergence in the domain of integration available by piecewise polynomials.

Key words: weakly singular integral equation, collocation method.

1. INTRODUCTION

We consider the nonlinear weakly singular integral equation

$$u(x) = \int_G K(x,y,u(y))dy + f(x), \qquad x \in G, \tag{1}$$

where $G \equiv \{x = (x_1, \ldots, x_n) : 0 < x_k < b_k,\ k = 1, \ldots, n\}$ is an n-dimensional parallelepiped. For the solution of Eq. (1) we shall use the piecewise polynomial collocation method (cf. [1]): the set $\overline{G}$ will be partitioned into small parallelepipeds (cells) and the approximate solution will be searched in the form of a function which is on every cell a polynomial of the same degree. It is shown in [1] how to choose the nonuniform grid so that the method might have the best convergence rate in

$L^\infty(G)$-norm. The purpose of the present paper is to study the convergence rate of the piecewise polynomial collocation method at the collocation points.

Note that the one-dimensional case $n = 1$ is studied, for example, in [2–9]. We refer also to [10–14] where the case $n \geq 2$ is discussed.

2. INTEGRAL EQUATION

We shall make the following assumptions (A1)–(A3).

(A1) The kernel $K(x, y, u)$ is m times ($m \geq 1$) continuously differentiable with respect to x, y, and u for $x, y \in G$, $x \neq y$, $u \in \mathbb{R}$, whereby there exists a real number $\nu \in (-\infty, n)$ such that, for any nonnegative integer $l \in \mathbb{Z}_+$ and multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$ with $l + |\alpha| + |\beta| \leq m$, the following inequalities hold:

$$\left| D_x^\alpha D_{x+y}^\beta \left(\frac{\partial}{\partial u}\right)^l K(x, y, u)\right| \leq \psi_1(|u|) \left\{ \begin{array}{ll} 1, & \nu + |\alpha| < 0 \\ 1 + |\log|x-y||, & \nu + |\alpha| = 0 \\ |x-y|^{-\nu-|\alpha|}, & \nu + |\alpha| > 0 \end{array} \right\}, \tag{2}$$

$$\left| D_x^\alpha D_{x+y}^\beta \left(\frac{\partial}{\partial u}\right)^l K(x, y, u_1) - D_x^\alpha D_{x+y}^\beta \left(\frac{\partial}{\partial u}\right)^l K(x, y, u_2)\right|$$

$$\leq \psi_2(\max\{|u_1|, |u_2|\})|u_1 - u_2| \left\{ \begin{array}{ll} 1, & \nu + |\alpha| < 0 \\ 1 + |\log|x-y||, & \nu + |\alpha| = 0 \\ |x-y|^{-\nu-|\alpha|}, & \nu + |\alpha| > 0 \end{array} \right\}. \tag{3}$$

Here,

$$\begin{aligned} |\alpha| &= \alpha_1 + \ldots + \alpha_n \text{ for } \alpha \in \mathbb{Z}_+^n, \\ |x| &= (x_1^2 + \ldots + x_n^2)^{1/2} \text{ for } x \in \mathbb{R}^n, \\ D_x^\alpha &= \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \\ D_{x+y}^\beta &= \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n} + \frac{\partial}{\partial y_n}\right)^{\beta_n}, \end{aligned}$$

and the functions $\psi_1 \colon [0, \infty) \to [0, \infty)$ and $\psi_2 \colon [0, \infty) \to [0, \infty)$ are assumed to be monotonically increasing.

(A2) $f \in C^{m,\nu}(G)$. The space $C^{m,\nu}(G)$ is defined as the collection of all m times continuously differentiable functions $u \colon G \to \mathbb{R}$ such that the estimates

$$|D^\alpha u(x)| \leq \text{const} \left\{ \begin{array}{ll} 1, & |\alpha| < n - \nu \\ 1 + |\log \rho(x)|, & |\alpha| = n - \nu \\ \rho(x)^{n-\nu-|\alpha|}, & |\alpha| > n - \nu \end{array} \right\},$$

$$\left|\frac{\partial^l u(x)}{\partial x_k^l}\right| \le \text{const}\begin{Bmatrix} 1, & l < n-\nu \\ 1+|\log\rho_k(x)|, & l = n-\nu \\ \rho_k(x)^{n-\nu-l}, & l > n-\nu \end{Bmatrix}$$

hold for $x \in G$, $|\alpha| \le m$, $l = 1,\dots,m$, $k = 1,\dots,n$, where $\rho_k(x) = \min\{x_k, b_k - x_k\}$ and $\rho(x) = \min\limits_{1\le k\le n}\rho_k(x)$ is the distance from x to ∂G, the boundary of G.

(A3) Integral equation (1) has a solution $u_0 \in L^\infty(G)$ and the linearized integral equation

$$v(x) = \int\limits_G K_0(x,y)v(y)dy,\;\; K_0(x,y) = \left[\frac{\partial K(x,y,u)}{\partial u}\right]_{u=u_0(y)},$$

has only the trivial solution $v = 0$ in $L^\infty(G)$.

Note that the assumption (A1) holds, for example, for the kernels $K(x,y,u) = K_1(x,y,u)|x-y|^{-\nu}$ $(0 < \nu < n)$ and $K(x,y,u) = K_1(x,y,u)\log|x-y|$ $(\nu = 0)$, where $K_1(x,y,u)$ is an $m+1$ times continuously differentiable function with respect to x, y, u for $x, y \in \overline{G}, u \in I\!R$.

From (A1)–(A3) it follows that, for the solution $u_0 \in L^\infty(G)$ of Eq. (1), we actually have $u_0 \in C^{m,\nu}(G)$ [1,12].

3. COLLOCATION METHOD

We use the same nonuniform grid as in [1,13]. To define the partition of $\overline{G}$ into cells, we choose a vector $N = (N_1,\dots,N_n)$ of natural numbers and introduce in the intervals $[0,b_k]$, $k = 1,\dots,n$, the following $2N_k + 1$ grid points:

$$\begin{aligned} x_{k,N}^{j_k} &= \frac{b_k}{2}\left(\frac{j_k}{N_k}\right)^r, \;\; j_k = 0,1,\dots,N_k, \\ x_{k,N}^{N_k+j_k} &= b_k - x_{k,N}^{N_k-j_k}, \;\; j_k = 1,\dots,N_k. \end{aligned} \tag{4}$$

Here $r \in I\!R$, $r \ge 1$, characterizes the nonuniformity of the grid. If $r = 1$, then the grid points (4) are uniformly located. Using the points (4), we introduce the partition of $\overline{G}$ into closed cells G_N^j:

$$G_N^j \equiv \{x = (x_1,\dots,x_n) : x_{k,N}^{j_k-1} \le x_k \le x_{k,N}^{j_k},\; k = 1,\dots,n\} \subset \overline{G}, \tag{5}$$

$$j \in J_N \equiv \{j = (j_1,\dots,j_n) :\; j_k = 1,\dots,2N_k,\; k = 1,\dots,n\}. \tag{6}$$

We determine the collocation points in the following way. We choose m points $\eta_1,\dots,\eta_m$ in the interval $[-1,1]$: $-1 \le \eta_1 < \eta_2 < \dots < \eta_m \le 1$. By affine transformations we transfer them into the interval $\left[x_{k,N}^{j_k-1}, x_{k,N}^{j_k}\right]$ ($j_k = 1, \ldots, 2N_k$, $k = 1, \ldots, n$):

$$\xi_{k,N}^{j_k,q_k} = x_{k,N}^{j_k-1} + \frac{\eta_{q_k}+1}{2}\left(x_{k,N}^{j_k} - x_{k,N}^{j_k-1}\right),\ q_k = 1, \ldots, m.$$

We assign the collocation points

$$\xi_N^{j,q} = \left(\xi_{1,N}^{j_1,q_1}, \ldots, \xi_{n,N}^{j_n,q_n}\right),\ q \in Q, \tag{7}$$

$$Q \equiv \{q = (q_1, \ldots, q_n) : q_k = 1, \ldots, m,\ k = 1, \ldots, n\}$$

to the cells G_N^j, $j \in J_N$. We define the interpolation operator $\mathcal{P}_N$ by the formula

$$(\mathcal{P}_N u)(x) = \sum_{q \in Q} u(\xi_N^{j,q})\varphi_N^{j,q}(x),\ x \in G_N^j,\ j \in J_N, \tag{8}$$

where $\varphi_N^{j,q}(x) = \varphi_{1,N}^{j_1,q_1}(x_1) \cdots \varphi_{n,N}^{j_n,q_n}(x_n)$ and $\varphi_{k,N}^{j_k,q_k}(x_k)$, $k = 1, \ldots, n$, are the polynomials of one variable of degree $m-1$ such that

$$\varphi_{k,N}^{j_k,q_k}\left(\xi_{k,N}^{j_k,p_k}\right) = \left\{\begin{array}{ll} 1 & \text{if } p_k = q_k \\ 0 & \text{if } p_k \neq q_k \end{array}\right\},\ p_k = 1, \ldots, m. \tag{9}$$

Let us denote by E_N the range of the operator $\mathcal{P}_N$. This is the finite-dimensional space of piecewise polynomial functions u_N on $\overline{G}$ which on any cell G_N^j, $j \in J_N$, are polynomials of the degree not exceeding $m-1$ with respect to any of arguments $x_1, \ldots, x_n$.

We determine the approximate solution $u_N \in E_N$ of the integral equation (1) by the collocation method from the following conditions:

$$\left[u_N(x) - \int_G K(x, y, u_N(y))dy - f(x)\right]_{x=\xi_N^{i,p}} = 0,\ p \in Q,\ i \in J_N. \tag{10}$$

We can present $u_N \in E_N$ in the form

$$u_N(x) = \sum_{q \in Q} c^{j,q}\varphi_N^{j,q}(x) \text{ if } x \in G_N^j,\ j \in J_N,$$

where, as it follows from (9), $c^{j,q} = u_N(\xi_N^{j,q})$. Now the collocation conditions (10) will take a form of a nonlinear system which determines the coefficients $c^{j,q}$.

The convergence of the collocation method described above under assumptions (A1)–(A3) is discussed in [1]. It is shown in [1] how to choose r so that this method would have the optimal convergence rate:

$$\max_{x \in \overline{G}} |u_N(x) - u_0(x)| \leq \text{const}\, h_N^m \text{ for } \left\{\begin{array}{ll} r > \dfrac{m}{n-\nu} & \text{if } n-\nu \leq m \\ r \geq 1 & \text{if } n-\nu > m \end{array}\right\}, \tag{11}$$

$$\varepsilon_N \le \mathrm{const}\, h_N^m \quad \text{for} \left\{ \begin{array}{ll} r > \dfrac{m}{2(n-\nu)},\ r \ge 1 & \text{if } n-\nu \le 1 \\ r > \dfrac{m}{n-\nu+1} & \text{if } 1 < n-\nu \le m-1 \\ r \ge 1 & \text{if } n-\nu > m-1 \end{array} \right\}, \qquad (12)$$

where

$$h_N = \max\left\{\frac{b_1}{N_1}, \ldots, \frac{b_n}{N_n}\right\} \qquad (13)$$

and

$$\varepsilon_N = \max_{p\in Q, i\in J_N} \left| u_N(\xi_N^{i,p}) - u_0(\xi_N^{i,p}) \right| \qquad (14)$$

is the maximal error of u_N at collocation points (7).

In the following section we shall show how to choose the grading exponent r and the collocation points (7) so that the method (10) would converge with the rate $\varepsilon_N = o(h_N^m)$.

4. SUPERCONVERGENCE AT THE COLLOCATION POINTS

Assume that $\eta_1, \ldots, \eta_m$ are the knots of the quadrature formula

$$\int_{-1}^{1} g(\xi)d\xi \approx \sum_{q=1}^{m} w_q g(\eta_q), \quad -1 \le \eta_1 < \ldots < \eta_m \le 1, \qquad (15)$$

which is exact for all polynomials of degree $m+\mu$, $0 \le \mu \le m-1$. Using (15), we introduce the cubature formula

$$\int_{G_N^j} g(x)dx \approx 2^{-n}(\mathrm{meas}\, G_N^j) \sum_{q_1=1}^{m} \cdots \sum_{q_m=1}^{m} w_{q_1} \cdots w_{q_n} g\left(\xi_N^{j,q}\right), \qquad (16)$$
$$q = (q_1, \ldots, q_m),\ j \in J_N,$$

which is exact for all polynomials of degree $m+\mu$ with respect to any of arguments $x_1, \ldots, x_n$ (see [1], p. 126). Actually, the weights $w_1, \ldots, w_m$ will not be used in our algorithms; the existence of the cubature formula (16) which is exact for polynomials of degree $m+\mu$ $(0 \le \mu \le m-1)$ is used in the proof of the following theorems.

Theorem 1. *Assume that* (A3) *and the following conditions* (A1'), (A2'), (A4) *are fulfilled.*

(A1') *The kernel* $K(x,y,u)$ *and* $\tilde{K}(x,y,u) \equiv \partial K(x,y,u)/\partial u$ *are* $m+\mu+1$ *times continuously differentiable with respect to* x, y, *and* u *for* $x, y \in G$, $x \ne y$, $u \in \mathbb{R}$, *whereby there exists a real number* $\nu \in (-\infty, n)$ *such that* $K(x,y,u)$ *and* $\tilde{K}(x,y,u)$ *satisfy* (2) *and* (3) *for* $|\alpha|+|\beta|+l \le m+\mu+1$, $m \ge 1$, $0 \le \mu \le m-1$.

(A2') $f \in C^{m+\mu+1,\nu}(G)$.

(A4) *The collocation points* (7) *are generated by the knots* $\eta_1, \ldots, \eta_m$ *of the quadrature formula* (15) *which is exact for polynomials of degree* $m+\mu,\ 0 \le \mu \le m-1$; *the scaling parameter* $r = r(m,\nu,\mu) \ge 1$ *satisfies the following conditions*:

$$\begin{aligned} &r > \frac{m}{n-\nu}, \quad r \ge \frac{m+n-\nu}{n-\nu+1} \quad if\ n-\nu < \mu+1; \\ &r > \frac{m}{n-\nu}, \quad r > \frac{m+\mu+1}{n-\nu+1} \quad if\ \ \mu+1 \le n-\nu \le m; \\ &r \ge 1, \qquad\quad r > \frac{m+\mu+1}{n-\nu+1} \quad if\ \ m < n-\nu. \end{aligned} \tag{17}$$

Then there exist N_k^0 $(k = 1, \ldots, n)$ *and* $\delta_0 > 0$ *such that, for* $N_k \ge N_k^0$ $(k = 1, \ldots, n)$, *the collocation method* (10) *determines a unique approximation* $u_N \in E_N$ *to* u_0 *satisfying* $||u_N - u_0||_{L^\infty(G)} \le \delta_0$. *The following estimates hold*:

$$\varepsilon_N \le \text{const}\ h_N^m[\sigma_{n,\nu,\mu}(h_N) + \tau_{n,\nu}(h_N)], \tag{18}$$

where h_N *and* ε_N *are defined in* (13) *and* (14) *and*

$$\sigma_{n,\nu,\mu}(h_N) = \left\{ \begin{array}{ll} h_N^{\mu+1} & \text{if}\ \ n-\nu > \mu+1 \\ h_N^{\mu+1}(1+|\log h_N|) & \text{if}\ \ n-\nu = \mu+1 \\ h_N^{n-\nu} & \text{if}\ \ n-\nu < \mu+1 \end{array} \right\},$$

$$\tau_{n,\nu}(h_N) = \left\{ \begin{array}{ll} h_N^n & \text{if}\ \ \nu < 0 \\ h_N^n(1+|\log h_N|) & \text{if}\ \ \nu = 0 \\ h_N^{n-\nu} & \text{if}\ \ \nu > 0 \end{array} \right\}.$$

Remark 1. If $\mu+1 \le n$ or if $\nu > 0$, then under conditions of Theorem 1 we have $\varepsilon_N \le \text{const}\ h_N^m \sigma_{n,\nu,\mu}(h_N)$.

Theorem 2. *Let the conditions of Theorem 1 be fulfilled. We assume additionally that* $\nu \le 0$, $\mu+1 > n$ *and for* $|\alpha| \le \min\{\mu+1-n, -\nu\}$, $0 \le k \le \min\{\mu+1-n, -\nu\}$, *the derivatives* $D_y^\alpha \partial^{k+1} K(x,y,u)/\partial u^{k+1}$ *are bounded and continuous on* $G \times G \times (-\rho, \rho)$ *with any* $\rho > 0$, *including the diagonal* $x = y$. *Then*

$$\varepsilon_N \le \text{const}\ h_N^m \sigma_{n,\nu,\mu}(h_N). \tag{19}$$

Remark 2. Under conditions of Theorem 1, for the iterated approximation $\tilde{u}_N(x) = \int_G K(x,y,u_N(y))dy + f(x)$, $\quad x \in G$, we have

$$\max_{x \in \overline{G}} |\tilde{u}_N(x) - u_0(x)| \le \text{const}\ h_N^m[\sigma_{n,\nu,\mu}(h_N) + \tau_{n,\nu}(h_N)].$$

Under conditions of Theorem 2,

$$\max_{x\in\overline{G}} |\tilde{u}_N(x) - u_0(x)| \le \text{const}\, h_N^m \sigma_{n,\nu,\mu}(h_N).$$

5. PROOF OF THEOREMS 1 AND 2

Under conditions of Theorem 1 we have (see [1], p. 144)

$$\varepsilon_N \le c \sup_{x\in G} \left| \int_G \frac{\partial K(x,y,u_0(y))}{\partial u} [(P_N u_0)(y) - u_0(y)]\, dy \right| + c' \|u_0 - P_N u_0\|^2_{L^\infty(G)}, \tag{20}$$

where c and c' are positive constants not depending on N.

Fix $x \in G$. We shall use the notation

$$\begin{aligned} B(x,h_N) &= \{y \in I\!R^n : |y-x| \le h_N\}, \\ \Gamma_{h_N} &= \{y \in G : \rho(y) \le h_N\}, \\ S(x,h_N) &= G \cap \{B(x,h_N) \cup \Gamma_{h_N}\}, \end{aligned}$$

and denote by

$$K^{(s)}_{y_N^j}(x,y,u_0(y)) = \sum_{|\alpha|\le s} c_\alpha \left[D_y^\alpha \frac{\partial K(x,y,u_0(y))}{\partial u} \right]_{y=y_N^j} (y - y_N^j)^\alpha$$

the Taylor expansion of $[\partial K(x,y,u)/\partial u]_{u=u_0(y)}$ with respect to y at the centre $y_N^j = (y_{1,N}^{j_1}, \ldots, y_{n,N}^{j_n})$ of the cell G_N^j, $j \in J_N$, $y_{k,N}^{j_k} = (x_{k,N}^{j_k-1} + x_{k,N}^{j_k})/2$; $c_\alpha = 1/\alpha! = 1/(\alpha_1!)\cdots(\alpha_n!)$, $(y-y_N^j)^\alpha = (y_1 - y_{1,N}^{j_1})^{\alpha_1} \cdots (y_n - y_{n,N}^{j_n})^{\alpha_n}$; the value $s \in Z\!\!Z$, $0 \le s \le \mu$, will be chosen later (cf. [1], p. 128). Using the sharpness of the cubature formula (16) for the polynomials of degree $m+\mu$, we have, for $x \in G_N^j$,

$$\int_{G_N^j} \frac{\partial K(x,y,u_0(y))}{\partial u} [u_0(y) - (P_N u_0)(y)]\, dy = \gamma_{1,N}^j(x) + \gamma_{2,N}^j(x), \tag{21}$$

where

$$\gamma_{1,N}^j(x) = \int_{G_N^j} \left[\frac{\partial K(x,y,u_0(y))}{\partial u} - K^{(s)}_{y_N^j}(x,y,u_0(y)) \right] [u_0(y) - (P_N u_0)(y)]\, dy,$$

$$\gamma^j_{2,N}(x) = \sum_{|\alpha|\le s} c_\alpha \left[D^\alpha_y \frac{\partial K(x,y,u_0(y))}{\partial u}\right]_{y=y^j_N} \times \int_{G^j_N} (y-y^j_N)^\alpha [u_0(y) - (P^{(\alpha)}_N u_0)(y)]dy.$$

Here $P^{(\alpha)}_N$ is an interpolation operator similar to P_N but corresponding to the space of piecewise polynomials of degree $m+\mu-|\alpha|$ and the interpolation knots generated by $m+\mu+1-|\alpha|$ knots in $[-1,1]$ – the knots $\eta_1,\dots,\eta_m$ of the quadrature formula (15) and additional knots $\eta_{m+1},\dots,\eta_{m+1+\mu-|\alpha|}$ (cf. Section 2); the choice of the last ones in $[-1,1]$ is arbitrary but we assume that they are somehow fixed. Summing up over the cells G^j_N, $j\in J_N$, we obtain from (21), for $x\in G$,

$$\int_G \frac{\partial K(x,y,u_0(y))}{\partial u}[u_0(y) - (P_N u_0)(y)]dy = \eta_N(x) + \eta_{1,N}(x) + \eta_{2,N}(x), \quad (22)$$

where

$$\eta_N(x) = \sum_{j\in J_N: G^j_N\cap S(x,h)\ne\emptyset} \int_{G^j_N} \frac{\partial K(x,y,u_0(y))}{\partial u}[u_0(y) - (P_N u_0)(y)]dy,$$

$$\eta_{k,N}(x) = \sum_{j\in J_N: G^j_N\cap S(x,h)=\emptyset} \gamma^j_{k,N}(x), \quad k=1,2.$$

We get from (2)

$$|\eta_N(x)| \le \text{const}\ \tau_{n,\nu}(h_N)||u_0 - P_N u_0||_{L^\infty(G)} \quad (x\in G). \quad (23)$$

Putting $s=\mu$ if $n-\nu\ge\mu+1$ and $s=[n-\nu]$, the integer part of $n-\nu$, if $n-\nu<\mu+1$, due to $u_0\in C^{m+\mu+1,\nu}(G)$ and (17), we have (cf. [9,11])

$$|\eta_{k,N}(x)| \le \text{const}\ \sigma_{n,\nu,\mu}(h_N)||u_0 - P_N u_0||_{L^\infty(G)} \quad (x\in G; k=1,2). \quad (24)$$

Since $||u_0 - P_N u_0||_{L^\infty(G)} \le \text{const}\ h^m_N$ (see [1], p. 115), the estimate (18) follows from (20), (22)–(24). The proof of Theorem 1 is completed.

Now assume the conditions of Theorem 2. To establish (19), we only have to prove that

$$|\eta_N(x)| \le \text{const}\ \sigma_{n,\nu,\mu}(h_N)||u_0 - P_N u_0||_{L^\infty(G)} \quad (x\in G). \quad (25)$$

Let us divide $\eta_N(x)$ into two parts: $\eta_N(x) = \delta^{(1)}_N(x) + \delta^{(2)}_N(x)$, where $x\in G$,

$$\delta^{(k)}_N(x) = \sum_{j\in J^{(k)}_N} \int_{G^j_N} \frac{\partial K(x,y,u_0(y))}{\partial u}[u_0(y) - (P_N u_0)(y)]dy, \ k=1,2,$$

$J_N^{(1)} = \{j \in J_N: G_N^j \cap B(x, h_N) \neq \emptyset\}$, $J_N^{(2)} = \{j \in J_N: G_N^j \cap \Gamma_{h_N} \neq \emptyset, j \notin J_N^{(1)}\}$. For $j \in J_N^{(2)}$ we apply the estimates derived on the basis of the expansion (21):

$$|\delta_N^{(2)}(x)| \leq \text{const}\, \sigma_{n,\nu,\mu}(h_N) \|u_0 - P_N u_0\|_{L^\infty(G)} \; (x \in G). \tag{26}$$

For $j \in J_N^{(1)}$ we apply (cf. [1], p. 121; [9]) a representation of type (21) using now the point x as the centre of the Taylor expansion:

$$\int_{G_N^j} \frac{\partial K(x, y, u_0(y))}{\partial u} [u_0(y) - (P_N u_0)(y)] dy$$

$$= \int_{G_N^j} \left[\frac{\partial K(x, y, u_0(y))}{\partial u} - K_x^{(s)}(x, y, u_0(y)) \right] [u_0(y) - (P_N u_0)(y)] dy$$

$$+ \sum_{|\alpha| \leq s} c_\alpha \left[D_y^\alpha \frac{\partial K(x, y, u_0(y))}{\partial u} \right]_{y=x} \int_{G_N^j} (y - x)^\alpha [u_0(y) - (P_N^{(\alpha)} u_0)(y)] dy,$$

where $G_N^j \cap B(x, h_N) \neq \emptyset$ and

$$K_x^{(s)}(x, y, u_0(y)) = \sum_{|\alpha| \leq s} c_\alpha \left[D_y^\alpha \frac{\partial K(x, y, u_0(y))}{\partial u} \right]_{y=x} (y - x)^\alpha.$$

We put

(i) $s = \mu - n$ if $|\nu| > \mu + 1 - n$;
(ii) $s = |\nu| - 1$ if $|\nu| \leq \mu + 1 - n, \nu \in \mathbb{Z}$;
(iii) $s = [|\nu|]$, the integer part of $|\nu|$, if $|\nu| \leq \mu + 1 - n, \nu \notin \mathbb{Z}$.

Then, respectively,

(i) $\nu + s + 1 < 0$,
(ii) $\nu + s + 1 = 0$,
(iii) $0 < \nu + s + 1 < 1$.

Using the integral form of remainder $\partial K(x, y, u_0(y))/\partial u - K_x^{(s)}(x, y, u_0(y))$, in all three cases (i)–(iii) we obtain

$$|\delta_N^{(1)}(x)| \leq \text{const}\, \sigma_{n,\nu,\mu}(h_N) \|u_0 - P_N u_0\|_{L^\infty(G)} \; (x \in G). \tag{27}$$

We refer to ([1], p. 131) for details concerning the estimation of $\delta_N^{(1)}$. Using (26) and (27), we obtain the estimate (25). This completes the proof of Theorem 2.

ACKNOWLEDGEMENT

This investigation was partly supported by the Estonian Science Foundation (grant No. 2999).

REFERENCES

1. Vainikko, G. Multidimensional weakly singular integral equations. *Lecture Notes Math.*, 1993, **1549**.
2. Vainikko, G. and Uba, P. A piecewise polynomial approximation to the solution of an integral equation with weakly singular kernel. *J. Austral. Math. Soc. Ser. B*, 1981, **22**, 431–438.
3. Vainikko, G., Pedas, A. and Uba, P. *Methods for Solving Weakly Singular Integral Equations.* University of Tartu, 1984 (in Russian).
4. Schneider, C. Product integration for weakly singular integral equations. *Math. Comp.*, 1981, **36**, 207–213.
5. Hackbusch, W. *Integralgleichungen.* Teubner, Stuttgart, 1989.
6. Kress, R. *Linear Integral Equations.* Springer-Verlag, Berlin–Heidelberg, 1989.
7. Sloan, I. Superconvergence. In *Numerical Solution of Integral Equations* (Golberg, M., ed.). Plenum Press, New York, 1990, 35–70.
8. Kaneko, H., Noren, R. and Xu, Y. Numerical solutions for weakly singular Hammerstein equations and their superconvergence. *J. Integral Equations Appl.*, 1992, **4**, 391–407.
9. Pedas, A. and Vainikko, G. Superconvergence of piecewise polynomial collocations for nonlinear weakly singular integral equations. *J. Integral Equations Appl.*, 1997, **9**, 4, 379–406.
10. Graham, I. G. Collocation methods for two dimensional weakly singular integral equations. *J. Austral. Math. Soc. Ser. B*, 1981, **22**, 456–473.
11. Pedas, A. Superconvergence of the spline collocation method for nonlinear two dimensional weakly singular integral equations. *Differentsial'nye Uravneniya*, 1997, **33**, 9, 1–8 (in Russian).
12. Pedas, A. and Vainikko, G. Tangential derivatives of solutions to nonlinear multidimensional weakly singular integral equations. In *Beiträge zur angewandten Analysis und Informatik* (Schock, E., ed.). Shaker Verlag, Aachen, 1994, 271–287.
13. Tamme, E. Two-grid method for the solution of weakly singular integral equations by piecewise polynomial approximation. *Proc. Estonian Acad. Sci. Phys. Math.*, 1997, **46**, 4, 241–250.
14. Atkinson, K. E. The numerical solution of integral equations of second kind. *Cambridge Monogr. Appl. Comput. Math.*, 1997, 4.

SPLAIN-KOLLOKATSIOONIMEETOD NÕRGALT SINGULAARSETE INTEGRAALVÕRRANDITE LAHENDAMISEKS

Arvet PEDAS ja Gennadi VAINIKKO

On vaadeldud mitmemõõtmelise nõrgalt singulaarse tuumaga mittelineaarse teist liiki integraalvõrrandi lahendamist kollokatsioonimeetodiga tükiti polünomiaalsete baasfunktsioonide korral. Võttes aluseks integraalvõrrandi lahendi ja selle tuletiste käitumise integreerimispiirkonna raja lähedal on uuritud vaadeldava meetodi ülikoonduvust (superkoonduvust) sobivalt valitud kollokatsioonipunktide korral.