Proc. Estonian Acad. Sci. Phys. Math., 1999, **48**, 2, 133–145
https://doi.org/10.3176/phys.math.1999.2.07

ON THE A POSTERIORI PARAMETER CHOICE IN REGULARIZATION METHODS

Uno HÄMARIK and Toomas RAUS

Institute of Applied Mathematics, University of Tartu, Liivi 2, 50409 Tartu, Estonia; uno_h@ut.ee; toomas_r@ut.ee

Received 15 October 1998

Abstract. We consider parameter choice in regularization methods for solving linear ill-posed problems in Hilbert spaces. Recently, U. Tautenhahn proposed a new rule for parameter choice in the Tikhonov method. We consider a similar rule for the iterated Tikhonov method and show that it leads to a smaller error than the parameter choice rules of Raus (*Acta Comment. Univ. Tartuensis*, 1985, 715, 12–20) and Gfrerer (*Math. Comput.*, 1987, **49**, 507–522).

Key words: ill-posed problems, regularization methods, parameter choice.

1. PROBLEM AND METHODS

Consider the linear ill-posed problem

$$Au = f, \tag{1}$$

where $A \in \mathcal{L}(H, F)$ is a linear bounded operator between infinite dimensional Hilbert spaces H and F with the nonclosed range $\mathcal{R}(A)$ of A. Assume that instead of $f \in \mathcal{R}(A)$ only the element $f_\delta \in F$ with $||f_\delta - f|| \leq \delta$ is available.

For solving ill-posed problems regularization methods are necessary (see [1–5]). To generate a regularization method, we use a differentiable and Borel measurable function $g_\alpha(\lambda) : [0, a] \to \mathbb{R}$ ($\alpha > 0$, $||A||^2 \leq a$) satisfying the conditions

$$\sup_{0 \leq \lambda \leq a} |g_\alpha(\lambda)| \leq \frac{\gamma}{\alpha} \qquad (\alpha > 0), \tag{2}$$

$$\sup_{0 \leq \lambda \leq a} \lambda^p |1 - \lambda g_\alpha(\lambda)| \leq \gamma_p \alpha^p \qquad (\alpha > 0,\ 0 \leq p \leq p_0). \tag{3}$$

Here γ, γ_p, and p_0 are positive constants.

Let $\overline{u} \in H$ be initial approximation. The nearest to $\overline{u}$ solution u_* of (1) is approximated by

$$u_\alpha = \overline{u} + g_\alpha(A^*A)A^*(f_\delta - A\overline{u}). \tag{4}$$

The constant p_0 in (3) is called the qualification of the method (4). The class of regularization methods (4) was introduced in [6,7] and has often been used later (see, e.g., [1–5,8–16]).

2. TWO WELL-KNOWN A POSTERIORI PARAMETER CHOICE RULES

The main problem in applying regularization methods is the right choice of the regularization parameter $\alpha > 0$ in dependence on the noise level δ. This choice must be a compromise between accuracy and stability: if parameter α decreases, the accuracy in case of exact data ($\delta = 0$) increases, but the stability decreases.

The first prominent parameter choice rule is the discrepancy principle of Morozov [17], where the parameter $\alpha = \alpha_M$ is found as the solution of

$$d_M(\alpha) = b\delta, \qquad d_M(\alpha) \equiv \|Au_\alpha - f_\delta\|, \qquad b = \text{const} \geq 1. \tag{5}$$

This parameter choice has the following properties:

1) $||u_{\alpha_M} - u_*|| \to 0 \quad \text{for} \quad \delta \to 0$,

2) if

$$u_* - \overline{u} = |A|^p v, \;\; v \in H, \;\; \|v\| \leq \rho, \tag{6}$$

where $|A| \equiv (A^*A)^{1/2}$, then

$$||u_{\alpha_M} - u_*|| \leq c\rho^{1/(p+1)}\delta^{p/(p+1)} \tag{7}$$

for $p \leq 2p_0 - 1$.

The drawback of the a posteriori parameter choice by the discrepancy principle is that by the assumption (6) the order optimal error estimate (7) holds only for $p \leq 2p_0 - 1$, but not for all $p \leq 2p_0$ as by a priori choice of α. To overcome this drawback, one may find $\alpha = \alpha_{RG}$ as the solution of

$$\begin{aligned} d_{RG}(\alpha) &= b\delta, \\ d_{RG}(\alpha) &\equiv \|(I - AA^*g_\alpha(AA^*))^{1/(2p_0)}(Au_\alpha - f_\delta)\|, \end{aligned} \tag{8}$$

with $b = \text{const} \geq 1$. For methods with infinite qualification $p_0 = \infty$, this rule coincides with the discrepancy principle (7). The rule (8) was proposed by Raus [9], a similar rule by Gfrerer [10] (see also [18]). This parameter choice has property 1), the desired property

2') if (6) holds, the estimate (7) is true for $p \leq 2p_0$, and property

3) for compact A with eigenvalues $\lambda_k \in \delta(A^*A)$, $\lambda_k \to 0$ $(k \to \infty)$, by the condition $\lambda_k/\lambda_{k+1} \leq \text{const}$ $(k \to \infty)$, it holds that (see [9,10])

$$\|u_{\alpha_{RG}} - u_*\| \leq c \sup_{f_\delta \in F, \|f_\delta - f\| \leq \delta} \inf_{\alpha \geq 0} \|u_\alpha - u_*\|.$$

Note that the rule (8) is often recommended (see, e.g., [4,5,19]).

3. IDEA OF THE NEW PARAMETER CHOICE RULE

In [20] U. Tautenhahn proposed a new parameter choice rule for the Tikhonov method and proved that it leads to a smaller error of the approximative solution than the parameter choice (8). We consider the idea of the new rule for the class of methods (4). The idea is to find the least α_* for which it is known that the error of the approximation (4), $\|u_\alpha - u_*\|$, is increasing for $\alpha \geq \alpha_*$. We have

$$\begin{aligned}
\frac{d}{d\alpha}\|u_\alpha - u_*\|^2 &= 2\left(u_\alpha - u_*, \frac{d}{d\alpha}u_\alpha\right) \\
&= 2\left(u_\alpha - u_*, \frac{d}{d\alpha}g_\alpha(A^*A)A^*(f_\delta - A\overline{u})\right) \\
&= 2\left(Au_\alpha - f_\delta + f_\delta - f, \frac{dg_\alpha(AA^*)}{d\alpha}(f_\delta - A\overline{u})\right).
\end{aligned}$$

From $\|f_\delta - f\| \leq \delta$ we get

$$\begin{aligned}
&\frac{d}{d\alpha}\|u_\alpha - u_*\|^2 \\
&\geq 2\left[\left(Au_\alpha - f_\delta, \frac{dg_\alpha(AA^*)}{d\alpha}(f_\delta - A\overline{u})\right) - \delta\left\|\frac{dg_\alpha(AA^*)}{d\alpha}(f_\delta - A\overline{u})\right\|\right],
\end{aligned}$$

hence $\frac{d}{d\alpha}\|u_\alpha - u_*\| \geq 0$ for $d(\alpha) \geq \delta$, where

$$d(\alpha) \equiv \frac{(Au_\alpha - f_\delta, \frac{dg_\alpha(AA^*)}{d\alpha}(f_\delta - A\overline{u}))}{\|\frac{dg_\alpha(AA^*)}{d\alpha}(f_\delta - A\overline{u})\|}. \tag{9}$$

Let us introduce the following

Condition C. The function $d(\alpha)\colon [0,\infty) \to \mathbb{R}$ is continuous and strong, monotonically increasing, $\lim_{\alpha\to 0} d(\alpha) \leq \delta$, $\lim_{\alpha\to\infty} d(\alpha) = \|f_\delta - A\overline{u}\|$.

Note that the functions $d_M(\alpha)$ and $d_{RG}(\alpha)$, instead of $d(\alpha)$, fulfill condition C under the assumptions (see [1,2,18])

$$|1 - \lambda g_{\alpha_1}(\lambda)| \leq |1 - \lambda g_{\alpha_2}(\lambda)| \quad (0 \leq \lambda \leq \alpha, \quad 0 < \alpha_1 \leq \alpha_2), \tag{10}$$

$$g_{\alpha_n}(\lambda) \to g_\alpha(\lambda) \quad \text{for} \quad \alpha_n \to \alpha. \tag{11}$$

If condition C holds for $d(\alpha)$ too, then in case $b\delta \le \|f_\delta - A\overline{u}\|$, one may find α_* as the unique solution of

$$d(\alpha) = b\delta, \qquad b \ge 1. \tag{12}$$

Then $\frac{d}{d\alpha}\|u_\alpha - u_*\| > 0$ for $\alpha > \alpha_*$, hence:

1)

$$\|u_{\alpha_*} - u_*\| < \|u_\alpha - u_*\| \quad \text{for} \quad \alpha > \alpha_*; \tag{13}$$

2) if $d(\alpha_1) = b_1\delta$, $d(\alpha_2) = b_2\delta$ with $1 \le b_1 < b_2$, then from monotonicity of the function $d(\alpha)$ follows $\alpha_1 < \alpha_2$, and due to 1), $\|u_{\alpha_1} - u_*\| < \|u_{\alpha_2} - u_*\|$. Hence the best choice for the constant b in (12) is $b = 1$.

4. APPLICATION OF THE NEW RULE TO THE ITERATED TIKHONOV METHOD

The iterated Tikhonov method is a method of the class (4), with the generating function $g_\alpha(\lambda) = \lambda^{-1}[1 - (1+\lambda/\alpha)^{-m}]$, $r = 1/\alpha$, $p_0 = m$, where m is the fixed natural number, $m \ge 1$. Starting from arbitrary initial approximation $u_{0,\alpha} = \overline{u} \in H$, we compute iteratively $u_{1,\alpha}, \ldots, u_{m,\alpha}$ from the equations

$$(A^*A + \alpha I)u_{k,\alpha} = \alpha u_{k-1,\alpha} + A^* f_\delta \qquad (k = 1, \ldots, m)$$

and take $u_\alpha = u_{m,\alpha}$ for the approximative solution of (1). In case $m = 1$, this method is the Tikhonov method.

Theorem 1. *Let* $A^*(f_\delta - A\overline{u}) \ne 0$ *and*

$$\|Qf_\delta\| < b\delta < \|A\overline{u} - f_\delta\|, \tag{14}$$

where Q *denotes the orthoprojection of* F *onto* $\mathcal{N}(A^*) = \overline{\mathcal{R}(A)}^\perp$. *Then the function* $d(\alpha) : [0, \infty) \to \mathbb{R}$ *in the iterated Tikhonov method fulfills condition* C, $d(0) = \|Qf_\delta\|$, *there holds*

$$\frac{d}{d\alpha}(d^2(\alpha)) \ge \frac{2m\|R^{m+1/2}z\|^2}{\alpha^2\|R^{m+1}z\|^2}\|A^*R^{m+1}z\|^2 \tag{15}$$

with $R = (I + \alpha^{-1}AA^*)^{-1}$, $z = A\overline{u} - f_\delta$, *and Eq.* (12) *has the unique solution* $\alpha = \alpha_*$.

Proof. It may be shown by induction that $Au_{k,\alpha} - f_\delta = R^k z$ for all $k = 0, 1, \dots$. From the relation $\frac{dg_\alpha(AA^*)}{d\alpha} = -\frac{m}{\alpha^2}(I + \alpha^{-1}AA^*)^{-m-1}$ we get that the discrepancy function $d(\alpha)$ for the iterated Tikhonov method has the form

$$d(\alpha) = \frac{(Au_\alpha - f_\delta, Au_{m+1,\alpha} - f_\delta)}{\|Au_{m+1,\alpha} - f_\delta\|} = \frac{\|R^{m+1/2}z\|^2}{\|R^{m+1}z\|}, \tag{16}$$

where $u_{m+1,\alpha}$ is the approximation of the iterated Tikhonov method with $m+1$ instead of m. It is clear that $d(\alpha)$ is continuous; we prove the monotonicity. Using the relations

$$\frac{d}{d\alpha}\|R^k z\|^2 = \frac{d}{d\alpha}(R^{2k}z, z) = \frac{2k}{\alpha^2}(AA^*R^{2k+1}z, z) = \frac{2k}{\alpha^2}\|A^*R^{k+1/2}z\|^2,$$

with $k = m + \frac{1}{2}$ and $k = m+1$, for differentiating the square of (16), we get

$$\begin{aligned}\frac{d}{d\alpha}(d^2(\alpha)) = \ & \frac{1}{\|R^{m+1}z\|^4}\Big[2\frac{d}{d\alpha}(\|R^{m+1/2}z\|^2)\|R^{m+1/2}z\|^2\|R^{m+1}z\|^2 \\ & -\|R^{m+1/2}z\|^4\frac{d}{d\alpha}(\|R^{m+1}z\|^2)\Big] \\ = \ & \frac{2\|R^{m+1/2}z\|^2}{\alpha^2\|R^{m+1}z\|^4}\Big[(2m+1)\|A^*R^{m+1}z\|^2\|R^{m+1}z\|^2 \\ & -(m+1)\|R^{m+1/2}z\|^2\|A^*R^{m+3/2}z\|^2\Big].\end{aligned} \tag{17}$$

So, as using

$$\begin{aligned}\|A^*R^k z\|^2 &= \alpha((\alpha^{-1}AA^* + I - I)(I + \alpha^{-1}AA^*)^{-1}R^{k-1}z, R^k z) \\ &= \alpha\|R^{k-1/2}z\|^2 - \alpha\|R^k z\|^2,\end{aligned}$$

with $k = m+1$, $k = m + 3/2$, gives

$$\begin{aligned}&\|A^*R^{m+1}z\|^2\|R^{m+1}z\|^2 - \|R^{m+1/2}z\|^2\|A^*R^{m+3/2}z\|^2 \\ &\quad = \alpha[\|R^{m+1/2}z\|^2\|R^{m+3/2}z\|^2 - \|R^{m+1}z\|^4] \\ &\quad = \alpha[\|R^{m+1/2}z\|^2\|R^{m+3/2}z\|^2 - (R^{m+1/2}z, R^{m+3/2}z)^2] \geq 0,\end{aligned}$$

from (17) we obtain (15).

Taking into account that

$$\begin{aligned}d_{RG}(\alpha) &= \|(I - AA^*g_\alpha(AA^*))^{1/2m}(Au_\alpha - f_\delta)\| = \|R^{1/2}(Au_\alpha - f_\delta)\| \\ &= \|R^{m+1/2}z\| \leq \|R^{m+1/2}z\|^2/\|R^{m+1}z\| = d(\alpha),\end{aligned}$$

from (5), (16) we have

$$d_{RG}(\alpha) \le d(\alpha) \le d_M(\alpha) \quad (\forall \alpha > 0). \tag{18}$$

From (12), (14) and the relation $d_{RG}(0) = d_M(0) = \|Qf_\delta\|$ (see [7,18]) it follows that $d(0) = \|Qf_\delta\|$, condition C is fulfilled, and Eq. (12) has the unique solution $\alpha = \alpha_*$. Theorem 1 is proved.

Remark 1. From the increase of three functions in (18) we get

$$\alpha_M \le \alpha_* \le \alpha_{RG};$$

from (13) we obtain

$$\|u_{\alpha_*} - u_*\| \le \|u_{\alpha_{RG}} - u_*\|. \tag{19}$$

Hence, the new parameter choice rule is "better" than the Raus–Gfrerer rule [9,18] and yields order optimal error estimates. The last result follows directly from the inequality (19) and from the following theorem.

Theorem 2. *Let* $A \in \mathcal{L}(H, F)$, $\|f_\delta - f\| \le \delta$, $\overline{u} - u_* = |A|^p v$, $\|v\| \le \rho$, $p > 0$, *and let* $\alpha = \alpha_{RG}$ *be the solution of Eq. (8). Then, for the Tikhonov and iterated Tikhonov methods the following inequality holds*:

$$\|u_\alpha - u_*\| \le \left\{ (b+1)^{p/(p+1)} + \gamma_* \left(\frac{\overline{\gamma}_{(p+1)/2}}{b-1} \right)^{1/(p+1)} \right\} \rho^{1/(p+1)} \delta^{p/(p+1)},$$

$$0 \le p \le 2p_0, \tag{20}$$

where

$$\overline{\gamma}_{(p+1)/2} = \left(\gamma_{p_0(p+1)/(2p_0+1)}\right)^{1+1/(2p_0)}$$

and $\gamma_* = 1/2$ *for the Tikhonov method and* $\gamma_* = \sqrt{m}$ *for the iterated Tikhonov method.*

Proof. We introduce the operators

$$K_\alpha = I - A^* A g_\alpha(A^* A), \quad \tilde{K}_\alpha = I - AA^* g_\alpha(AA^*).$$

Then

$$u_\alpha - u_* = K_\alpha(\overline{u} - u_*) + g_\alpha(A^* A) A^* (f_\delta - Au_*), \tag{21}$$

$$\tilde{K}_\alpha^{1/(2p_0)}(Au_\alpha - f_\delta) = \tilde{K}_\alpha^{1+1/(2p_0)} A(\overline{u} - u_*) + \tilde{K}_\alpha^{1+1/(2p_0)}(Au_* - f_\delta). \tag{22}$$

If the parameter $\alpha = \alpha_{RG}$ is chosen according to the rule (8), then from (22) and the inequalities $\|f_\delta - f\| \le \delta$, $\|\tilde{K}_\alpha\| \le 1$ it follows that

$$\|\tilde{K}_\alpha^{1+1/(2p_0)} A(\overline{u} - u_*)\| \le (b+1)\delta, \tag{23}$$

$$\|\tilde{K}_\alpha^{1+1/(2p_0)} A(\overline{u} - u_*)\| \geq (b-1)\delta. \tag{24}$$

It is easy to show that for the Tikhonov and iterated Tikhonov methods the inequality

$$\sup_{0\leq\lambda\leq a} \sqrt{\lambda}|g_\alpha(\lambda)| \leq \gamma_*/\sqrt{\alpha} \qquad (\alpha > 0) \tag{25}$$

holds. Now, from (21) and (25) it follows that

$$\|u_\alpha - u_*\| \leq \|K_\alpha(\overline{u} - u_*)\| + \gamma_*\delta/\sqrt{\alpha}. \tag{26}$$

Using (23) and the inequality of the moments

$$\|B^s x\| \leq \|B^t x\|^{s/t}\|x\|^{1-s/t}, \qquad 0 < s \leq t,$$

with $B = K_\alpha^{1/(2p_0)}|A|$, $x = K_\alpha^{1-p/(2p_0)}v$, $s = p$, $t = p+1$, we can estimate

$$\begin{aligned}
\|K_\alpha(u_* - \overline{u})\| &= \|K_\alpha|A|^p v\| \\
&\leq \|K_\alpha^{1+1/(2p_0)}|A|^{1+p}v\|^{p/(p+1)}\|K_\alpha^{1-p/(2p_0)}v\|^{1/(p+1)} \\
&\leq \|\tilde{K}_\alpha^{1+1/(2p_0)} A(\overline{u} - u_*)\|^{p/(p+1)}\rho^{1/(p+1)} \\
&\leq (b+1)^{p/(p+1)}\rho^{1/(p+1)}\delta^{p/(p+1)}.
\end{aligned} \tag{27}$$

Using the condition (3), we obtain

$$\begin{aligned}
\|\tilde{K}_\alpha^{1+1/(2p_0)} A(\bar{u} - u_*)\| &= \|\tilde{K}_\alpha^{1+1/(2p_0)}|A|^{p+1}v\| \\
&\leq \rho \sup_{0\leq\lambda\leq\alpha} \lambda^{(1+p)/2}(1-\lambda g_\alpha(\lambda))^{1+1/(2p_0)} \\
&= \rho\Big(\sup_{0\leq\lambda\leq\alpha} \lambda^{p_0(1+p)/(2p_0+1)}(1-\lambda g_\alpha(\lambda))\Big)^{1+1/(2p_0)} \\
&\leq \rho\overline{\gamma}_{(1+p)/2}\alpha^{(1+p)/2},
\end{aligned}$$

from which, together with (24), it follows that

$$\alpha \geq ((b-1)/\overline{\gamma}_{(1+p)/2})^{2/(p+1)}\rho^{-2/(p+1)}\delta^{2/(p+1)}. \tag{28}$$

Now the assertion of the theorem follows from (26)–(28).

Remark 2. For the Tikhonov method,

$$\overline{\gamma}_{(p+1)/2} = (\gamma_{(p+1)/3})^{3/2} = 3^{-3/2}(p+1)^{(p+1)/2}(2-p)^{(2-p)/2} \leq 1,$$

$0 \leq p \leq 2$, and the estimation (20) has the form

$$\|u_\alpha - u_*\| \leq [(b+1)^{p/(p+1)} + \tfrac{1}{2}(b-1)^{-1/(p+1)}]\rho^{1/(p+1)}\delta^{p/(p+1)}.$$

5. APPLICATION OF THE NEW RULE TO THE METHODS WITH INFINITE QUALIFICATION

In this section we compare the new parameter choice (12) and the discrepancy principle (5) for some well-known methods with the infinite qualification. We consider the following methods.

1. The Richardson iteration method. It has the form

$$u_r = u_{r-1} - \mu A^*(Au_{r-1} - f_\delta), \, r = 1, 2 \ldots, \quad 0 < \mu < 1/\|A^* A\|.$$

This method is of the form (4), with

$$r = 1/\alpha, \qquad g_r(\lambda) = (1 - (1 - \mu\lambda)^r)/\lambda.$$

2. The implicit iteration method. Let $\rho > 0$ be a constant. We determine consequtively u_r as the solutions of the equations

$$\rho u_r + A^* A u_r = \rho u_{r-1} + A^* f_\delta, \qquad r = 1, 2, \ldots.$$

For this method the generating function has the form

$$g_r(\lambda) = (1 - (\rho/(\rho + \lambda))^r)/\lambda, \qquad r = 1/\alpha.$$

3. The method of the Cauchy problem (a continuous analogue of iterative methods). We take as approximation u_r the solution of the Cauchy problem

$$u'(r) + A^* A u(r) = A^* f_\delta, \qquad u(0) = \overline{u}.$$

This method is of the form (4), with

$$r = 1/\alpha, \qquad g_r(\lambda) = (1 - e^{-r\lambda})/\lambda.$$

Using the inequalities

$$\lambda/(\lambda + \rho) \leq \ln(1 + \lambda/\rho) \leq \lambda/\rho,$$

$$\mu\lambda \leq -\ln(1 - \mu\lambda) \leq \mu\lambda/(1 - \mu\lambda),$$

it is easy to show that, for methods 1–3, it holds that

$$\gamma(1 - \lambda g_{r+k_1}(\lambda)) \leq \frac{dg_r(\lambda)}{dr} \leq \gamma(1 - \lambda g_{r+k_2}(\lambda)),$$

where $\gamma = \mu$, $k_1 = 0$, $k_2 = -1$ for method 1; $\gamma = 1/\rho$, $k_1 = 1$, $k_2 = 0$ for method 2; and $\gamma = 1$, $k_1 = k_2 = 0$ for method 3. Using the last inequalities, we get

$$\left\| \frac{dg_r(AA^*)}{dr}(f_\delta - A\overline{u}) \right\| \leq \gamma \|Au_{r+k_2} - f_\delta\|,$$

$$\left(Au_r - f_\delta, \frac{dg_r(AA^*)}{dr}(f_\delta - A\overline{u})\right) \geq \gamma(Au_r - f_\delta, Au_{r+k_1} - f_\delta)$$

$$= \gamma\|Au_{r+k_1/2} - f_\delta\|^2,$$

from which it follows that

$$\frac{\|Au_{r+k_1/2} - f_\delta\|^2}{\|Au_{r+k_2} - f_\delta\|} \leq d(r) \leq \|Au_r - f_\delta\|.$$

From this result we see that for the method of the Cauchy problem the function $d(r)$ coincides with the discrepancy function $d_M(r)$ and that for iterative methods the function $d(r)$ is close (at least for a small error level δ) to the function $d_M(r)$.

6. SOME REMARKS

Remark 3. It is not clear how to apply the idea of the new parameter choice rule to self-adjoint problems. Note that other parameter choices for self-adjoint problems were studied, e.g., in [2,7,8,13,16,21], for non-self-adjoint problems in [1–7,9–15,17–19,22–24] (whereas [2,5,7,10,14,17,18] deal also with problems of quasisolution ($f \notin \mathcal{R}(A)$, $Qf \in \mathcal{R}(A)$) as well; [4,5,22–24] treat the case of the unknown error level δ).

Remark 4. Consider the case when the problem (1) is at first discretized by the projection method $A_h u_h = Q_h f_\delta$, $A_h = Q_h A P_h$, $u_h \in \mathcal{R}(P_h)$, where $h > 0$ is the discretization step and P_h, Q_h are orthoprojectors in H, F, respectively, with $\|A(I - P_h\| \to 0$, $\|(I - Q_h)A\| \to 0\ (h \to 0)$. Regularizing the discretized problem by the method (4), we get the approximation

$$u_{h,\alpha} = P_h\overline{u} + g_\alpha(A_h^* A_h)A_h^*(f_\delta - A_h\overline{u}). \tag{29}$$

If $\alpha = \alpha_{RG}$ is chosen from the discrete version of (8),

$$\|(I - A_h A_h^* g_\alpha(A_h A_h^*))^{1/(2p_0)}(A_h\overline{u} - Q_h f_\delta)\| = b\delta, \qquad b \geq 1,$$

then, under the assumptions $\left.\frac{d(g_s(\lambda))}{ds}\right|_{s=\alpha} \leq \mathrm{const}(1 - \lambda g_\alpha(\lambda))^{1+1/p_0}$, $u_* = |A|^p w$, $w \in H$, $\|w\| \leq \rho$, (2), (3), (6), (10), (11), it holds for $p \leq 2p_0$ (see [13]) that

$$\|u_{h,\alpha_{RG}} - u_*\| \leq c\{\rho^{1/(p+1)}\delta^{p/(p+1)} + \rho\|(I - P_h)|A|^p\| + \rho e(Q_h)\}, \tag{30}$$

where

$$e(Q_h) = \begin{cases} \|(I - Q_h)|A^*|^\mu\|^{\min\{p/\mu, 2\}}\ \forall\mu,\ 0 < \mu \leq 1,\ \mu \neq p/2 & \text{for } p \leq 2, \\ (1 + \ln\|(I - Q_h)|A^*|^{p/2}\|)\|(I - Q_h)|A^*|^{p/2}\|^2 & \text{for } p \leq 2, \\ \|(I - Q_h)A\|\,\|(I - Q_h)|A^*|^{p-1}\| & \text{for } p \geq 2, \end{cases}$$

and $|A^*| \equiv (AA^*)^{1/2}$. A reasonable choice of h in (29) is determined by the rule

$$\|(I-P_h)|A|^{2p_0/(2p_0+1)}\|^{1+1/(2p_0)} + \|(I-Q_h)|A^*|^{p_0/(2p_0+1)}\|^{2+1/p_0} \approx \delta,$$

guaranteeing the error estimate $\|u_{h,\alpha}-u_*\| \le c\delta^{p/(p+1)}$ for $p \le 2p_0$ (see [13]).

If for regularization the iterated Tikhonov method is used, then we find $u_{h,0,\alpha} = P_h\overline{u}$, compute $u_{h,1,\alpha},\dots,u_{h,m,\alpha}$ from the equations

$$(A_h^*A_h+\alpha I)u_{h,k,\alpha} = \alpha u_{h,k-1,\alpha} + A_h^* f_\delta \qquad (k=1,\dots,m),$$

and take $u_{h,\alpha} = u_{h,m,\alpha}$. If the least error method ($A_h = Q_hA$) is regularized by the iterated Tikhonov method and the parameter $\alpha = \alpha_*$ is chosen from the equation (compare (8), (16))

$$(A_hu_{h,\alpha} - Q_hf_\delta, A_hu_{h,m+1,\alpha} - Q_hf_\delta) = b\|A_hu_{h,m+1,\alpha} - Q_hf_\delta\|\delta, \qquad b \ge 1,$$

then we may get the analogue of (19): $\|u_{h,\alpha_*} - u_*\| \le \|u_{h,\alpha_{RG}} - u_*\|$. Hence, for $\|u_{h,\alpha_*} - u_*\|$, the estimate (30) holds under the assumption (6) (with $p \le 2p_0$) as well. Other error estimates for regularized projection methods are given in [12,13,16,25].

7. NUMERICAL EXAMPLES

Consider the Fredholm integral equation $\int_0^1 K(t,s)u(s)ds = f(t)$, with the kernel

$$K(t,s) = \begin{cases} t(1-s) & \text{if} \quad t \le s \\ s(1-t) & \text{if} \quad t > s, \end{cases}$$

as the equation $Au = f$ with the operator $A\colon L_2(0,1) \to L_2(0,1)$.

The equation was discretized by the Galerkin method using 1200 piecewise constant basis functions and after that solved by the Tikhonov method 20 times adding to $f(t)$ a random noise with a relative error $\delta_r = \delta/\|f\|$. Three equations with the following forms of $u(s)$ and $f(t)$ were solved (examples 2–4 from [18]):

1) $u(s) = s - s^2$, $f(t) = (t^4 - 2t^3 + t)/12$, $\|f\| = 0.0185$;

2) $u(s) = s - 2s^3 + 3s^4$, $f(t) = (-t^6 + 3t^5 - 5t^3 + 3t)/30$, $\|f\| = 0.2248$;

3) $u(s) = \sin(\pi s)$, $f(t) = \sin(\pi t)/\pi^2$, $\|f\| = 0.0716$.

The averages and maximums of errors for three parameter choices (discrepancy principle, Raus–Gfrerer rule, new rule; in all cases $b = 1$ was used) are presented in Table 1.

Table 1. Errors for three parameter choice rules

δ_r	Averages			Maximums		
	$\|u_{\alpha_M}-u\|$	$\|u_{\alpha_{RG}}-u\|$	$\|u_{\alpha}-u\|$	$\|u_{\alpha_M}-u\|$	$\|u_{\alpha_{RG}}-u\|$	$\|u_{\alpha}-u\|$
Equation (1)						
10^{-1}	5.15×10^{-3}	6.72×10^{-3}	5.55×10^{-3}	6.57×10^{-3}	8.31×10^{-3}	7.25×10^{-3}
10^{-2}	1.97×10^{-3}	2.03×10^{-3}	1.66×10^{-3}	2.34×10^{-3}	2.51×10^{-3}	2.11×10^{-3}
10^{-3}	7.55×10^{-4}	6.23×10^{-4}	5.11×10^{-4}	8.68×10^{-4}	7.38×10^{-4}	6.15×10^{-4}
10^{-4}	2.90×10^{-4}	1.91×10^{-4}	1.57×10^{-4}	3.22×10^{-4}	2.21×10^{-4}	1.86×10^{-4}
10^{-5}	1.07×10^{-4}	6.17×10^{-5}	5.11×10^{-5}	1.16×10^{-4}	6.61×10^{-5}	5.55×10^{-5}
Equation (2)						
10^{-1}	6.86×10^{-3}	7.59×10^{-3}	6.22×10^{-3}	9.29×10^{-3}	8.69×10^{-3}	7.29×10^{-3}
10^{-2}	2.62×10^{-3}	1.86×10^{-3}	1.54×10^{-3}	3.12×10^{-3}	2.10×10^{-3}	1.76×10^{-3}
10^{-3}	1.01×10^{-3}	4.53×10^{-4}	3.80×10^{-4}	1.08×10^{-3}	5.06×10^{-4}	4.28×10^{-4}
10^{-4}	3.70×10^{-4}	1.11×10^{-4}	9.31×10^{-5}	4.03×10^{-4}	1.22×10^{-4}	1.03×10^{-4}
10^{-5}	1.30×10^{-4}	2.67×10^{-5}	2.24×10^{-5}	1.42×10^{-4}	2.92×10^{-5}	2.48×10^{-5}
Equation (3)						
10^{-1}	2.50×10^{-2}	2.10×10^{-2}	1.80×10^{-2}	2.99×10^{-2}	2.57×10^{-2}	2.23×10^{-2}
10^{-2}	9.47×10^{-3}	5.38×10^{-3}	4.63×10^{-3}	1.14×10^{-2}	6.72×10^{-3}	5.81×10^{-3}
10^{-3}	3.37×10^{-3}	1.38×10^{-3}	1.18×10^{-3}	3.76×10^{-3}	1.63×10^{-3}	1.40×10^{-3}
10^{-4}	1.20×10^{-3}	3.49×10^{-4}	3.00×10^{-4}	1.33×10^{-3}	3.93×10^{-4}	3.35×10^{-4}
10^{-5}	4.25×10^{-4}	8.32×10^{-5}	7.12×10^{-5}	4.69×10^{-4}	9.43×10^{-5}	8.08×10^{-5}

ACKNOWLEDGEMENT

This work was supported by the Estonian Science Foundation (grant No. 3026). We thank Reimo Palm, University of Tartu, for providing a MATLAB-code for the numerical computations.

REFERENCES

1. Groetsch, C. W. *The Theory of Tikhonov Regularization for Fredholm Equations of the First Kind*. Pitman, Boston, 1984.
2. Vainikko, G. M. and Veretennikov, A. Y. *Iteration Procedures in Ill-Posed Problems*. Nauka, Moscow, 1986 (in Russian).

3. Louis, A. K. *Inverse und schlecht gestellte Probleme*. Teubner, Stuttgart, 1989.
4. Hanke, M. and Hansen, P. C. Regularization methods for large-scale problems. *Surv. Math. Ind.*, 1993, **4**, 3, 253–315.
5. Engl, H. W., Hanke, M. and Neubauer, A. *Regularization of Inverse Problems*. Kluwer, Dordrecht, 1996.
6. Bakushinskii, A. B. A general method of constructing regularizing algorithms for a linear ill-posed equation in Hilbert space. *Zh. Vychisl. Mat. Mat. Fiz.*, 1967, **7**, 672–677 (in Russian).
7. Vainikko, G. The discrepancy principle for a class of regularization methods. *U.S.S.R. Comput. Math. Math. Phys.*, 1982, **22**, 3, 1–19.
8. Raus, T. On the discrepancy principle for the solution of ill-posed problems. *Acta Comment. Univ. Tartuensis*, 1984, 672, 16–26 (in Russian).
9. Raus, T. On the discrepancy principle for solution of ill-posed problems with non-selfadjoint operator. *Acta Comment. Univ. Tartuensis*, 1985, 715, 12–20.
10. Engl, H. W. and Gfrerer, H. A posteriori parameter choice for general regularization methods for solving linear ill-posed problems. *Appl. Numer. Math.*, 1988, **4**, 395–417.
11. Groetsch, C. W. and Neubauer, A. Regularization of ill-posed problems: optimal parameter choice in finite dimensions. *J. Approx. Theory*, 1989, **58**, 184–200.
12. Plato, R. and Vainikko, G. On the regularization of projection methods for solving ill-posed problems. *Numer. Math.*, 1990, **57**, 1, 63–79.
13. Hämarik, U. On the discretization error in regularized projection method with parameter choice by discrepancy principle. In *Ill-Posed Problems in Natural Sciences. Proc. Int. Conf.* (Tikhonov, A. N. et al., eds.). VSP, Utrecht/TVP. Sci. Publ., Moscow, 1992, 24–28.
14. Raus, T. On the regularization parameter choice for ill-posed problems with a quasi-solution. In *Ill-posed Problems in Natural Sciences. Proc. Int. Conf.* (Tikhonov, A. N. et al., eds.). VSP, Utrecht/TVP. Sci. Publ., Moscow, 1992, 144–148.
15. Raus, T. About regularization parameter choice in case of approximately given error bounds of data. *Acta Comment. Univ. Tartuensis*, 1992, 937, 77–89.
16. Hämarik, U. On the parameter choice in the regularized Ritz–Galerkin method. *Proc. Estonian Acad. Sci. Phys. Math.*, 1993, **42**, 2, 133–143.
17. Morozov, V. A. On the solution of functional equations by the method of regularization. *Soviet Math. Dokl.*, 1966, **7**, 414–417.
18. Gfrerer, H. An a posteriori parameter choice for ordinary and iterated Tikhonov regularization of ill-posed problems leading to optimal convergence rates. *Math. Comput.*, 1987, **49**, 507–522.
19. Lukas, M. A. Asymptotic behaviour of the minimum bound method for choosing the regularization parameter. *Inverse Probl.*, 1998, **14**, 1, 149–159.
20. Tautenhahn, U. On a new parameter choice rule for ill-posed inverse problems. In *Proc. Int. Conf. HERCMA'98, Athens* (Lipitakis, E. A., ed.). Lea Publishers, 1999 (in press).
21. Plato, R. and Hämarik, U. On pseudo-optimal parameter choices and stopping rules for regularization methods in Banach spaces. *Numer. Funct. Anal. Optim.*, 1996, **17**, 1/2, 181–195.
22. Hanke, M. and Raus, T. A general heuristic for choosing the regularization parameter in ill-posed problems. *SIAM J. Sci. Comput.*, 1996, **17**, 4, 956–972.
23. Wahba, G. Practical approximate solutions to linear operator equations when the data are noisy. *SIAM J. Numer. Anal.*, 1977, **14**, 651–667.
24. Hansen, P. C. and O'Leary, D. The use of the L-curve in the regularization of discrete ill-posed problems. *SIAM J. Sci. Comput.*, 1993, **14**, 6, 1487–1503.
25. King, J. T. and Neubauer, A. A variant of finite dimensional Tikhonov regularization with a-posteriori parameter choice. *Computing*, 1988, **40**, 91–109.

APOSTERIOORSEST PARAMEETRIVALIKUST REGULARISATSIOONIMEETODITES

Uno HÄMARIK ja Toomas RAUS

U. Tautenhahn pakkus hiljuti välja uue aposterioorse reegli regularisatsiooniparameetri valikuks mittekorrektse ülesande lahendamisel Tihhonovi meetodiga. Siinses artiklis on üldistatud see parameetrivaliku reegel itereeritud Tihhonovi meetodile ning näidatud, et see valikureegel garanteerib väiksema vea kui Rausi–Gfrereri parameetrivaliku reegel.