Proc. Estonian Acad. Sci. Phys. Math., 1999, **48**, 2, 123–132
https://doi.org/10.3176/phys.math.1999.2.06

COMPLEXITY FOR SOME CLASSES OF WELL-POSED PROBLEMS

Sergei G. SOLODKY

Institute of Mathematics of the Ukrainian Academy of Sciences, Tereschenkivska St. 3, 252601 Kiev, Ukraine; solodky@imath.kiev.ua

Received 25 September 1998, in revised form 18 December 1998

Abstract. For some classes of equations of the second kind in Hilbert space the exact power order of complexity of the approximate solution is found. It is established that the optimal power order is realized by iterative methods which use the Galerkin information with indexes from the hyperbolic crosses.

Key words: complexity, Galerkin information, hyperbolic cross.

Let $\{e_i\}_{i=1}^{\infty}$ be some orthonormal basis in a Hilbert space X, and let P_n be the orthogonal projector on $\text{span}\{e_1, e_2, \ldots, e_n\}$, i.e. $P_n\varphi = \sum_{i=1}^{n} e_i(e_i, \varphi)$, where $(\cdot,\cdot)$ is an inner product in X. We denote by X^r, $r = 1, 2, \ldots$, a linear subspace of X such that for any $m = 1, 2, \ldots$, $\|I - P_m\|_{X^r \to X} \leq \beta_r m^{-r}$, where I is the identity operator and the constant $\beta_r \geq 1$ is independent of m, and for all $\varphi \in X^r$, $\|\varphi\|_X \leq \|\varphi\|_{X^r}$.

Moreover, let $\mathcal{L}(X, X^r)$ be the space of linear and continuous operators H acting from X into X^r with usual norm. We denote by $\Psi^r_{\mathcal{H},\Phi}$ the class of uniquely solvable equations of the second kind

$$z = Hz + f, \tag{1}$$

where $H \in \mathcal{H} \subset \mathcal{L}(X, X^r)$ and $f \in \Phi \subset X^r$.

We shall investigate the complexity of finding an approximate solution to Eq. (1) for some classes $\Psi^r_{\mathcal{H},\Phi}$. The formulation of the problem and terminology are borrowed from [1,2].

Let $T = \{\delta_i\}_{i=1}^m$ be a collection of continuous functionals δ_i, of which $\delta_1, \delta_2, \ldots, \delta_\mu$ are defined on the set $\mathcal{H}$ and $\delta_{\mu+1}, \ldots, \delta_m$ on the set Φ, $\mathrm{card}(T) = m$,

$$\mathcal{T}_M = \{T : \mathrm{card}(T) \le M\}.$$

To each Eq. (1) of $\Psi^r_{\mathcal{H},\Phi}$ we assign the numerical vector

$$T(H, f) = \Big(\delta_1(H), \delta_2(H), \ldots, \delta_\mu(H), \delta_{\mu+1}(f), \ldots, \delta_m(f)\Big), \tag{2}$$

which we call the information about Eq. (1); the collection of functionals T will be called a method of specifying information.

By the algorithm A for approximate solving of equations from $\Psi^r_{\mathcal{H},\Phi}$ we mean the operator assigning an element $A(T, H, f) \in X$ to information (2) as an approximate solution of Eq. (1). We assume that any algorithm A is connected with the parametric set of elements

$$F_A = \{\varphi_{\xi_1,\xi_2,\ldots,\xi_n} : \varphi_{\xi_1,\xi_2,\ldots,\xi_n} \in X, \quad \xi_i \in R_1, \quad i = 1, 2, \ldots, n\}$$

and $A(T, H, f) = \varphi_{\xi_1,\xi_2,\ldots,\xi_n} \in F_A$, where each value ξ_i, $i = 1, 2, \ldots, n$, depends on the components of the vector $T(H, f)$. For calculation of these values it is required to execute only a finite number of arithmetic operations (AOs) on $\delta_1(H), \delta_2(H), \ldots \delta_\mu(H), \delta_{\mu+1}(f), \ldots, \delta_m(f)$. We denote by $\mathcal{A}_N(T)$ the set of algorithms A in which it is required to perform no more than N AOs on the components of the vector (2) to determine $A(T, H, f) \in F_A$. For considering algorithms from $\mathcal{A}_N(T)$ it is natural to suppose that $T \in \mathcal{T}_M$ at $M \le N$. Otherwise, no algorithm of $\mathcal{A}_N$ can utilize all information represented by the components of the vector (2). The error of the algorithm A on the class $\Psi^r_{\mathcal{H},\Phi}$ is defined as

$$e(\Psi^r_{\mathcal{H},\Phi}, A) = \sup_{\substack{z = Hz + f \\ H \in \mathcal{H}, f \in \Phi}} \|z - A(T, H, f)\|_X.$$

The quantity

$$E_N(\Psi^r_{\mathcal{H},\Phi}) = \inf_{\substack{T \in \mathcal{T}_M \\ M \le N}} \inf_{A \in \mathcal{A}_N(T)} e(\Psi^r_{\mathcal{H},\Phi}, A)$$

is the minimal error, which we are able to guarantee on the class $\Psi^r_{\mathcal{H},\Phi}$ after the execution of N AOs on the informational functionals δ_i. Thus, the quantity E_N characterizes the complexity of approximate solving of equations from $\Psi^r_{\mathcal{H},\Phi}$.

Let Π_N be the set of all continuous maps π from X^r into the N-dimensional Euclidean space R_N. Moreover, let $\pi^{-1} \circ \pi(\varphi)$ be an inverse image of the element $\pi(\varphi) \in R_N$, $\varphi \in X^r$. The quantity

$$\Delta_N(X^r_d, X) = \inf_{\pi \in \Pi_N} \sup_{\varphi \in X^r_d} \sup_{f, g \in \pi^{-1} \circ \pi(\varphi)} \|f - g\|_X$$

is called [3] a pretabulated width of the set

$$X_d^r = \{\varphi : \varphi \in X^r, \ \|\varphi\|_{X^r} \le d\}.$$

The next lemma ascertains a connection between $E_N(\Psi^r_{\mathcal{H},\Phi})$ and the pretabulated width $\Delta_N(X_d^r, X)$.

Lemma 1 [4]. *Let $\mathcal{H}$ be the following set of linear operators:*

$$\mathcal{H} = \mathcal{H}_\gamma^{r,0} =: \{H : H \in \mathcal{L}(X, X^r), \ \|(I-H)^{-1}\|_{X\to X} \le \gamma_0, \ \|H\|_{X\to X^r} \le \gamma_1\},$$

$$\gamma = (\gamma_0, \gamma_1).$$

Then

$$E_N(\Psi^r_{\mathcal{H},\Phi}) \ge \frac{1}{2}\Delta_N(X_d^r, X),$$

where $d = (1+\gamma_1)^{-1}$.

Now consider the set of methods for specifying information which we call the Galerkin information. The so-called Galerkin method of approximate solution of Eq. (1) reduces to the situation where a uniquely solvable equation

$$z_G = P_n H z_G + P_n f,$$

is assigned to Eq. (1) and z_G is taken as an approximate solution of Eq. (1). It is clear that $z_G = \sum_{i=1}^n \eta_i e_i$, where unknown coefficients η_i will be found from the following system of linear equations:

$$\eta_i = \sum_{j=1}^n \eta_j (e_i, He_j) + (e_i, f).$$

Thus, in the Galerkin method, to construct the approximate solution z_G it is necessary to have the information (2), where $\delta_1(H) = (e_{i_1}, He_{j_1}), \ldots, \delta_\mu(H) = (e_{i_\mu}, He_{j_\mu}), \delta_{\mu+1}(f) = (e_{i_{\mu+1}}, f), \ldots, \delta_m(f) = (e_{i_m}, f)$. Information of this type we call the Galerkin information.

Denote by $\Psi_\gamma^{r,s}$, $r, s = 1, 2, \ldots$, the class of Eq. (1) whose free terms f belong to the ball X_1^r and operators H belong to the class

$$\mathcal{H}_\gamma^{r,s} = \{H : H \in \mathcal{H}_\gamma^{r,0}, \quad H^* \in \mathcal{L}(X, X^s), \quad \|H\|_{X\to X^r} + \|H^*\|_{X\to X^s} \le \gamma_1\}.$$

Note that the complexity of the classes $\Psi_\gamma^{r,s}$ was investigated already in [4], but there it was assumed that $s \le r \le 2s$. Our purpose is to find the exact power order of the complexity for the whole scale of the classes $\Psi_\gamma^{r,s}$.

Let us introduce some notation: we write $a_n \prec b_n$ if there is a constant c_0 such that for all $n \geq n_0$, $a_n \leq c_0 b_n$; we write $a_n \asymp b_n$ if $a_n \prec b_n$ and $b_n \prec a_n$.

Let Γ_n be the planar set having the form of a hyperbolic cross

$$\Gamma_n = \bigcup_{j=1}^{2n} [1, 2^{2n-j}] \times (2^{j-1}, 2^j] \bigcup [1, 2^{2n}] \times \{1\}.$$

We consider the method of specifying information T_n^Γ, determined by Galerkin functionals (e_i, He_j) with indexes from Γ_n. Namely, $T_n^\Gamma(H, f)$ is the Galerkin information of the form

$$T_n^\Gamma(H, f) = \big((e_i, He_j), \quad (e_k, f); \quad (i, j) \in \Gamma_n, \quad k = 1, 2, \ldots, 2^{2n}\big).$$

Let us assign to each operator $H \in \mathcal{H}_\gamma^{r,s}$ the finite-dimensional operator

$$\begin{aligned} H^\Gamma &= H^\Gamma(H) := \sum_{j=1}^{2n} P_{2^{2n-j}} H (P_{2^j} - P_{2^{j-1}}) + P_{2^{2n}} H P_1 \\ &= \sum_{j=1}^{2n} (P_{2^j} - P_{2^{j-1}}) H P_{2^{2n-j}} + P_1 H P_{2^{2n}}. \end{aligned} \tag{3}$$

Now we give some subsidiary results which will be used later on.

Lemma 2. *Let $r, s = 1, 2, \ldots$ Then, for $H \in \mathcal{H}_\gamma^{r,s}$*

$$\|H - H^\Gamma(H)\|_{X \to X} \leq \gamma_1 c_1 2^{-2rsn/(r+s)} \tag{4}$$

and for $H \in \mathcal{H}_\gamma^{r,0}$

$$\|H - H^\Gamma(H)\|_{X^r \to X} \leq 2^{r+1} \gamma_1 \beta_r^2 2^{-2rn} \sqrt{n}, \tag{5}$$

where

$$c_1 = \frac{2^r}{\sqrt{2^{2r} - 1}} \beta_r + 2^s \Big(1 + \frac{2^r}{\sqrt{2^{2r} - 1}}\Big) \beta_s.$$

Proof. First of all we note that for $H \in \mathcal{H}_\gamma^{r,s}$ and for any $k = 1, 2, \ldots$

$$\begin{aligned} \|H - P_{2^k} H\|_{X \to X} &\leq \gamma_1 \beta_r 2^{-kr}, \\ \|H - H P_{2^k}\|_{X \to X} &\leq \gamma_1 \beta_s 2^{-ks}. \end{aligned} \tag{6}$$

Taking into account (3) and (6), we have for any $f \in X_1^r$ ($\|f\|_{X^r} \leq 1$)

$$
\begin{aligned}
&\|(H - H^\Gamma)f\|_X \\
&= \|(H - P_{2^{2n}}H)f + (P_{2^{2n}}H - H^\Gamma)f\|_X \\
&= \sup_{\|g\|_X \leq 1} \Big| \Big(g, (I - P_{2^{2n}})Hf + P_1 H(I - P_{2^{2n}})f \\
&\quad + \sum_{k=1}^{2n} (P_{2^k} - P_{2^{k-1}})H(I - P_{2^{2n-k}})f\Big)\Big| \\
&\leq \sup_{\|g\|_X \leq 1} \Big(|((I - P_{2^{2n}})g, (I - P_{2^{2n}})Hf)| + |(P_1 g, P_1 H(I - P_{2^{2n}})f)| \\
&\quad + \sum_{k=1}^{2n} |((P_{2^k} - P_{2^{k-1}})g, (P_{2^k} - P_{2^{k-1}})H(I - P_{2^{2n-k}})f)|\Big) \\
&\leq \sup_{\|g\|_X \leq 1} \Big(\|(I - P_{2^{2n}})g\|_X \|(I - P_{2^{2n}})Hf\|_X + \|P_1 g\|_X \|P_1 H(I - P_{2^{2n}})f\|_X \\
&\quad + \sum_{k=1}^{2n} \|(P_{2^k} - P_{2^{k-1}})g\|_X \|(I - P_{2^{k-1}})H(I - P_{2^{2n-k}})f\|_X \Big) \\
&\leq \sup_{\|g\|_X \leq 1} \Big(\|(I - P_{2^{2n}})g\|_X^2 + \|P_1 g\|_X^2 + \sum_{k=1}^{2n} \|(P_{2^k} - P_{2^{k-1}})g\|_X^2 \Big)^{1/2} \\
&\quad \times \Big(\|(I - P_{2^{2n}})Hf\|_X^2 + \|H\|_{X \to X}^2 \|(I - P_{2^{2n}})f\|_X^2 \\
&\quad + \sum_{k=1}^{2n} \|(I - P_{2^{k-1}})H\|_{X \to X}^2 \|(I - P_{2^{2n-k}})f\|_X^2 \Big)^{1/2} \\
&\leq \Big(2\gamma_1^2 \beta_r^2 2^{-4rn} + \gamma_1^2 \beta_r^4 \sum_{k=1}^{2n} 2^{-2r(k-1)} 2^{-2r(2n-k)} \Big)^{1/2} \\
&\leq 2^r \gamma_1 \beta_r^2 2^{-2rn} \sqrt{2n+2}.
\end{aligned} \tag{7}
$$

The estimate (5) follows from (6), (7), and from the inequality

$$\|H - H^\Gamma\|_{X^r \to X} \leq \|H(I - P_{2^{2n}})\|_{X^r \to X} + \|HP_{2^{2n}} - H^\Gamma\|_{X^r \to X}.$$

In a similar manner, we obtain (4). The Lemma is proved. □

Lemma 3. *For $H \in \mathcal{H}_\gamma^{r,0}$, $H^\Gamma = H^\Gamma(H)$ in* (3), *and any $m < 2n$ we have*

$$\|H^\Gamma - P_{2^m} H^\Gamma\|_{X \to X} \leq 2\gamma_1 \beta_r 2^{-rm}. \tag{8}$$

Proof. Using the definition (3) of the operator H^Γ we obtain for $m < 2n$

$$\begin{aligned} P_{2^m} H^\Gamma &= P_{2^m} H P_1 + \sum_{k=1}^{2n-m-1} P_{2^m} H(P_{2^k} - P_{2^{k-1}}) \\ &+ \sum_{k=2n-m}^{2n} P_{2^{2n-k}} H(P_{2^k} - P_{2^{k-1}}) \\ &= P_{2^m} H P_{2^{2n-m-1}} + \sum_{k=2n-m}^{2n} P_{2^{2n-k}} H(P_{2^k} - P_{2^{k-1}}), \end{aligned}$$

$$\begin{aligned} H^\Gamma - P_{2^m} H^\Gamma &= (HP_{2^{2n-m-1}} P_{2^m} H^\Gamma) + (H^\Gamma - HP_{2^{2n-m-1}}) \\ &= (I - P_{2^m}) H P_{2^{2n-m-1}} - \sum_{k=1}^{2n-m-1} (I - P_{2^{2n-k}}) H (P_{2^k} - P_{2^{k-1}}) \\ &- (I - P_{2^{2n}}) H P_1 . \end{aligned}$$

As is easy to calculate,

$$\begin{aligned} \|H^\Gamma - P_{2^m} H^\Gamma\|_{X\to X} &\le \|(I - P_{2^m})H\|_{X\to X} + \sum_{k=0}^{2n-m-1} \|(I - P_{2^{2n-k}})H\|_{X\to X} \\ &\le \gamma_1 \beta_r 2^{-rm} + \gamma_1 \beta_r 2^{-2rn} \sum_{k=0}^{2n-m-1} 2^{kr} \\ &= \gamma_1 \beta_r \left(2^{-rm} + 2^{-2rn} \frac{2^{(2n-m)r} - 1}{2^r - 1} \right). \end{aligned}$$

This implies the desired estimate. □

Lemma 4 [4]. *Let*

$$g = \sum_{i=1}^{2^{2n}} \alpha_i e_i$$

be an arbitrary element of the subspace $\mathrm{span}\{e_1, e_2, \ldots, e_{2^{2n}}\}$. *To represent the element* $H^\Gamma g$ *in the standard form*

$$H^\Gamma g = \sum_{j=1}^{2^{2n}} \beta_j e_j, \tag{9}$$

it suffices to perform no more than $O(n2^{2n})$ *AOs on the components of the vector* $T_n^\Gamma(H, f)$ *and coefficients* $\alpha_i,\ i = 1, \ldots, 2^{2n}$.

For each Eq. (1) we determine the sequence of elements

$$z_0 = 0,\ z_k = z_{k-1} + (I - P_{2^m} H^\Gamma)^{-1}(H^\Gamma z_{k-1} - z_{k-1} + P_{2^{2n}} f),$$
$$k = 1,2,3; \quad m = \left[\frac{2}{3}n\right] + 1. \tag{10}$$

All these elements belong to the subspace $\operatorname{span}\{e_1, e_2, \dots, e_{2^{2n}}\}$. To construct the elements z_k, $k = \overline{1,3}$, it suffices to have the information $T_n^\Gamma(H, f)$ and to solve the equations

$$\varepsilon_k = P_{2^m} H^\Gamma \varepsilon_k + (H^\Gamma z_{k-1} - z_{k-1} + P_{2^{2n}} f),$$
$$z_k = z_{k-1} + \varepsilon_k. \tag{11}$$

Now we consider the algorithm A_n for which $A_n(T_n^\Gamma, H, f) = z_3$.

Lemma 5. *Within the framework of the algorithm A_n, to represent the approximate solution $A_n(T_n^\Gamma, H, f) = z_3$ of any Eq. (1) from the class $\Psi_\gamma^{r,s}$ in the form*

$$\sum_{i=1}^{2^{2n}} \alpha_i e_i,$$

it suffices to perform no more than $O(n2^{2n})$ AOs on the components of the vector $T_n^\Gamma(H, f)$.

Proof. We are looking for the element ε_k in (11) in the form

$$\varepsilon_k = \sum_{i=1}^{2^m} q_i e_i + g_k, \tag{12}$$

where

$$g_k = H^\Gamma z_{k-1} - z_{k-1} + P_{2^{2n}} f.$$

Then the unknown coefficients q_i, $i = 1, 2, \dots, 2^m$, will be found from the following system of linear equations

$$q_i = \sum_{j=1}^{2^m} q_j (e_i, H e_j) + (e_i, H^\Gamma g_k), \quad i = 1, 2, \dots, 2^m. \tag{13}$$

Observe that if the values of the functionals from $T_n^\Gamma(H, f)$ are known, then it is necessary to represent the element $H^\Gamma g_1 = H^\Gamma P_{2^{2n}} f$ in the standard form (9). According to Lemma 4, the fulfilment of this procedure requires no more than

$O(n2^{2n})$ AOs. Then, to solve the system of equations (13) at $k = 1$, it suffices to perform $c(2^m)^3 = O(2^{2n})$ AOs. By virtue of (12), 2^m operations of addition provide for the element $z_1 = \varepsilon_1$ the representation (9), where

$$\alpha_i = q_i + (e_i, f), \quad i = 1, \ldots, 2^m,$$
$$\alpha_i = (e_i, f), \quad i = 2^m + 1, \ldots, 2^{2n}.$$

As follows from Lemma 4, for the representation of elements $g_2 = H^\Gamma z_1 - \sum_{i=1}^{2m} q_i e_i$ and $H^\Gamma g_2$ in the form (9) it is necessary to perform $O(n2^{2n})$ AOs. To find the element z_3, it suffices to repeat the scheme described above for $k = 2, 3$. Thus the inclusion

$$A_n \in \mathcal{A}_N(T_n^\Gamma), \quad N \asymp n2^{2n},$$

holds. □

Theorem. *If, for the pretabulated width of the ball X_d^r, we have the estimate*

$$\Delta_N(X_d^r, X) \succ N^{-r}, \tag{14}$$

then, for any $r, s = 1, 2, \ldots$

$$N^{-r} \prec E_N(\Psi_\gamma^{r,s}) \prec N^{-r} \log^{r+1/2} N. \tag{15}$$

The algorithm A_n and the Galerkin information $T_n^\Gamma(H, f)$, $n2^{2n} \asymp N$, are order-optimal in the power scale in the sense of the quantity $E_N(\Psi_\gamma^{r,s})$.

Proof. The required lower estimate (15) follows from Lemma 1 and (14).

To obtain the upper estimate (15), we calculate the error of the algorithm A_n on the class $\Psi_\gamma^{r,s}$.

From relations (4), (8), and from the theorem on the invertibility of a linear operator that is close to an invertible operator, it follows that for $H \in \mathcal{H}_\gamma^{r,s}$

$$\begin{aligned} \|(I - H^\Gamma)^{-1}\|_{X\to X} &\le \frac{\|(I - H)^{-1}\|_{X\to X}}{1 - \|(I - H)^{-1}\|_{X\to X}\|H - H^\Gamma\|_{X\to X}} \\ &\le \frac{\gamma_0}{1 - c_1\gamma_0 2^{-2rsn/(r+s)}} \le 2\gamma_0, \end{aligned} \tag{16}$$

where

$$n \ge n_1 = 1 + \left[(1 + |\log_2(c_1\gamma_0)|) \frac{r+s}{2rs} \right],$$

and

$$\begin{aligned}\|(I - P_{2^m} H^{\Gamma})^{-1}\|_{X\to X} &\le \frac{\|(I - H^{\Gamma})^{-1}\|_{X\to X}}{1 - \|(I - H^{\Gamma})^{-1}\|_{X\to X}\|H^{\Gamma} - P_{2^m} H^{\Gamma}\|_{X\to X}} \\ &\le \frac{2\gamma_0}{1 - 2c_2\gamma_0 2^{-rm}} \le 4\gamma_0, \end{aligned} \tag{17}$$

where

$$c_2 = 2\gamma_1\beta_r, \;\; m \ge m_1 = 1 + \left[\frac{2 + |\log_2(c_2\gamma_0)|}{r}\right].$$

In other words, the inequalities (16) and (17) are fulfilled for $n \ge \max\{n_1, [3m_1/2]\}$. Moreover, for any $H \in \mathcal{H}_{\gamma}^{r,s}$ and $f \in X_1^r$ there is the estimate

$$\|z\|_{X^r} = \|H(I - H)^{-1} f + f\|_{X^r} \le \gamma_0\gamma_1 + 1. \tag{18}$$

Let us assign to each Eq. (1) the equation

$$\tilde{z} = H^{\Gamma}(H)\tilde{z} + P_{2^{2n}} f. \tag{19}$$

Taking into account (5), (16), and (18), we have

$$\|z - \tilde{z}\|_X = \|(I - H^{\Gamma})^{-1}\left(f - P_{2^{2n}} f + (H - H^{\Gamma})z\right)\|_X \le c_3 2^{-2nr}\sqrt{n}, \tag{20}$$

where

$$c_3 = 2\gamma_0\beta_r\left(1 + 2^{r+1}(1 + \gamma_0\gamma_1)\gamma_1\beta_r\right).$$

Since for the solution $\tilde{z}$ of Eq. (19) at any $k = 1, 2, 3$ the representation

$$\tilde{z} = z_{k-1} + (I - H^{\Gamma})^{-1}(H^{\Gamma} z_{k-1} - z_{k-1} + P_{2^{2n}} f) \tag{21}$$

holds by virtue of (10), the following relation exists:

$$\begin{aligned}\tilde{z} - z_3 &= (I - P_{2^m} H^{\Gamma})^{-1}(H^{\Gamma} - P_{2^m} H^{\Gamma})(\tilde{z} - z_2) \\ &= \left((I - P_{2^m} H^{\Gamma})^{-1}(H^{\Gamma} - P_{2^m} H^{\Gamma})\right)^2(\tilde{z} - z_1).\end{aligned} \tag{22}$$

Again, using (10) and (21), we find

$$\tilde{z} - z_1 = (I - P_{2^m} H^{\Gamma})^{-1}(H^{\Gamma} - P_{2^m} H^{\Gamma})\tilde{z}. \tag{23}$$

Joining relations (22) and (23), we obtain

$$\tilde{z} - z_3 = \left((I - P_{2^m} H^{\Gamma})^{-1}(H^{\Gamma} - P_{2^m} H^{\Gamma})\right)^3 \tilde{z}. \tag{24}$$

Inserting the bounds (8), (17), (18), and (20) into (24), we have

$$\|\tilde{z} - z_3\|_X \le (8\gamma_0\gamma_1\beta_r)^3(1 + c_3 + \gamma_0\gamma_1)2^{-2rn}.$$

From the last inequality and (20) it follows

$$\begin{aligned}\|z - A_n(T_n^\Gamma, H, f)\|_X &= \|z - z_3\|_X \\ &\leq \|z - \tilde{z}\|_X + \|\tilde{z} - z_3\|_X \prec 2^{-2rn}\sqrt{n}. \quad (25)\end{aligned}$$

Using the relation

$$\operatorname{card}(\Gamma_n) = (n+1)2^{2n},$$

Lemma 5, and (25), we obtain the upper estimate (15) for $N \asymp n2^{2n}$

$$E_N(\Psi_\gamma^{r,s}) \leq e(\Psi_\gamma^{r,s}, A_n) \prec N^{-r} \log^{r+1/2} N.$$

Thus, the Theorem is proved. □

REFERENCES

1. Traub, J. F. and Wozniakowski, H. *General Theory of Optimal Algorithms*. Academic Press, New York, 1980.
2. Pereverzev, S. V. *Optimization of Methods for Approximate Solution of Operator Equations*. Nova, New York, 1995.
3. Babenko, K. I. Estimating the quality of computational algorithms: 1. *Comput. Methods Appl. Mech. Engrg.*, 1976, **7**, 47–63.
4. Pereverzev, S. V. and Scharipov, C. C. Information complexity of equations of the second kind with compact operators in Hilbert space. *J. Complexity*, 1992, **8**, 176–202.

MÕNEDE KORREKTSETE ÜLESANNETE KLASSIDE LAHENDAMISE KEERUKUS

Sergei G. SOLODKY

On käsitletud teist liiki operaatorvõrrandeid Hilberti ruumides ja uuritud nende lahendamise keerukust. On leitud selliste võrrandite ligikaudse lahendamise keerukuse täpne järk ja näidatud, et see on saavutatav, kui kasutada Galjorkini iteratsioonimeetodit, mille nn. informatsioonivektor kuulub eukleidilise ruumi hüperboolsesse risti.