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STABILITY ANALYSIS OF SOME TWO-DIMENSIONAL FINITE-DIFFERENCE SCHEMES

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Abstract. A stability analysis is given for two classes of odd-even finite-difference schemes, which approximate the two-dimensional variable coefficient heat conduction and the Schrödinger problems. Sufficient and necessary stability conditions are derived for the von Neumann stability for the case of constant coefficient problems. The case of variable coefficients is investigated by the discrete energy method.

Key words: stability analysis, odd-even schemes, the Schrödinger problem.

1. INTRODUCTION

In this paper we consider the two-dimensional variable coefficient diffusion equation with the initial and boundary conditions

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}\left(a_1(x)\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(a_2(x)\frac{\partial u}{\partial x_2}\right) + f(x,t) \text{ in } \Omega \times (0,T], \quad (1)$$

$$u(x,t) = \mu(x,t) \quad \text{on } \partial\Omega \times [0,T],$$
$$u(x,0) = u_0(x) \quad \text{in } \Omega,$$

where $x = (x_1, x_2)$, Ω is a unite square in R^2. Assume that the functions $a_1(x)$ and $a_2(x)$ are continuous and that there exist constants α_1 and α_2 such that $0 < \alpha_1 \leq a_1(x), a_2(x) \leq \alpha_2$ for $x \in \Omega$. Also assume that $f(x,t)$, $\mu(x,t)$, and $u_0(x)$ are such that the solution u is sufficiently smooth.

We also consider the Schrödinger equation

$$\frac{\partial u}{\partial t} = i\alpha\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) + f(x,t) \quad \text{in} \quad \Omega\times(0,T] \tag{2}$$

subject to the same initial and boundary conditions; here, $u(x,t)$ and $f(x,t)$ are complex functions.

We rewrite Eqs. (1) and (2) as one equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}\left(A_1(x)\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(A_2(x)\frac{\partial u}{\partial x_2}\right) + f(x,t), \tag{3}$$

where $A_j(x) = a_j(x)$ for (1) and $A_j(x) = i\alpha$ for (2), respectively.

The odd-even (OE) schemes are popular methods for the efficient numerical solution of advection-diffusion problems (see [1,2]). The OE schemes are also used as a basis for popular iterative methods in the case of elliptic two- and three-dimensional diffusion problems, for example the red-black SOR iteration. We note that OE schemes can be successfully applied on parallel computers (see [3]).

Convergence analysis of OE schemes is studied in [2,4]. In both papers stability analysis is given for the constant coefficient linear advection-diffusion model problem. The Fourier analysis is carried out for an associated scheme which is equivalent to the OE scheme. We also note the paper [5], where some stability results are proved in the energy norm.

Our purpose is to extend the stability analysis to a variable coefficient heat conduction problem and to the Schrödinger problem. We investigate two classes of OE schemes. The von Neumann stability definition is used for constant coefficient problems. We propose to carry out the Fourier analysis directly for the OE scheme. Sufficient and necessary conditions are derived for the von Neumann stability. The variable coefficient heat conduction problem is investigated by the discrete energy method.

2. THE ODD-EVEN METHOD

Let $\Omega_h = \Omega_{1h} \times \Omega_{2h}$ be a discretization of Ω, where Ω_{1h} and Ω_{2h} are uniform meshes obtained by dividing intervals $]0,1[$ into mesh intervals by the points $x_{kj} = jh,\ k = 1,2,\ j = 1,2,\dots,N-1$, where $Nh = 1$ and h denotes the spatial mesh size. Then, for any pair (l_j, l_k) with $0 < l_j,\ l_k < N$, we get a discrete point $X_{jk} = (x_{1j}, x_{2k}) \in \Omega_h$. Let $U_{jk}(t)$ denote the numerical approximation of $u(X_{jk}, t)$. We define the inner product between the mesh functions U and V and the discrete L_2 norm

$$(U,V) = h^2 \sum_{X\in\Omega_h} U(X)\overline{V(X)}, \quad \|V\| = \sqrt{(V,V)};$$

here, $\overline{V}$ is the complex conjugate function of V. These definitions are simplified for real functions.

We approximate Eq. (3) by the semidiscrete scheme

$$\frac{dU}{dt} = LU + f(X,t), \quad X \in \Omega_h,$$

where $LU = CU - DU$ and the difference operators C and D are defined as follows:

$$\begin{aligned} CU &= \frac{1}{h^2}(A_{j+1/2,k}U_{j+1,k} + A_{j-1/2,k}U_{j-1,k} \\ &\quad + A_{j,k+1/2}U_{j,k+1} + A_{j,k-1/2}U_{j,k-1}), \\ DU &= \frac{1}{h^2}(A_{j+1/2,k} + A_{j-1/2,k} + A_{j,k+1/2} + A_{j,k-1/2})U_{j,k}, \end{aligned}$$

with

$$A_{j+1/2,k} = A_1(X_{j+1/2,k}), \quad A_{j,k+1/2} = A_2(X_{j,k+1/2}).$$

Let ω_τ be the uniform time mesh, where τ is the discrete time step, and let $U_{jk}^n = U(X_{jk}, t_n),\ t_n = n\tau$. We also use the difference operator

$$U_t^n = \frac{U^{n+1} - U^n}{\tau}, \quad U_{\overset{\circ}{t}}^n = \frac{U^{n+1} - U^{n-1}}{2\tau}.$$

Then the OE method is a special combination of the two well-known methods, i.e. the explicit forward Euler method

$$U_t^n = LU_{jk}^n + f(X_{jk}, t_n)$$

and the implicit backward Euler method

$$U_t^n = LU_{jk}^{n+1} + f(X_{jk}, t_{n+1}).$$

The basic formula defining the OE method (see [1,2]) then reads

$$U_t^n = \Theta_{jk}^n(LU_{jk}^n + f_{jk}^n) + (1 - \Theta_{jk}^n)(LU_{jk}^{n+1} + f_{jk}^{n+1}), \tag{4}$$

where the *hopscotch parameter* Θ_{jk}^n is defined by

$$\Theta_{jk}^n = \begin{cases} 1 & \text{for odd values of } \nu(j,k,n), \\ 0 & \text{for even values of } \nu(j,k,n), \end{cases}$$

and $\nu(j,k,n)$ is a given integer function. By applying different functions $\nu(j,k,n)$, we derive various cases of the OE method.

Notice that the explicit forward Euler scheme is only conditionally stable under the restriction $\tau \le h^2/(4\alpha_2)$ for the parabolic problem (1) and it is unconditionally unstable for the Schrödinger problem (2). The implicit backward Euler method is unconditionally stable for the general equation (3). Both of these methods approximate the differential equation (3) with the order $O(\tau + h^2)$.

3. THE SPACE-DEPENDENT ODD-EVEN METHOD

In this section we consider the first example of the function $\nu(j,k,n)$. It is assumed that for any OE method the numbers of odd and even mesh points are equal, and the costs of realization of the OE scheme are equivalent to the costs of realization of the explicit forward Euler method. Such schemes are called *economical*.

Let us define the test function

$$\nu(j,k,n) = j + k,$$

where only space-dependent arguments j, k are involved in the definition. This OE space mesh coincides with the well-known red-black colouring scheme. We get the OE scheme

$$\begin{aligned} U_t^n &= LU_{jk}^n + f_{jk}^n \quad \text{for} \quad j+k = 2m+1, \\ U_t^n &= LU_{jk}^{n+1} + f_{jk}^{n+1} \quad \text{for} \quad j+k = 2m. \end{aligned} \tag{5}$$

In addition, we use the boundary conditions

$$U_{jk}^n = \mu(X_{jk}, t_n) \quad \text{for } X_{jk} \in \partial\Omega_h, \quad n \geq 0.$$

Let Z_{jk}^n denote the global error of the difference solution

$$Z_{jk}^n = u(X_{jk}, t_n) - U_{jk}^n.$$

The OE scheme has truncation errors

$$\begin{aligned} \Psi_{jk}^{n+1} &= u_t(X_{jk}, t_n) - Lu(X_{jk}, t_n) - f_{jk}^n \quad \text{for} \quad j+k = 2m+1, \\ \Phi_{jk}^{n+1} &= u_t(X_{jk}, t_n) - Lu(X_{jk}, t_{n+1}) - f_{jk}^{n+1} \quad \text{for} \quad j+k = 2m. \end{aligned}$$

Then Z_{jk}^n satisfies the difference problem

$$\begin{aligned} Z_t^n &= LZ_{jk}^n + \Psi_{jk}^{n+1} \quad \text{for } j+k = 2m+1, \\ Z_t^n &= LZ_{jk}^{n+1} + \Phi_{jk}^{n+1} \quad \text{for } j+k = 2m, \\ Z_{jk}^n &= 0 \quad \text{for } X_{jk} \in \partial\Omega_h, n \geq 0. \end{aligned} \tag{6}$$

In order to carry out the stability analysis, we propose to rewrite Eqs. (6). We introduce two error vectors

$$E_{jk}^n = \begin{cases} Z_{jk}^n & \text{for} \quad j+k = 2m, \\ W_{jk}^n & \text{for} \quad j+k = 2m+1, \end{cases}$$

$$O_{jk}^n = \begin{cases} W_{jk}^n & \text{for} \quad j+k=2m, \\ Z_{jk}^n & \text{for} \quad j+k=2m+1, \end{cases}$$

where W_{jk}^n are fictitious global-error components. The difference problem for E, O then reads

$$O_t^n = CE_{jk}^n - DO_{jk}^n + \Psi_{jk}^{n+1} \qquad \text{for } X_{jk} \in \Omega_h, \tag{7}$$

$$E_t^n = CO_{jk}^{n+1} - DE_{jk}^{n+1} + \Phi_{jk}^{n+1} \qquad \text{for } X_{jk} \in \Omega_h, \tag{8}$$

$$E_{jk}^n = 0, \quad O_{jk}^n = 0 \qquad \text{for } X_{jk} \in \partial\Omega_h, \ n \geq 0;$$

here, the definition of the functions Ψ and Φ is extended for all $X \in \Omega_h$.

We consider the Schrödinger problem. The stability analysis for the parabolic problem is given in [6]. We represent the global-error vectors in the form of the Fourier series

$$\begin{aligned} E_{jk}^n &= \sum_{l=1}^{N-1}\sum_{m=1}^{N-1} e_{lm}^n \sin(\pi l x_{1j}) \sin(\pi m x_{2k}) \text{ for } X_{jk} \in \Omega_h, \\ O_{jk}^n &= \sum_{l=1}^{N-1}\sum_{m=1}^{N-1} o_{lm}^n \sin(\pi l x_{1j}) \sin(\pi m x_{2k}) \text{ for } X_{jk} \in \Omega_h. \end{aligned} \tag{9}$$

The similar expansions are valid for the truncation error vectors

$$\begin{aligned} \Psi_{jk}^n &= \sum_{l=1}^{N-1}\sum_{m=1}^{N-1} \psi_{lm}^n \sin(\pi l x_{1j}) \sin(\pi m x_{2k}) \quad \text{for} \quad X_{jk} \in \Omega_h, \\ \Phi_{jk}^n &= \sum_{l=1}^{N-1}\sum_{m=1}^{N-1} \varphi_{lm}^n \sin(\pi l x_{1j}) \sin(\pi m x_{2k}) \quad \text{for} \quad X_{jk} \in \Omega_h. \end{aligned} \tag{10}$$

Substitution of (9) and (10) into the difference equations (7) and (8) leads to the difference scheme for the Fourier coefficients

$$\begin{pmatrix} o^{n+1} \\ e^{n+1} \end{pmatrix} = S \begin{pmatrix} o^n \\ e^n \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ \dfrac{i\gamma a}{1+i\gamma} & \dfrac{1}{1+i\gamma} \end{pmatrix} \begin{pmatrix} \tau\psi^{n+1} \\ \tau\varphi^{n+1} \end{pmatrix},$$

where S is the *amplification matrix*

$$S = \begin{pmatrix} 1 - i\gamma & i\gamma a \\ \dfrac{i\gamma a(1-i\gamma)}{1+i\gamma} & \dfrac{1-(\gamma a)^2}{1+i\gamma} \end{pmatrix}.$$

Here we denote

$$\gamma = \frac{4\tau\alpha}{h^2}, \quad a_{lm} = \frac{1}{2}(\cos(\pi l h) + \cos(\pi m h)).$$

To simplify notations, we will omit subscripts l and m in the remainder of this section. The von Neumann stability condition requires that all eigenvalues of the amplification matrix be in the closed unit disk and the ones on the unit circle be simple (see [7]). In particular, von Neumann's condition is necessary for the stability in the L_2 norm.

Theorem 1. *If* $\tau \le 5\pi h^3/(21\alpha)$ *and* N *is an odd number, then von Neumann's necessary stability condition is satisfied for the OE scheme* (5) *for solving the Schrödinger problem* (2).

Proof. We get the characteristic equation of the amplification matrix S

$$\lambda^2 - \frac{2+\gamma^2(1-a^2)}{1+i\gamma}\lambda + \frac{1-i\gamma}{1+i\gamma} = 0. \tag{11}$$

The eigenvalues of S are therefore

$$\lambda_{1,2} = \frac{2+\gamma^2(1-a^2) \pm \gamma\sqrt{\gamma^2(1-a^2)^2 - 4a^2}}{2(1+i\gamma)}.$$

For $\gamma > 2|a|/(1-a^2)$, there exists $|\lambda| > 1$ and hence von Neumann's condition is violated. If $\gamma \le 2|a|/(1-a^2)$, then we have $|\lambda_{1,2}| = 1$. If $N = 2M$, then $a_{MM} = 0$ and the OE scheme is unconditionally unstable. If $N = 2M+1$, then we get

$$\min_{l,m} |a_{lm}| = a_{MM} = \sin\left(\frac{\pi}{2}h\right).$$

As is easily proved, we have for $h \le \frac{1}{3}$

$$\sin\left(\frac{\pi}{2}h\right) \ge \frac{\pi}{2}h\left(1 - \frac{\pi^2}{216}\right) \ge \frac{10}{21}\pi h;$$

hence the OE scheme satisfies von Neumann's stability condition if

$$\gamma \le \frac{20}{21}\pi h \quad \text{or} \quad \tau \le \frac{5\pi h^3}{21\alpha}. \qquad \square$$

Now we consider the stability and convergence of the OE scheme (5) for the parabolic problem (1) with variable coefficients. The discrete energy method is used in our analysis.

We shall need the following lemmas.

Lemma 1. *If* $U = V = 0$ *for* $X \in \partial\Omega_h$, *then*

$$(CU, V) = (U, CV). \tag{12}$$

Let us denote $Y = (O,E)^T$. We define the discrete function

$$||Y||_A^2 = (DO,O) + (DE,E) - (CO,E) - (CE,O).$$

Lemma 2. *If* $Y = 0$ *for* $X \in \partial\Omega_h$, *then* $||Y||_A^2$ *is the norm which can be written as*

$$\begin{aligned} ||Y||_A^2 &= \sum_{k=1}^{N-1}\sum_{j=1}^{N} a_{j-1/2,k}\left(\left(\frac{O_{jk}-E_{j-1,k}}{h}\right)^2 + \left(\frac{E_{jk}-O_{j-1,k}}{h}\right)^2\right) \\ &\quad + \sum_{j=1}^{N-1}\sum_{k=1}^{N} a_{j,k-1/2}\left(\left(\frac{O_{jk}-E_{j,k-1}}{h}\right)^2 + \left(\frac{E_{jk}-O_{j,k-1}}{h}\right)^2\right). \end{aligned}$$

The proof of both lemmas follows from the definition of the operators C and D and it requires only simple calculations.

Theorem 2. *If* $\tau \le (1-\varepsilon)h^2/(2\alpha_2),\ \ 0 \le \varepsilon \le 1$, *then the solution of the OE scheme* (5) *converges to the solution of the parabolic problem* (1) *and the global error satisfies the inequality*

$$||Y^n||_A^2 \le ||Y^0||_A^2 + \frac{\tau}{2}\sum_{j=1}^{n}\left(\frac{1}{\varepsilon}||\Psi^j||^2 + ||\Phi^j||^2\right). \tag{13}$$

Proof. We multiply (7) by $2\tau O_t$ and (8) by $2\tau E_t$ to obtain

$$\begin{aligned} &2\tau||O_t||^2 - \tau^2(DO_t,O_t) + \tau^2(CE_t,O_t) \\ &\quad = (C(E^{n+1}+E^n), O^{n+1}-O^n) - (DO^{n+1},O^{n+1}) \\ &\quad\quad + (DO^n,O^n) + 2\tau(\Psi^{n+1},O_t), \end{aligned} \tag{14}$$

$$\begin{aligned} &2\tau||E_t||^2 + \tau^2(DE_t,E_t) - \tau^2(CO_t,E_t) \\ &\quad = (C(O^{n+1}+O^n), E^{n+1}-E^n) - (DE^{n+1},E^{n+1}) \\ &\quad\quad + (DE^n,E^n) + 2\tau(\Phi^{n+1},E_t). \end{aligned} \tag{15}$$

Then we add (14) and (15), use Lemma 1, and get

$$\begin{aligned} &2\tau(||O_t||^2 + ||E_t||^2) + \tau^2((DE_t,E_t) - (DO_t,O_t)) + ||Y^{n+1}||_A^2 \\ &\quad = ||Y^n||_A^2 + 2\tau((\Psi^{n+1},O_t) + (\Phi^{n+1},E_t)). \end{aligned} \tag{16}$$

We estimate the last two terms in (16)

$$\begin{aligned}(\Psi^{n+1}, O_t) &\le \varepsilon\|O_t\|^2 + \frac{1}{4\varepsilon}\|\Psi^{n+1}\|^2,\\ (\Phi^{n+1}, E_t) &\le \|E_t\|^2 + \frac{1}{4}\|\Phi^{n+1}\|^2.\end{aligned}$$

Now we substitute these inequalities into (16) and get

$$2\tau\Big((1-\varepsilon)\|O_t\|^2 - \frac{\tau}{2}(DO_t, O_t)\Big) + \tau^2(DE_t, E_t) + \|Y^{n+1}\|_A^2 \le \|Y^n\|_A^2 + \frac{\tau}{2}\Big(\frac{1}{\varepsilon}\|\Psi^{n+1}\|^2 + \|\Phi^{n+1}\|^2\Big). \tag{17}$$

It is easy to see that when $\tau \le (1-\varepsilon)h^2/(2\alpha_2)$, we have

$$(1-\varepsilon)\|O_t\|^2 - \frac{\tau}{2}(DO_t, O_t) \ge 0.$$

Finally, we add inequalities (17) and obtain (13). □

4. THE TIME-DEPENDENT ODD-EVEN SCHEME

Let us define the following test function [1,2]:

$$\nu(j,k,n) = j + k + n.$$

By first applying the explicit forward Euler method at all odd points

$$U_{jk}^{n+1} = U_{jk}^n + \tau(LU_{jk}^n + f_{jk}^n) \tag{18}$$

and subsequently applying the implicit backward Euler method at all even points

$$(I + \tau D)U_{jk}^{n+1} = U_{jk}^n + \tau(CU_{jk}^{n+1} + f_{jk}^{n+1}), \tag{19}$$

we carry out one step with the OE method.

The difference problem for the global error functions O and E then reads

$$\begin{aligned}\frac{O_{jk}^{n+1} - E_{jk}^n}{\tau} &= CO_{jk}^n - DE_{jk}^n + \Psi_{jk}^{n+1} \quad \text{for } X_{jk} \in \Omega_h,\\ \frac{E_{jk}^{n+1} - O_{jk}^n}{\tau} &= CO_{jk}^{n+1} - DE_{jk}^{n+1} + \Phi_{jk}^{n+1} \quad \text{for } X_{jk} \in \Omega_h,\\ E_{jk}^n &= 0, \qquad O_{jk}^n = 0 \quad \text{for } X_{jk} \in \partial\Omega_{jk}.\end{aligned} \tag{20}$$

First we consider the parabolic equation (1) with constant coefficients $a_1(x) = a_2(x) = \alpha$ and the Schrödinger problem (2). After computations, similar to Section 3, we obtain the difference equation for the Fourier coefficients

$$\begin{pmatrix} o^{n+1} \\ e^{n+1} \end{pmatrix} = S \begin{pmatrix} o^n \\ e^n \end{pmatrix} + \begin{pmatrix} \frac{1}{1+\beta} & \frac{a\beta}{1+\beta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau\psi^{n+1} \\ \tau\varphi^{n+1} \end{pmatrix},$$

where S is the amplification matrix

$$S = \begin{pmatrix} \frac{a\beta(1-\beta)}{1+\beta} & \frac{1+(a\beta)^2}{1+\beta} \\ 1-\beta & a\beta \end{pmatrix},$$

and $\beta = \gamma$ for the parabolic problem, and $\beta = i\gamma$ for the Schrödinger problem; here, $\gamma = 4\tau\alpha/h^2$.

Theorem 3. *The von Neumann necessary stability condition is satisfied unconditionally for the OE scheme* (20).

Proof. We get the characteristic equation of the amplification matrix S

$$\lambda^2 - \frac{2a\beta}{1+\beta}\lambda - \frac{1-\beta}{1+\beta} = 0.$$

Application of the Hurwitz criterion gives that von Neumann's stability condition is satisfied for the parabolic problem if

$$\frac{1-\gamma}{1+\gamma} \le 1 \quad \text{and} \quad \frac{2|a|\gamma}{1+\gamma} \le 1 - \frac{1-\gamma}{1+\gamma}.$$

We have that both of these inequalities are satisfied unconditionally. In the case of the Schrödinger problem the eigenvalues of S are given by

$$\lambda_{1,2} = \frac{a\gamma i \pm \sqrt{1+(1-a^2)\gamma^2}}{1+i\gamma}. \tag{21}$$

As is easily seen from (21) we have $|\lambda_{1,2}| = 1$ unconditionally. □

Next we consider the convergence of the OE scheme (18), (19) for the parabolic problem (1) with variable coefficients. First we derive from (20) the difference problem which involves only the function O. Taking (20) for the nth time step, we have the equation

$$\frac{E_{jk}^n - O_{jk}^{n-1}}{\tau} = CO_{jk}^n - DE_{jk}^n + \Phi_{jk}^n. \tag{22}$$

Adding (20) and (22), we get

$$\frac{O_{jk}^{n+1}-O_{jk}^{n-1}}{2\tau}=CO_{jk}^{n}-DE_{jk}^{n}+\frac{1}{2}(\Phi_{jk}^{n}+\Psi_{jk}^{n+1}). \tag{23}$$

Subtracting (22) from (20), we find

$$E_{jk}^{n}=\frac{1}{2}(O_{jk}^{n+1}+O_{jk}^{n-1})+\frac{1}{2}\tau(\Phi_{jk}^{n}-\Psi_{jk}^{n+1}). \tag{24}$$

Finally, substituting (24) into (23), we obtain the three-level difference scheme for the function O

$$\frac{O_{jk}^{n+1}-O_{jk}^{n-1}}{2\tau}=CO_{jk}^{n} \;-\; D\frac{O_{jk}^{n+1}+O_{jk}^{n-1}}{2}+\frac{\tau}{2}D(\Psi_{jk}^{n+1}-\Phi_{jk}^{n})$$
$$+\;\frac{1}{2}(\Phi_{jk}^{n}+\Psi_{jk}^{n+1}). \tag{25}$$

Lemma 3. *If the solution of the problem* (1) *is a sufficiently smooth function, then we have the following estimates of the truncation error*:

$$\|\Psi^{n+1}+\Phi^{n}\| \;\le\; C_1(\tau^2+h^2),$$
$$\|D(\Psi^{n+1}-\Phi^{n})\| \;\le\; C_2(\tau/h^2+h^2).$$

Proof. Using a Taylor series expansion, we get from (18), (19)

$$\begin{aligned}\Psi_{jk}^{n+1} &= \frac{\tau}{2}\frac{\partial^2 u(X_{jk},t_n)}{\partial t^2}+\frac{h^2}{24}T(u(X_{jk},t_n))+O(\tau^2+h^4),\\ \Phi_{jk}^{n} &= -\frac{\tau}{2}\frac{\partial^2 u(X_{jk},t_n)}{\partial t^2}+\frac{h^2}{24}T(u(X_{jk},t_n))+O(\tau^2+h^4),\end{aligned} \tag{26}$$

where

$$T(u(X,t))=-\sum_{l=1}^{2}\left(\frac{\partial^3}{\partial x_l^3}\left(a_l(X)\frac{\partial u}{\partial x_l}\right)+\frac{\partial}{\partial x_l}\left(a_l(X)\frac{\partial^3 u}{\partial x_l^3}\right)\right).$$

From (26) and from the definition of D we obtain immediately the required estimates. □

The difference problem (25) defines O^n for $n\ge 2$. Hence we must estimate O^1 separately.

Lemma 4. *Assume that the solution of the problem* (1) *has continuous bounded partial derivatives. Then*

$$(DO^1, O^1) \le C_3 \tau^2 (\tau + h^2)^2 / h^2. \tag{27}$$

Proof. The initial condition is satisfied exactly by the solution of the OE scheme (18), (19); hence we have

$$E^0_{jk} = 0, \quad O^0_{jk} = 0 \quad \text{for} \quad X_{jk} \in \Omega_h.$$

Then it follows from (20)

$$O^1_{jk} = \tau \Psi^1_{jk}.$$

Taking into account (26) and using the definition of D, we get (27). □

Theorem 4. *The global error O of the solution of the OE scheme* (18), (19) *satisfies the inequality*

$$(DO^{n+1}, O^{n+1}) + (DO^n, O^n) - 2(CO^{n+1}, O^n)$$
$$\le (DO^1, O^1) + 2\tau \sum_{j=2}^{n+1} \left(\left\| \frac{\Psi^j + \Phi^{j-1}}{2} \right\|^2 + \frac{\tau^2}{4} \| D(\Psi^j - \Phi^{j-1}) \|^2 \right). \tag{28}$$

Proof. We multiply (25) by $4\tau O_{\overset{\circ}{t}}$ to obtain

$$4\tau \|O_{\overset{\circ}{t}}\|^2 + (DO^{n+1}, O^{n+1}) - 2(CO^n, O^{n+1})$$
$$= (DO^{n-1}, O^{n-1}) - 2(CO^n, O^{n-1}) + 4\tau (F^{n+1}, O_{\overset{\circ}{t}}), \tag{29}$$

where we denote

$$F^{n+1} = \frac{1}{2} \tau D(\Psi^{n+1} - \Phi^n) + \frac{1}{2}(\Psi^{n+1} + \Phi^n).$$

Then we estimate the last term in (29)

$$4\tau (F^{n+1}, O_{\overset{\circ}{t}}) \le 4\tau \|O_{\overset{\circ}{t}}\|^2 + \tau \|F^{n+1}\|^2,$$

add (DO^n, O^n) to both sides of (29) and get

$$(DO^{n+1}, O^{n+1}) + (DO^n, O^n) - 2(CO^n, O^{n+1})$$
$$\le (DO^n, O^n) + (DO^{n-1}, O^{n-1}) - 2(CO^n, O^{n-1}) + \tau \|F^{n+1}\|^2. \tag{30}$$

Finally we add inequalities (30), use Lemma 1 and equality $O^0 \equiv 0$, and obtain (28). It remains to note that the function on the left side of (28) is a norm, as it follows from Lemma 2. □

As a consequence, from Theorem 4, Lemma 3, and Lemma 4 we have that if $\tau \le C_4 h^{1+p}$ with $0 < p \le 1$, then

$$(DO^{n+1}, O^{n+1}) + (DO^n, O^n) - 2(CO^{n+1}, O^n) \le Ch^{4p}.$$

The case of the variable coefficient Schrödinger problem is investigated in [8].

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KAHEDIMENSIONAALSETE DIFERENTSSKEEMIDE STABIILSUSE ANALÜÜS

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On käsitletud kahedimensionaalse difusioonivõrrandi diskretiseerimise meetodite kahe klassi arvutuslikku stabiilsust, sealhulgas nn. von Naumanni stabiilsust. Tulemused on sõnastatud teoreemidena, mis on ka tõestatud. Eraldi on vaadeldud konstantsete kordajatega juhtu ning Schrödingeri võrrandit.