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SINGULAR SOLUTIONS OF SINGULAR INTEGRAL EQUATIONS AND THEIR APPLICATIONS

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Abstract. Recently, in aerodynamics much attention has been paid to the investigation of the wing with the energy high-lift devices, including the device for external flow suction as a constituent part. This problem is reduced to the solution of the first-kind singular integral equations on a segment or with Holder kernel in the class of singular solutions, i.e. the solutions which have the singularity of the type $1/x$ at some point. As a rule, these equations have no unique solution. There are some restrictions on the right-hand side in usual spaces that are not comfortable for numerical solutions. Therefore we constructed wider spaces in which these equations, together with some conditions, have a unique solution and no restrictions occur on the right-hand side. A numerical method of the solution of these equations in wider spaces was proposed.

Key words: singular integral equations, singular solutions, widening spaces.

1. Let an airfoil (with the contour L) be flowed around by an ideal incompressible fluid of the velocity $\overline{V}_0 = \operatorname{grad} U_0$, where U_0 is a harmonic function in the whole (x, y)-plane. Let on the airfoil, at the point M_Q, be the device for external flow suction (see Fig. 1). We simulate this device by the source which gives the velocity field by the formula

$$\overline{V}_Q(M) = \frac{Q}{2\pi} \frac{\overline{r}_{MM_Q}}{r^2_{MM_Q}}, \quad M \neq M_Q, \quad M \in R^2, \tag{1}$$

$$\overline{r}_{MM_Q} = \overline{MM_Q}.$$

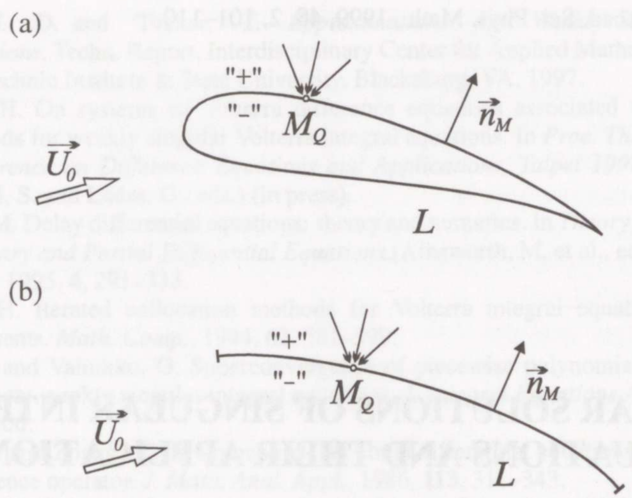

Fig. 1. (a) A thick airfoil. (b) A thin airfoil. "+", the positive side on the contour L; "–", the negative side on the contour L; $\vec{n}_m$, the ort.

The contour L we simulate by the vorticity layer of the density $\gamma(M) = \gamma(t)$ at the point $M(t) = M(x(t), y(t))$ on L, which gives the velocity field by the formula

$$\overline{V}_\gamma(M_0) = \int_L \overline{\omega}_\gamma(M_0, M(t))\gamma(t)ds,$$

$$\overline{\omega}_\gamma(M_0, M(t)) = \frac{1}{2\pi}\frac{(y_0 - y(t))\,\vec{i} - (x_0 - x(t))\,\vec{j}}{(x_0 - x(t))^2 + (y_0 - y(t))^2}, \quad M_0 \notin L. \tag{2}$$

The density $\gamma(M)$ must satisfy the following conditions:

1) non-penetration of the contour L,

$$\overline{V}(M_0)\overline{n}_{M_0} = 0, \quad M_0 \in L, \quad M_0 \neq M_Q,$$

$$\overline{V}(M_0) = \overline{V}_\gamma(M_0) + \overline{V}_Q(M_0) + \overline{V}_0, \tag{3}$$

$$(\overline{V}_\gamma(M_0)\overline{n}_{M_0} \text{ is continuous on } L);$$

2) the velocities $\overline{V}_\gamma(M_0)$ have the form of the type (1) on the positive side of the contour L in the neighbourhood of the point M_Q and are smooth on the negative side.

It may be shown (see [1]) that Eq. (3) has the form

$$\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{t_0-t}{2}\gamma(t)dt+\int_0^{2\pi}K(t_0,t)\gamma(t)dt=f(t_0),\qquad t_0\in[2,\pi],\tag{4}$$

$$f(t_0)=-a(t_0)\left(\overline{V}_Q(M_0)+\overline{V}_0\right)\overline{n}_{M_0},$$

where $a(t_0)$ is a smooth function if L is smooth closed and has the form

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\gamma(t)dt}{t_0-t}+\int_{-1}^{1}K(t_0,t)\gamma(t)\,dt=f(t_0),\qquad t_0\in(-1,1)\tag{5}$$

if L is open.

In order that the function $\gamma(t)$ satisfies conditions 1) and 2), we must find the solution to Eqs. (4) and (5) in the form

$$\gamma(t)=\frac{\psi(t)}{t-t_Q},\tag{6}$$

where $\psi(t)$ is smooth and $\psi(t_Q)\neq 0$ if $Q\neq 0$.

The solutions of the form (6) to Eqs. (4) or (5) we call the singular solutions and will denote by $\gamma_s(t)$.

Now let us consider some approaches to numerical determination of the singular solutions on the example of the characteristic equations

$$\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{t_0-t}{2}\gamma(t)dt=f(t_0),\qquad t_0\in[0,2\pi],\tag{7}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\gamma(t)dt}{t_0-t}=f(t_0),\qquad t_0\in[-1,1].\tag{8}$$

2. We shall begin with Eq. (7) and give some formulas from [2]:

$$\cot\frac{t-t_0}{2}\cot\frac{t_Q-t}{2}=-\cot\frac{t_0-t_Q}{2}\left(\cot\frac{t_Q-t}{2}+\cot\frac{t-t_0}{2}\right)+1,\tag{9}$$

$$\int_0^{2\pi}\cot\frac{t-t_0}{2}\cot\frac{t_Q-t}{2}dt=2\pi,\qquad t_0\in[0,2\pi],\qquad t_0\neq t_Q,\tag{10}$$

$$\int_0^{2\pi}\cot^2\frac{t-t_0}{2}dt=-2\pi,\qquad t_0\in[0,2\pi],\tag{11}$$

where, according to Hadamard [1], the integral is treated as a finite part. Therefore one may take $t_0=t_Q$ in (10)

$$\int_0^{2\pi}\cot\frac{t_0-t}{2}dt\int_0^{2\pi}\cot\frac{t-\tau}{2}f(\tau)dt=-4\pi^2 f(t_0)+2\pi\int_0^{2\pi}f(\tau)d\tau. \tag{12}$$

From the formulas (10) and (12) it follows that the function

$$\gamma(t)=-\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{t-\tau}{2}f(\tau)d\tau+C+\cot\frac{t-t_Q}{2}\int_0^{2\pi}f(\tau)d\tau \tag{13}$$

is the solution to Eq. (7) for any function $f(t_0)$ belonging to Holder class (H class) on $[0,2\pi]$.

Further, we shall write the solutions of the type (13)

$$\gamma_s(t)=\gamma(t)+\cot\frac{t-t_Q}{2}B, \tag{14}$$

where $\gamma(t)\in H$ or L_2 on $[0,2\pi]$.

Below we apply some operator method for the construction and justification of the numerical solution of the type (14) to Eq. (7).

At first we consider Eq. (7) in X^* class ($X^* = H$ or L_2). In this case the solution to this equation exists if and only if the condition

$$\frac{1}{2\pi}\int_0^{2\pi}f(t)dt=0 \tag{15}$$

is valid.

The condition (15) is not comfortable for numerical solution as we must observe this condition for $f_n(t)$. Therefore we consider the following construction. Let $\gamma(t)\in X^*$ and the number γ_0 belong to R^1 (number axis) and $x=(\gamma(t),\gamma_0)\in X^*\times R^1$ (direct product) with the norm $\|x\|_X = \sqrt{\|\gamma(t)\|^2_{X^*}+\|\gamma_0\|^2_{R^1}}$. Now, let $f(t)\in Y^*=X^*$ and the number C belong to R^1, and $y=(f(t),C)\in Y=Y^*\times R^1$. Let the operator $A_1 : X\to Y$ be given by the following rules:

$$\gamma_0+\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{t_0-t}{2}\gamma(t)dt=f(t_0),\qquad t_0\in[0,2\pi],$$

$$\frac{1}{2\pi}\int_0^{2\pi}\gamma(t)dt=C. \tag{16}$$

Sometimes the variable number γ_0 is called the "regularization variable" or "Lifanian". From the system (16) we have

$$\gamma_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt, \quad \gamma = -\frac{1}{2\pi}\int_0^{2\pi} \cot\frac{t-t_0}{2} f(t_0)\,dt_0 + C. \tag{17}$$

Thus the operator A_1 gives a one-to-one map X on Y and $\|Ax\|_Y \le \|A\|\,\|x\|_X$, i.e. A_1 is continuously invertible. In this case, if for the function $f(t)$ the condition (15) is fulfilled, we have $\gamma_0 = 0$ and $(\gamma(t),0) = \gamma(t)$ is the usual solution to Eq. (8). In the following case the operator $A_2 : X \to Y$ is defined by the rules

$$\gamma_0 + \frac{1}{2\pi}\int_0^{2\pi} \cot\frac{t_0 - t}{2}\gamma(t)\,dt = f(t_0), \qquad t_0 \in [0, 2\pi],$$
$$\gamma_0 + \frac{1}{2\pi}\int_0^{2\pi} \gamma(t)\,dt = C. \tag{18}$$

Then we have

$$\gamma_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt,$$
$$\gamma(t) = \frac{1}{2\pi}\int_0^{2\pi} \cot\frac{t-t_0}{2} f(t_0)\,dt_0 + C - \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt, \tag{19}$$

i.e. the operator A_2 is continuously invertible.

Now we shall consider another case. Let the value $\gamma(t_T)$, $t_T \in [0, 2\pi]$, $(X^* = H)$, be known. Then the operator $A_3 : X \to Y$ is defined by the following rules:

$$\gamma_0 + \frac{1}{2\pi}\int_0^{2\pi} \cot\frac{t_0 - t}{2}\gamma(t)\,dt = f(t_0), \qquad t_0 \in [0, 2\pi],$$
$$\gamma(t_T) = \lambda. \tag{20}$$

Then we have

$$\gamma_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt,$$
$$\gamma(t) = -\frac{1}{2\pi}\int_0^{2\pi} \left[\cot\frac{t-t_0}{2} - \cot\frac{t_T - t_0}{2}\right] f(t_0)\,dt_0 + \lambda. \tag{21}$$

Thus the operator A_3 is continuously invertible.

Now we consider Eq. (7) in the class of singular solutions (14). Let the intensity Q of the source be known (see (4)), and as Eq. (7) corresponds to the unit circle, then

$$\overline{V}_Q(M_0)\overline{n}_{M_0} = \frac{Q}{4\pi}, \quad M_0 \in L, \quad M_0 \neq M_Q.$$

In this case the function $f(t_0)$ is known. The function $\gamma_s(t)$ we shall consider as the point $x = (\gamma(t), B)$ in the space $X = X^* \times R_s^1, R_s^1 = R^1$. Thus, the operator $A_{s,1}: X \to Y = X^* \times R^1$ takes the form

$$\frac{1}{2\pi}\int_0^{2\pi} \cot\frac{t_0 - t}{2}\gamma_s(t)dt = f(t_0), \quad t_0 \in [0, 2\pi],$$
$$\frac{1}{2\pi}\int_0^{2\pi} \gamma_s(t)dt = C. \tag{22}$$

From the formula (10) it follows that the system (22) may be rewritten in the form (16) by replacing γ_0 by B and $\gamma_s(t)$ by $\gamma(t)$. Now we see that the number B is equal to γ_0 in (17) and $\gamma(t)$ is given in this formula too. Thus, the solution $\gamma_s(t)$ to the system (22) is given by the formula (13) and the operator $A_{s,1}$ is continuously invertible.

Now let the intensity Q of the source be unknown (see (4)). In this case the function $f(t_0)$ may be represented as $f_1(t_0) - \gamma_0$, where γ_0 is unknown and $f_1(t_0)$ is known. Therefore we take the space $X = X^* \times R_s^1 \times R^1$ which has the elements $x = (\gamma(t), B, \gamma_0)$ and the space $Y = Y^* \times R^1 \times R^1 \ni y = (f_1(t_0), C, \lambda)$ and define the operator $A_{s,2}: X \to Y$ by the following rules:

$$\gamma_0 + \frac{1}{2\pi}\int_0^{2\pi} \cot\frac{t_0 - t}{2}\gamma_s(t)dt = f_1(t_0), \quad t_0 \in [0, 2\pi],$$
$$\frac{1}{2\pi}\int_0^{2\pi} \gamma_s(t)dt = C, \tag{23}$$
$$\gamma_s(t_T) = \lambda, \quad t_T \neq t_Q.$$

The operator $A_{s,2}$ is continuously invertible because the equalities

$$B=\tan\frac{t_T-t_Q}{2}\left(\lambda+\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{t_T-t_0}{2}f(t_0)\,dt_0-C\right),$$

$$\gamma_0=\frac{1}{2\pi}\int_0^{2\pi}f(t)\,dt-B,$$

$$\gamma(t)=-\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{t-t_0}{2}f(t_0)\,dt_0+C, \tag{24}$$

$$\gamma_s(t)=\gamma(t)+\cot\frac{t-t_Q}{2}B$$

are valid.

Note 1. Analogous results can be received for the operator $A_{s,3}$ which is defined by the following rule: in the system (23), instead of the second and third equations, take the equations

$$\gamma_s(t_{T_1})=\lambda_1,\quad t_{T_1}\neq t_Q,$$
$$\gamma_s(t_{T_2})=\lambda_2,\quad t_{T_2}\neq t_Q,\quad t_{T_2}\neq t_{T_1}. \tag{25}$$

Now let us consider the following case. We have at the point t_{Q_1} the source of the known intensity Q_1, and at the point t_{Q_2}, $t_{Q_2}\neq t_{Q_1}$, we have the source of the unknown intensity Q_2. In this case we get the solution in the form

$$\gamma_s(t)=\gamma(t)+\cot\frac{t-t_{Q_1}}{2}B_1+\cot\frac{t-t_{Q_2}}{2}B_2, \tag{26}$$

which will represent the point of the space $X^*\times R^1_{1,s}\times R^1_{2,s}$, and define the operator $A_{s,4}$ from the space $X=X^*\times R^1_{1,s}\times R^1_{2,s}\times R^1$ in the space $Y=Y^*\times R^1\times R^1\times R^1$ by the following rules:

$$\gamma_0+\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{t_0-t}{2}\gamma_1(t)\,dt=f_1(t_0),\quad t_0\in[0,2\pi],$$

$$\frac{1}{2\pi}\int_0^{2\pi}\gamma_s(t)\,dt=C, \tag{27}$$

$$\gamma_s(t_T)=\lambda,\quad t_T\neq t_{Q_1},t_{Q_2},$$

$$\lim_{\alpha\to 0}\alpha\left[\gamma_s\left(t_{Q_1}+\frac{\alpha}{2}\right)-\gamma_s\left(t_{Q_1}-\frac{\alpha}{2}\right)\right]=\frac{4Q_1}{\pi}.$$

It may be shown that

$$B_1=\frac{Q_1}{2\pi},\quad \gamma_0=\frac{1}{2\pi}\int_0^{2\pi} f_1(t)\,dt-B_1-B_2,$$

$$B_2=\tan\frac{t_T-t_{Q2}}{2}\left(\lambda-\gamma(t_T)-\cot\frac{t_T-t_{Q1}}{2}B_1\right),$$

and $\gamma(t)$ is given by the formula (17). Therefore the operator $A_{s,4}$ is continuously invertible.

Note 2. Finally we consider the following case. Let the intensities $Q_m, m=1,\ldots,k,$ which are placed at the points $t_{Q_m}, m=1,\ldots,k,$ be known but the intensity Q_{k+1} be unknown and placed at the point $t_{Q_{k+1}}$ $(t_Q\neq t_{Q_j}, \ i\neq j, \ i,j=1,\ldots,k+1)$. In this case we seek the solution to Eq. (7) in the form

$$\gamma_s(t)=\gamma(t)+\sum_{m=1}^{k+1}\cot\frac{t-t_{Q_m}}{2}B_m,\quad t\in[0,2\pi], \tag{28}$$

and the operator $A_{s,k,1}$ is defined by the following rules:

$$\gamma_0+\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{t_0-t}{2}\gamma_s(t)\,dt=f_1(t_0),\quad t_0\in[0,2\pi],$$

$$\frac{1}{2\pi}\int_0^{2\pi}\gamma_s(t)\,dt=C, \tag{29}$$

$$\gamma_s(t_T)=\lambda,\quad t_T\neq t_{Q_m},\quad m=1,\ldots,k+1,$$

$$\lim_{\alpha\to0}\alpha\left[\gamma_s\left(t_{Q_m}+\frac{\alpha}{2}\right)-\gamma_s\left(t_{Q_m}-\frac{\alpha}{2}\right)\right]=\frac{4Q_1}{\pi},\quad m=1,\ldots,k.$$

3. The numerical solution of Eq. (7). Let us take in the system (7) instead of $f(t_0)$ a trigonometric polynomial $f_n(t_0)$ of degree *n*. Then the solution to this equation in the space *X** is a trigonometric polynomial $\gamma_n(t)$ of degree *n*. Let us take the points

$$t_i=\beta+i\frac{2\pi}{2n+1},\quad 0\le\beta\le\frac{\pi}{2n+1},\quad i=0,1,\ldots,2n\quad\text{and}\quad t_{0i}=t_i+\frac{\pi}{2n+1}.$$

Then the equality (see [1])

$$\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{t_{0j}-t}{2}\gamma_n(t)\,dt=\frac{1}{2\pi}\sum_{i=0}^{2n}\cot\frac{t_{0j}-t_i}{2}\gamma_n(t_i)\frac{2\pi}{2n+1}, \quad j=0,1,\ldots,2n, \quad (30)$$

is valid. Now, for numerical solution of the system (16) we get the following system of linear algebraic equations (s.l.a.e.):

$$\gamma_{n0}+\frac{1}{2\pi}\sum_{i=0}^{2n}\cot\frac{t_{0j}-t_i}{2}\gamma_n(t_i)\frac{2\pi}{2n+1}=f_n(t_{0j}), \quad j=0,1,\ldots,2n,$$
$$\frac{1}{2\pi}\sum_{i=0}^{2n}\gamma_n(t_i)\frac{2\pi}{2n+1}=C. \quad (31)$$

The solution to s.l.a.e. (31) is given by the following formulas:

$$\gamma_{n0}=\frac{1}{2\pi}\sum_{j=0}^{2n}f_n(t_{0j})\frac{2\pi}{2n+1},$$
$$\gamma_n(t_i)=-\frac{1}{2\pi}\sum_{j=0}^{2n}\cot\frac{t_i-t_{0j}}{2}f_n(t_{0j})\frac{2\pi}{2n+1}+C. \quad (32)$$

Note 3. If $f(t)\in H$ on $[0,2\pi]$, then we may take $f_n(t_{0j})=f(t_{0j})$, $j=0,1,\ldots,2n$.

For the systems (18) and (20) the s.l.a.e. are taken in an analogous way, only in (20) we must take $t_T=t_{i_T}$ for some i_T.

Now let us consider numerical solution of Eq. (7) in the class of singular solutions. Let $t_Q=t_{0j_Q}$. From the singularities of the choice of the points t_i and t_{0j}, $i,j=0,1,\ldots,2n$, and from the formulas we have

$$\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{t-t_{0j}}{2}\cot\frac{t_{0j_Q}-t}{2}dt=1=\frac{1}{2\pi}\sum_{i=0}^{2n}\cot\frac{t_i-t_{0j}}{2}\cot\frac{t_{0j_Q}-t_i}{2}\frac{2\pi}{2n+1}, \quad j\neq j_Q. \quad (33)$$

Now, for numerical solution of the system (20) we get the following s.l.a.e.:

$$\frac{1}{2\pi}\sum_{i=0}^{2n}\cot\frac{t_{0j}-t_i}{2}\gamma_{n,s}(t_i)\frac{2\pi}{2n+1}=f_n(t_{0j}), \quad j=0,1,\ldots,2n, \quad j\neq j_Q,$$
$$\frac{1}{2\pi}\sum_{i=0}^{2n}\gamma_{n,s}(t_i)\frac{2\pi}{2n+1}=C, \quad (34)$$

which has the solution

$$\gamma_{n,s}(t_i) = -\frac{1}{2\pi}\sum_{i=0}^{2n} \cot\frac{t_i - t_{0j}}{2} f_n(t_{0j})\frac{2\pi}{2n+1} + C + \cot\frac{t_i - t_{0j_Q}}{2} B_n, \qquad i = 0,1,\dots,2n,$$

$$B_n = \frac{1}{2\pi}\sum_{i=0}^{2n} f_n(t_{0j})\frac{2\pi}{2n+1}. \tag{35}$$

The convergence of the method is evident.

Note 4. For Eq. (8) an analogous method of numerical solution may be constructed.

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SINGULAARSETE INTEGRAALVÕRRANDITE SINGULAARSED LAHENDID JA NENDE RAKENDUSED

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Viimasel ajal on aerodünaamikas pööratud suurt tähelepanu sellistele tiiva konstruktsioonidele, mis sisaldavad tiiva pinnal õhku imevaid seadeldisi. Matemaatiliselt taanduvad seda laadi ülesanded singulaarse tuumaga esimest liiki integraalvõrrandi lahendamisele 1/*x*-tüüpi singulaarsustega funktsioonide ruumides. Niisuguste võrrandite lahendid pole üheselt määratud ja võrrandi parem pool peab lahendi olemasoluks rahuldama numbrilise lahendamise seisukohast küllaltki ebamugavaid tingimusi. Artiklis on konstrueeritud funktsioonide ruumid, kus mainitud võrrandid teatud lisatingimustel on üheselt lahenduvad suvalise parema poole korral, ning on esitatud ka võrrandite praktiliseks lahendamiseks sobiv algoritm.