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THE DISCRETE KHARITONOV THEOREM REVISITED

Ülo NURGES

Institute of Cybernetics at Tallinn Technical University, Akadeemia tee 21, EE-0026 Tallinn, Estonia; e-mail: nurges@ioc.ee

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Abstract. Starting from the discrete analogue of the weak Kharitonov theorem for monic polynomials, a sufficient stability condition in terms of polytopes is obtained using a Schur invariant transformation on the coefficients space of polynomials. This approach enables also the construction of a stable n-dimensional simplex with all vertices on the stability boundary.

Key words: robust stability, discrete-time systems.

1. INTRODUCTION

In the field of robust control and stability robustness with parametric uncertainties the Kharitonov theorems which are mainly applicable to Hurwitz polynomials play the central role [1]. The results regarding Schur polynomials are less attractive. However, for special cases the discrete analogues of Kharitonov theorems hold [2]. The weak Kharitonov theorem holds for Schur polynomials in a region defined by rectangles with edges sloped by $\pi/4$ [3]. Recently, Perez et al. [4] relaxed the requirements for slopes of edges. Unfortunately, the both versions of the discrete analogue of the weak Kharitonov theorem give a degenerate stability region for monic polynomials because the coefficients a_n and a_0 have to be fixed. It means that only an $(n-1)$-dimensional polytope can be obtained for the nth-order monic polynomials.

The aim of this paper is to find an n-dimensional stable polytope of monic Schur polynomials starting from an $(n-1)$-dimensional polytope defined by the discrete Kharitonov theorem. The paper is organized as follows. First, we recall the discrete analogue of the weak Kharitonov theorem. Second, the Schur invariant transformation on the coefficients space of polynomials is recalled [5]. Third, applying the Schur invariant transformation to the vertex polynomials defined by the discrete Kharitonov theorem, the main result – a sufficient stability condition in terms of polytopes in the coefficients space of monic polynomials is obtained. By special restrictions a stable n-dimensional simplex will be found.

2. DISCRETE ANALOGUE OF THE WEAK KHARITONOV THEOREM

Consider the polynomials with real coefficients $a_i \in \mathcal{R}$

$$a(z) = \sum_{i=0}^{n} a_i z^i .$$

A polynomial $a(z)$ is said to be Schur stable if all its roots lie inside the unit circle. The following extreme point result for Schur polynomials $a(z)$ is called the discrete analogue of the weak Kharitonov theorem [3].

Theorem 1. *Consider the class of polynomials $a(z), a_i \in [\underline{a}_i, \bar{a}_i]$ in which, for each $i \neq n/2$, a_i and a_{n-i} vary inside a rectangle with edges sloped by $\pi/4$ (Fig.1). If n is even, $a_{n/2}$ varies in an interval $[\underline{a}_{n/2}, \bar{a}_{n/2}]$. Then all the polynomials $a(z)$ in this class are stable if and only if every member of the finite set of $a(z)$ defined by every possible combination of corner points (and interval end points in case $i = n/2$) is stable.*

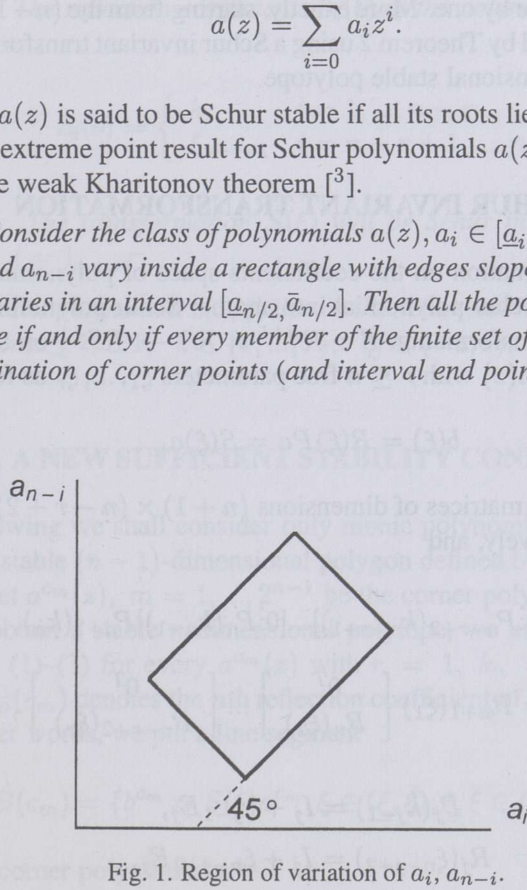

Fig. 1. Region of variation of a_i, a_{n-i}.

Recently, Perez et al. relaxed the requirements for slopes of edges and coupling of coordinates for the class of polynomials $a(z)$ and obtained a more general version of the weak Kharitonov theorem for discrete polynomials [4].

Theorem 2. *Consider the polytope in the coefficients space where each pair (a_i, a_k), $0 \leq i \leq n$, $n - i \leq k \leq n$, is varying inside a polytope with edges* *sloped in the closed interval $[\pi/4, 3\pi/4]$ and where each a_i can only be combined with one a_k and vice versa. Then every polynomial in the polygon will be stable if and only if all the polynomials obtained by combining all the polygon corners are stable.*

For monic polynomials we have $a_n = 1$ and by assumptions of Theorems 1 and 2 $a_0 = \text{const}$. So, all the stable polygons in the coefficients space of monic polynomials $a = (a_0, ..., a_{n-1}) \in \mathcal{R}^n$ obtained by Theorems 1 and 2 are of the dimension $n - 1$. It means that the discrete analogue of the Kharitonov theorem is considerably weaker for monic polynomials. To overcome this drawback, we recall a Schur invariant transformation which enables us to increase the dimension of the stability polytope by one. More exactly, starting from the $(n-1)$-dimensional stable polytope defined by Theorem 2 using a Schur invariant transformation, we are looking for an n-dimensional stable polytope.

3. SCHUR INVARIANT TRANSFORMATION

We call a transformation on the coefficients space of polynomials $a(z)$ Schur invariant if it maps a Schur polynomial into another Schur polynomial.

Let us define a transformation $S : \mathcal{R}^{n+1} \times \mathcal{R}^r \to \mathcal{R}^{n+1}$ on the coefficients space of polynomials $a(z)$ with $r \leq n$ free parameters $\xi_1, ..., \xi_r$ as follows:

$$b(\xi) = R(\xi)Pa = S(\xi)a, \tag{1}$$

where $R(\xi)$ and P are matrices of dimensions $(n+1) \times (n-r+2)$ and $(n-r+2) \times (n+1)$, respectively, and

$$P = [0\vdots P_{n-r+2}(k_{n-r+1})]...[0\vdots P_n(k_{n-1})]P_{n+1}(k_n), \tag{2}$$

$$R(\xi) = R_{n+1}(\xi_1)\begin{bmatrix} 0^T \\ R_n(\xi_2) \end{bmatrix} \cdots \begin{bmatrix} 0^T \\ R_{n-r+2}(\xi_r) \end{bmatrix}, \tag{3}$$

$$P_j(k_{j-1}) = I_j + k_{j-1}E_j,$$
$$R_j(\xi_{n-j+2}) = I_j + \xi_{n-j+2}E_j,$$

where I_n is an $n \times n$ unit matrix and $E_n = [e_n\vdots...\vdots e_1]$, $\quad e_i = (\underbrace{0...0}_{i-1}10...0)^T$.

Now let us recall the recursive definition of reflection coefficients k_i of a polynomial $a(z)$ [6]:

$$\bar{a}_i^{(n)} = \frac{a_{n-i}}{a_n}, \qquad i = 1, ..., n;$$

$$\bar{a}_j^{(i-1)} = \frac{\bar{a}_j^{(i)} + k_i \bar{a}_{i-j}^{(i)}}{1 - k_i^2}, \qquad j = 1, \ldots, i-1;$$

$$k_i = -\bar{a}_i^{(i)}.$$

The following theorem and corollaries hold [5].

Theorem 3. *The polynomial $b(z, \xi)$ will be Schur if and only if*

1) the reflection coefficients $k_1, \ldots, k_{n-r}$ of the polynomial $a(z)$ lie in the interval $(-1, 1)$;

2) the free parameters $\xi_1, \ldots, \xi_r$ lie in the interval $(-1, 1)$.

Corollary 3.1. *The reflection coefficients $k_i(b)$ of the polynomial $b(z, \xi)$ have the following values:*

$$k_i(b) = \begin{cases} k_i & i = 1, \ldots, n-r \\ \xi_{n-i+1} & i = n-r+1, \ldots, n. \end{cases}$$

Corollary 3.2. *The transformation $S(\xi)$ will be Schur invariant if and only if $-1 < \xi_j < 1$, $j = 1, \ldots, r$.*

Corollary 3.3. *The polynomial $b(z, \xi)$ lies on the Schur stability boundary if $a(z)$ is Schur and if some $\xi_k = \pm 1$, $k \in \{1, \ldots, r\}$; $\xi_j \in (-1, 1)$, $j \neq k$, $j = 1, \ldots, r$.*

4. A NEW SUFFICIENT STABILITY CONDITION

In the following we shall consider only monic polynomials, i.e., $a_n = 1$. Let us start from a stable $(n-1)$-dimensional polygon defined by Theorem 1 for some $a_0 = a_0^*$, and let $a^{c_m}(z)$, $m = 1, \ldots, 2^{n-1}$ be the corner polynomials of this stable polygon. To obtain a stable n-dimensional polytope, we use the Schur invariant transformation (1)–(3) for every $a^{c_m}(z)$ with $r = 1$, $k_n = k_n(c_m)$, and $\xi_1 \in (\underline{\xi}, \bar{\xi})$, where $k_n(c_m)$ denotes the nth reflection coefficient of the corner polynomial $a^{c_m}(z)$. In other words, we put a line segment

$$B(c_m) = \{b^{c_m} = S(\xi) a^{c_m}, \xi \in (\underline{\xi}, \bar{\xi}); \underline{\xi}, \bar{\xi} \in (-1, 1)\}$$

through every corner polynomial a^{c_m}.

Theorem 4. *Consider the polytope*

$$A = \mathrm{conv}\{a^{c_m}, m = 1, \ldots, 2^{n-1}\}$$

in the coefficients space of monic polynomials, where each pair (a_i, a_k), $1 \leq i \leq n-1$, $n-i \leq k \leq n-1$ is varying inside a polytope with edges sloped by $\pi/4$ (or

$3\pi/4$) and where each a_i can only be combined with one a_k and vice versa. Then every polynomial in the polytope

$$B = \mathrm{conv}\{b^{c_m}(\xi) \mid b^{c_m}(\xi) = S(\xi)a^{c_m},\ \ m = 1, ..., 2^{n-1};\ \ \xi \in (-1,1)\}$$

will be stable if all the corner polynomials a^{c_m} of the polytope A are stable.

Proof. By Theorem 1 the polytope A will be stable. For $r = 1$ the transformation (1)–(3) is linear in respect of the free parameter ξ. Hence, the transformation $S(\xi)$ maps the polytope A into another polytope

$$B(\xi) = \mathrm{conv}\{b^{c_m}, m = 1, ..., 2^{n-1}\}$$

and if $\xi \in (-1,1)$, the polytope $B(\xi)$ will be stable by Theorem 3. So, both the polytopes $B(\underline{\xi})$ and $B(\bar{\xi})$ will be stable. The corner polynomials $b^{c_m*}(\underline{\xi}) \in B(\underline{\xi})$ and $b^{c_m*}(\bar{\xi}) \in B(\bar{\xi})$ are the end points of a line segment

$$B(c_m) = \{b^{c_m}(\xi) = S(\xi)a^{c_m}, \xi \in (\underline{\xi}, \bar{\xi})\}$$

which is stable by Theorem 3 if a^{c_m} is stable. Therefore, all the exposed edges of the polytope B will be stable if a^{c_m}, $m = 1, ..., 2^{n-1}$, are stable and by the Edge theorem the polytope B will be stable. □

The polytope $B(\xi)$ has the maximal volume if $\underline{\xi} = -1$ and $\bar{\xi} = 1$. Then all the vertices of the polytope $B(\xi)$ are placed on the stability boundary. The number N of vertices $b^{c_m}(\xi = \pm 1)$ increases rapidly by increasing the degree n of polynomials, $N \leq 2^n$. The next theorem allows us to generate an n-dimensional stable simplex starting from an $(n-1)$-dimensional stable polytope of monic polynomials.

Theorem 5. *Consider the intersection points $a^{c_j} \in \mathcal{R}^{n-1}$ of hyperplanes*

$$a_i + a_{n-i} = \alpha_i, \qquad \alpha_i \in \{\underline{\alpha}_i, \bar{\alpha}_i\} \tag{4}$$

and

$$a_i - a_{n-i} = \beta_i, \qquad \beta_i \in \{\underline{\beta}_i, \bar{\beta}_i\}, \qquad i = 1, ..., n-1. \tag{5}$$

The n-dimensional polytope

$$S^n = \mathrm{conv}\{a^{c_j}(\xi) \mid \xi = \pm 1, j = 1, ..., N\}$$

• *is stable if and only if the $(n-1)$-dimensional polytope*

$$A^{n-1} = \mathrm{conv}\{a^{c_j}, j = 1, ..., N\},$$

where

$$a^{c_j}(\xi) = R(\xi)\begin{bmatrix} 0 \\ a^{c_j} \end{bmatrix},$$

is stable;

- *is a stable simplex if*

$$N = \mu\nu,$$

where μ is the number of hyperplanes (4) *considered*

$$\mu = \begin{cases} n/2+1 & \text{if } n \text{ even}, \\ (n+1)/2 & \text{if } n \text{ odd}, \end{cases}$$

and ν is the number of hyperplanes (5) *considered*

$$\nu = \begin{cases} n/2 & \text{if } n \text{ even}, \\ (n+1)/2 & \text{if } n \text{ odd}. \end{cases}$$

Proof. Necessity is quite obvious because $a^{c_j}(\xi)$ will be unstable if a^{c_j} is unstable by Theorem 3.

To prove the sufficiency, we consider two different cases for n even and odd because the structure of matrices $R(\xi = \pm 1)$ depends on n.

It is easy to see that the hyperplanes (4) and (5) have only one intersection point for some fixed α_i and β_i. Without loss of generality we can assume $a_0^* = 0$.

Let n be even. Then, for $\xi = 1$, we have from (3)

$$a(1) = R(1)\begin{bmatrix} 0 \\ a \end{bmatrix} = \begin{bmatrix} 1 & 0 & ... & 0 & 1 \\ 0 & 1 & ... & 1 & 0 \\ . & . & . & . & . \\ 0 & \dots & 2 & \dots & 0 \\ . & . & . & . & . \\ 0 & 1 & ... & 1 & 0 \\ 1 & 0 & ... & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a_1 \\ \vdots \\ a_{n-1} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha_1 \\ \vdots \\ \alpha_{n/2} \\ \vdots \\ \alpha_1 \\ 1 \end{bmatrix}.$$

Similarly, for $\xi = -1$, we have from (3)

$$a(-1) = R(-1)\begin{bmatrix} 0 \\ a \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & ... & 0 & -1 \\ 0 & 1 & ... & -1 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ . & . & . & . & . \\ 0 & -1 & ... & 1 & 0 \\ -1 & 0 & ... & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a_1 \\ \vdots \\ a_{n-1} \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ \beta_1 \\ \vdots \\ \beta_{n/2-1} \\ 0 \\ -\beta_{n/2-1} \\ \vdots \\ -\beta_1 \\ 1 \end{bmatrix}.$$

It means that all ν points $a^{c_j} \in \mathcal{R}^{n-1}$ of the hyperplane (4) with some fixed $\alpha_i,\ i = 1, ..., n/2$, will be transformed to the point $a(1) \in \mathcal{R}^n$ and all μ points of the hyperplane (5) with some fixed $\beta_i,\ i = 1, ..., n/2-1$, will be transformed to the point $a(-1) \in \mathcal{R}^n$. Obviously, all the vectors $a^{c_j}(1) \in \mathcal{R}^n,\ j = 1, ..., \mu,\ \mu \leq n/2+1$, with some different $\alpha_i \in \{\underline{\alpha}_i, \bar{\alpha}_i\},\quad i \in \{1, ..., n/2\}$, are linearly independent as much as the vectors $a^{c_j}(-1) \in \mathcal{R}^n,\quad j = 1, ..., \nu,\quad \nu \leq n/2$, with some different $\beta_i \in \{\underline{\beta}_i, \bar{\beta}_i\},\quad i \in \{1, ..., n/2-1\}$. Thus, for n even, $N = \mu + \nu = n + 1$ points $a(\pm 1) \in \mathcal{R}^n$ form an n-dimensional simplex S^n.

Now let n be odd. From (3) we have for $\xi = 1$

$$a(1) = R(1)\begin{bmatrix} 0 \\ a \end{bmatrix} = \begin{bmatrix} 1 & 0 & ... & ... & 0 & 1 \\ 0 & 1 & ... & ... & 1 & 0 \\ . & . & . & . & . & . \\ 0 & \dots & 1 & 1 & \dots & 0 \\ 0 & \dots & 1 & 1 & \dots & 0 \\ . & . & . & . & . & . \\ 0 & 1 & ... & ... & 1 & 0 \\ 1 & 0 & ... & ... & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a_1 \\ \vdots \\ a_{n-1} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha_1 \\ \vdots \\ \alpha_{(n-1)/2} \\ \alpha_{(n-1)/2} \\ \vdots \\ \alpha_1 \\ 1 \end{bmatrix}.$$

It means that all ν points $a^{c_j} \in \mathcal{R}^{n-1}$ of the hyperplane (4) with some fixed $\alpha_i,\ i = 1, ..., n/2$, will be transformed to the point $a(1) \in \mathcal{R}^n$. Similarly, for $\xi = -1$, we have from (3)

$$a(-1) = R(-1)\begin{bmatrix} 0 \\ a \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & ... & ... & 0 & -1 \\ 0 & 1 & ... & ... & -1 & 0 \\ . & . & . & . & . & . \\ 0 & \dots & 1 & -1 & \dots & 0 \\ 0 & \dots & -1 & 1 & \dots & 0 \\ . & . & . & . & . & . \\ 0 & -1 & ... & ... & 1 & 0 \\ -1 & 0 & ... & ... & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a_1 \\ \vdots \\ a_{n-1} \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ \beta_1 \\ \vdots \\ \beta_{(n-1)/2} \\ -\beta_{(n-1)/2} \\ \vdots \\ -\beta_1 \\ 1 \end{bmatrix}$$

and all μ points of the hyperplane (5) with some fixed $\beta_i,\ i = 1, ..., n/2-1$, will be transformed to the point $a(-1) \in \mathcal{R}^n$. Obviously, all the vectors $a^{c_j}(1) \in \mathcal{R}^n,\ j = 1, ..., \mu,\ \mu \leq (n+1)/2$, with some different $\alpha_i \in \{\underline{\alpha}_i, \bar{\alpha}_i\}$, $i \in \{1, ..., (n-1)/2\}$, are linearly independent as much as the vectors $a^{c_j}(-1) \in \mathcal{R}^n,\ j = 1, ..., \nu,\quad \nu \leq (n+1)/2$, with some different $\beta_i \in \{\underline{\beta}_i, \bar{\beta}_i\}, i \in \{1, ..., (n-1)/2\}$. It means that for n odd $N = \mu + \nu = n + 1$ points $a(\pm 1) \in \mathcal{R}^n$ form an n-dimensional simplex S^n.

Consider now the edges of the simplex S^n. By assumption all the points $a^{c_j} \in \mathcal{R}^{n-1}$, $j = 1, ..., N$, are stable. Hence, by Theorem 3, the line segments $a^{c_j}(\xi) = R(\xi) \begin{bmatrix} 0 \\ a^{c_j} \end{bmatrix}$, $j = 1, ..., N$; $\xi \in (-1, 1)$, will be stable.

By assumption all the line segments $\text{conv}(a^{c_j}, a^{c_k}) \in A^{n-1}; j, k \in \{1, ..., N\}$, are stable. Hence, by Theorem 3, the line segments $\text{conv}\{a^{c_j}(\xi), a^{c_k}(\xi) \mid \xi = \pm 1\}$ will lie on the stability boundary. So, all the edges of the n-dimensional simplex S^n will be stable or placed on the stability boundary. By the Edge theorem all the inside points of the simplex S^n will be stable. □

Example 1. Let $n = 3$ and $a_0 = 0$. The vertices of the tilted square (ABCD in Fig. 2) $a^{c_1} = (0, 0.5, 0)^T$, $a^{c_2} = (0, 0, 0.5)^T$, $a^{c_3} = (0, -0.5, 0)^T$, and $a^{c_4} = (0, 0, -0.5)^T$ are Schur stable. By Theorem 1 every polynomial in the square is stable.

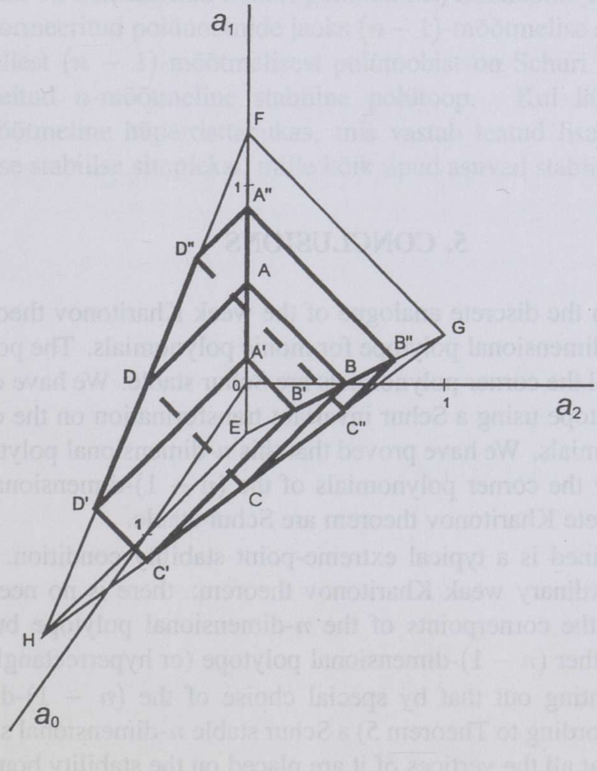

Fig. 2. Stable polytope of monic polynomials, $n = 3$.

Using the Schur invariant transformation (1)–(3), we find

$$\begin{aligned} b^{c_1} &= (\xi, 0.5, 0.5\xi)^T, \\ b^{c_2} &= (\xi, 0.5\xi, 0.5)^T, \\ b^{c_3} &= (\xi, -0.5, -0.5\xi)^T, \\ b^{c_4} &= (\xi, -0.5\xi, -0.5)^T. \end{aligned}$$

For $\underline{\xi} = -0.5$ and $\bar{\xi} = 0.5$, Theorem 4 claims that the polytope (A'B'C'D'A"B"C"D" in Fig. 2)

$$B = \mathrm{conv}\begin{pmatrix} 0.5 & 0.5 & 0.5 & 0.5 & -0.5 & -0.5 & -0.5 & -0.5 \\ 0.5 & 0.25 & -0.5 & -0.25 & 0.5 & -0.25 & -0.5 & 0.25 \\ 0.25 & 0.5 & -0.25 & -0.5 & -0.25 & 0.5 & 0.25 & -0.5 \end{pmatrix}$$

is stable. By Theorem 5 the simplex (EFGH in Fig. 2)

$$S^3 = \mathrm{conv}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{pmatrix}$$

is stable.

5. CONCLUSIONS

We started from the discrete analogue of the weak Kharitonov theorem which defines an $(n-1)$-dimensional polytope for monic polynomials. The polytope will be Schur stable if all the corner polynomials are Schur stable. We have obtained an n-dimensional polytope using a Schur invariant transformation on the coefficients space of the polynomials. We have proved that this n-dimensional polytope will be Schur stable if only the corner polynomials of the $(n-1)$-dimensional polytope defined by the discrete Kharitonov theorem are Schur stable.

The result obtained is a typical extreme-point stability condition. In fact, it is better than the ordinary weak Kharitonov theorem: there is no need to check the stability of all the cornerpoints of the n-dimensional polytope but only the cornerpoints of another $(n-1)$-dimensional polytope (or hyperrectangle).

It is worth pointing out that by special choise of the $(n-1)$-dimensional hyperrectangle (according to Theorem 5) a Schur stable n-dimensional simplex can be obtained such that all the vertices of it are placed on the stability boundary.

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DISKREETSE HARITONOVI TEOREEMI RAKENDUS

Ülo NURGES

Vaadeldud on normeeritud Schuri polünoome. Diskreetne Haritonovi teoreem defineerib normeeritud polünoomide jaoks $(n-1)$-mõõtmelise stabiilse polütoobi. Lähtudes sellest $(n-1)$-mõõtmelisest polütoobist on Schuri invariantse teisenduse abil leitud n-mõõtmeline stabiilne polütoop. Kui lähtepolütoobiks on $(n-1)$-mõõtmeline hüperristtahukas, mis vastab teatud lisatingimustele, saab n-mõõtmelise stabiilse simpleksi, mille kõik tipud asuvad stabiilsuspiiril.