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MERCER'S THEOREMS FOR GENERALIZED SUMMABILITY METHODS IN BANACH SPACES

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Abstract. Mercer's theorems, shortly, M-theorems, well-known for number sequences and summability methods given by scalar matrices, are generalized to larger classes of summability methods $\mathcal{B} = (B_{nk})$ and sequences of points in B-spaces X. The operators $B_{nk} : X \to X$ are continuous and linear on X. Seven M-theorems for generalized triangular methods and for generalized Euler–Knopp and Riesz methods of $c_X \to c_X$ type or $\ell_X \to \ell_X$ type are presented (c_X and ℓ_X being spaces of convergent sequences or absolutely convergent series). The applications of general results to scalar matrix methods and certain classical methods are also discussed.

Key words: Banach spaces, operators and generalized summability methods, methods of $\alpha \to \beta$ type, Mercer's theorems.

1. INTRODUCTION AND PRELIMINARIES

Let X and Y be Banach spaces (B-spaces) over the field $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$. For any two spaces X and Y the notation $\mathcal{F}: X \to Y$ denotes that the operator $\mathcal{F}$ maps X into Y, for which we could also use the notion "$\mathcal{F}$ is of $X \to Y$ type". The space $\mathcal{L}(X, Y)$ of all continuous linear operators from X into Y is known to be a B-space (see, e.g., [1],V; [2], IV). We denote further by I and θ the identity and the zero operator on any B-space, respectively. Let $\chi = (x_k)$ be a sequence of $x_k \in X$. The well-known sequence spaces are: $m_X = \{(x_k) : x_k \in X;\ \sup_k \|x_k\| < \infty\}$; $c_X = \{(x_k) : x_k \in X;\ \exists \ \lim_k x_k\}$; $\ell_X = \{(x_k) : x_k \in X;\ \sum_k \|x_k\| < \infty\}$. These are all B-spaces with the norm $\|\chi\| = \sup_k \|x_k\|$ in m_X and c_X, and the norm $\|\chi\| = \sum_k \|x_k\|$ in ℓ_X. Unless indicated otherwise, a sum $\sum x_k$ without limits will always be understood as follows: $\sum x_k = \sum_k x_k = \sum_{k=0}^{\infty} x_k$.

In this work the classical Mercer's theorems (M-theorems), well-known for number sequences and scalar matrix methods, are extended to sequences in B-spaces and generalized triangular summability methods $\mathcal{B} = (B_{nk})$ with $B_{nk} \in \mathcal{L}(X, X)$. More about matrix methods of this kind see, e.g. [3–8]. Our main results on the generalized M-theorems are given in Section 2. M-theorems for generalized Euler–Knopp and Riesz methods with some of classical cases are obtained in Section 3 and 4 as applications of the results of Section 2.

Let us fix some notations connected with the method $\mathcal{B}$, where the spaces s_X and s'_X will be m_X, c_X or ℓ_X and $\mathbf{N} := \{0, 1, 2, \dots\}$.

We define the operator $\mathcal{B}_n : s_X \to X$ by

$$\mathcal{B}_n \chi = \sum_{k=0}^{n} B_{nk} x_k \qquad (\chi \in s_X; n \in \mathbf{N}). \tag{1}$$

The special case of it as an operator of $X \to X$ type is also denoted by $\mathcal{B}_n$, so that

$$\mathcal{B}_n x = \sum_{k=0}^{n} B_{nk} x \qquad (x \in X; n \in \mathbf{N}). \tag{2}$$

Let the operator $\mathcal{B} : s_X \to s'_X$ be given by

$$\eta = \mathcal{B}\chi, \tag{3}$$

where $\eta = (y_n)$ and

$$y_n = \sum_{k=0}^{n} B_{nk} x_k \qquad (\chi \in s_X; n \in \mathbf{N}) \tag{4}$$

or, because of (1),

$$y_n = \mathcal{B}_n \chi \qquad (\chi \in s_X; n \in \mathbf{N}). \tag{5}$$

In view of (3)–(5) we have

$$\eta = \mathcal{B}\chi = (y_n) = (\mathcal{B}_n \chi) = \Big(\sum_{k=0}^{n} B_{nk} x_k \Big) \qquad (\chi \in s_X). \tag{6}$$

It is proved in [8] that the first of these operators $\mathcal{B}_n \in \mathcal{L}(s_X, X)$, the second operator $\mathcal{B}_n \in \mathcal{L}(X, X)$, and $\mathcal{B} \in \mathcal{L}(s_X, s'_X)$, whenever $B_{nk} \in \mathcal{L}(X, X)$ $(n, k \in \mathbf{N})$ and s'_X is any of m_X, c_X or $s'_X = s_X = \ell_X$.

In the sequel we need for generalized triangular methods $\mathcal{B} = (B_{nk})$ defined by (1)–(4) the following two theorems and the corollaries to them. Both theorems hold also for the case $B_{nk} \in \mathcal{L}(X, Y)$.

Theorem A. *Let the method $\mathcal{B} = (B_{nk})$ with $B_{nk} \in \mathcal{L}(X, X)$ be defined by* (6). *Then $\mathcal{B}$ is of $c_X \to c_X$ type if and only if*

1^0 *there exists* $\lim_n B_{nk}x = B_kx \qquad (x \in X; k \in \mathbf{N})$,

2^0 *there exists* $\lim_n \sum_{k=0}^{n} B_{nk}x = Bx \qquad (x \in X)$,

3^0 $\sup_{\|x_k\| \leq 1} \| \sum_{k=0}^{n} B_{nk}x_k\| = O(1) \qquad (n \in \mathbf{N})$.

Moreover,

$$\lim_n y_n = \lim_n \sum_{k=0}^{n} B_{nk}x_k = Bx^* + \sum_k B_k(x_k - x^*) \tag{7}$$

whenever these conditions are satisfied and $x^* = \lim_k x_k$.

Theorem B. *Let the method $\mathcal{B} = (B_{nk})$ with $B_{nk} \in \mathcal{L}(X, X)$ be defined by* (6). *Then $\mathcal{B}$ is of $\ell_X \to \ell_X$ type if and only if*

$$\sum_{n=k}^{\infty} \|B_{nk}x\| \leq M\|x\| \qquad (x \in X; k \in \mathbf{N}), \tag{8}$$

the constant M being independent from x and k.

Moreover,

$$\sum_n y_n = \sum_k \mathcal{G}_k x_k, \tag{9}$$

with

$$\mathcal{G}_k x = \sum_{n=k}^{\infty} B_{nk}x \qquad (x \in X; k \in \mathbf{N}), \tag{10}$$

whenever the condition (8) *is satisfied and* $(x_k) \in \ell_X$.

Corollary A.1. *Let the method $\mathcal{B} = (B_{nk})$ with $B_{nk} \in \mathcal{L}(X, X)$ be defined by* (6). *Then $\mathcal{B}$ is regular if and only if the conditions of Theorem* A *with $B = I$ and $B_k = \theta$ $(k \in \mathbf{N})$ are satisfied.*

These sentences can be obtained as immediate corollaries of their analogues proved for generalized infinite matrix methods by Zeller [4], Kangro [5], and Robinson [3], respectively. It is also clear that the following Remarks A.1, B.1, and B.2 are true.

Remark A.1. The operators B_k and B, appearing in the conditions 1^0 and 2^0 of Theorem A, are linear and bounded or, precisely, $B, B_k \in \mathcal{L}(X, X)$ $(k \in \mathbf{N})$.

Remark B.1. The series $\sum \mathcal{G}_k x_k$ of Theorem B is absolutely convergent and it may be treated either in the form $\sum y_n$ or $\sum \mathcal{B}_n \chi$. Moreover, there exists a constant M such that

$$\sum_n \|y_n\| = \sum_n \|\mathcal{B}_n \chi\| = \sum_k \|\mathcal{G}_k x_k\| \le M\|\chi\|. \tag{11}$$

Remark B.2. Let the operators $\mathcal{G}_k : \ X \to X$ be defined by (9) and (10) with all premises of Theorem B. Then

(a) $\mathcal{G}_k \in \mathcal{L}(X, X) \qquad (k \in \mathbf{N})$;

(b) the sequence $\|\mathcal{G}_k\|$ is bounded, at which

$$\|\mathcal{G}_k\| \le M \qquad (k \in \mathbf{N}), \tag{12}$$

where M is the constant from (8).

Below we need for methods of $\ell_X \to \ell_X$ type the following corollary of Theorem B, which is a generalization of an analogous result of the classical case derived by Baron [9]. The proofs of both these results are fully similar.

Corollary B.1. *Let the method $\mathcal{B} = (B_{nk})$ with $B_{nk} \in \mathcal{L}(X, X)$ be defined by* (6) *and let $\mathcal{B} : \ \ell_X \to \ell_X$. Then $\mathcal{B}$ is absolutely regular if and only if*

$$\mathcal{G}_k x = x \qquad (x \in X; k \in \mathbf{N}) \tag{13}$$

for $\mathcal{G}_k : \ X \to X$ fixed by (9) *and* (10). *At that $\|\mathcal{B}\| = 1$ and*

$$\sum_n y_n = \sum_k x_k \qquad (\chi \in \ell_X). \tag{14}$$

Note. The results, obtained in Corollary B.1, and Remarks B.1 and B.2, improve our knowledge about the methods $\mathcal{B} : \ \ell_X \to \ell_X$ discussed in [8].

2. GENERAL THEOREMS

Suppose that $\chi = (x_k)$ is a sequence of elements in the B-space X. In accordance with the classical form of Mercer's theorems for number sequences, let us examine a transformation $\zeta = (z_n)$ of χ, where $z_n = \alpha x_n + (1 - \alpha) \sum_{k=0}^{n} B_{nk} x_k$ and $\alpha \in \mathbf{R} \setminus \{0\}$. In view of (1), we use further for ζ mostly the following expression:

$$z_n = \alpha x_n + (1 - \alpha) \mathcal{B}_n \chi \qquad (n \in \mathbf{N}). \tag{15}$$

We need also the notation $\frac{\alpha-1}{\alpha} = q$, $\frac{1}{\alpha} z_n = t_n$, and $\tau = (t_n) = \frac{1}{\alpha}\zeta$. Now, from (15) it follows that

$$t_n = x_n - q\mathcal{B}_n\chi \qquad (n \in \mathbf{N}). \tag{16}$$

Employing the operator $\mathcal{B}: \ s_X \to s_X$ given by (6), we can transform the relations (15) and (16) to

$$\zeta = \alpha\chi + (1-\alpha)\mathcal{B}\chi \tag{17}$$

and

$$\tau = \chi - q\mathcal{B}\chi, \tag{18}$$

respectively, which are fixed by operators $\alpha I + (1-\alpha)\mathcal{B}$ and $I - q\mathcal{B}$. Based on the results of [8] and the sense of $\mathcal{L}(X, X)$, we get that $I - q\mathcal{B}_n : \ s_X \to X$, $I - q\mathcal{B}_n \in \mathcal{L}(s_X, X)$, $I - q\mathcal{B}: \ s_X \to s_X$, and $I - q\mathcal{B} \in \mathcal{L}(s_X, s_X)$ whenever $B_{nk} \in \mathcal{L}(X, X)$.

Theorem 1. *Let the method $\mathcal{B}: \ s_X \to s_X$ be defined by* (6) *with $B_{nk} \in \mathcal{L}(X, X)$ $(n, k \in \mathbf{N})$ and let s_X be m_X, c_X or ℓ_X.*

If $\zeta = (z_n) \in s_X$ is given by (17) *and if $|\frac{1-\alpha}{\alpha}| < \|\mathcal{B}\|^{-1}$, then $\chi = (x_k) \in s_X$. At that*

$$\lim_n z_n = \alpha x^* + (1-\alpha)\left[Bx^* + \sum_k B_k(x_k - x^*)\right], \tag{19}$$

where $Bx = \lim_n \mathcal{B}_n x$, $B_k x = \lim_n B_{nk} x \quad (x \in X; n, k \in \mathbf{N})$, and $x^* = \lim_k x_k$, or

$$\sum_n z_n = \alpha \sum_k x_k + (1-\alpha)\sum_k \mathcal{G}_k x_k, \tag{20}$$

where $\mathcal{G}_k x = \sum_{n=k}^{\infty} B_{nk} x \quad (x \in X; k \in \mathbf{N})$, for the cases $s_X = c_X$ or $s_X = \ell_X$, respectively.

Proof. By the assumptions for α, $\|\mathcal{B}\|$ and the meaning of q we get $\|q\mathcal{B}\| < 1$. As, additionally, s_X is a B-space and every $B_{nk} \in \mathcal{L}(X, X)$, then $\mathcal{B} \in \mathcal{L}(s_X, s_X)$ (see above). All this will guarantee the invertibility of $I - q\mathcal{B}$ with $(I - q\mathcal{B})^{-1} \in \mathcal{L}(s_X, s_X)$ (see, e.g., [1], V; [2], IV). Therefore, and since $\zeta \in s_X$ yields $\tau \in s_X$, we obtain from (18) that $\chi = (I - q\mathcal{B})^{-1}\tau \in s_X$.

To prove the relations (19) and (20), we observe separately the following two cases.

First, let $s_X = c_X$. Since by now $\chi \in c_X$, there exists $\lim_k x_k = x^*$ with $x^* \in X$. As $\mathcal{B} : c_X \to c_X$, then, relying on Theorem A and Remark A.1, $\lim_n y_n = \lim_n \mathcal{B}_n\chi$ takes the form (7). Thus, starting from (15), we get (19).

Second, let $s_X = \ell_X$. According to the first part of the proof we have $\chi \in \ell_X$. As $\mathcal{B} : \ell_X \to \ell_X$, then $(y_n) = (\mathcal{B}_n\chi) \in \ell_X$. In view of Theorem B and Remark B.2, the equality (9) holds with operators $\mathcal{G}_k \in \mathcal{L}(X, X)$ being fixed by (10). Finally, from (15) we infer (20), which completes the proof.

Remark 1.1. The parameter $\alpha \neq 0$ which occurs in generalized M-theorems is commonly fixed by the condition

$$\left|\frac{1-\alpha}{\alpha}\right| < \|\mathcal{B}\|^{-1}. \tag{21}$$

In some cases we use the following equivalent relation: (21) $\Leftrightarrow \alpha > \frac{1}{2}$, if $\|\mathcal{B}\| = 1$.

The proofs of M-theorems for methods of $c_X \to c_X$ and $\ell_X \to \ell_X$ type, where respectively the validity of $\lim_n z_n = \lim_k x_k$ and $\sum z_n = \sum x_k$ would be proved, can be simplified by the next two lemmas.

Lemma 1. *Let $\mathcal{B} : c_X \to c_X$ and suppose that all assumptions for $\mathcal{B}$ and ζ are the same as in Theorem* 1. *If $\zeta \in c_X$ and if $|\frac{1-\alpha}{\alpha}| < \|\mathcal{B}\|^{-1}$, then*

$$\lim_n z_n = \lim_k x_k \tag{22}$$

if and only if $\mathcal{B}$ is regular.

Proof. The implication $\zeta \in c_X \Rightarrow \chi \in c_X$ is obvious by Theorem 1.

So, it remains to prove that for (22) the regularity of $\mathcal{B}$ is necessary and sufficient. For the sufficiency the regularity of $\mathcal{B}$ is evident.

For the necessity, let (22) be valid for each $\chi \in c_X$. Then (22) is true also for both sequences $\chi_x = (x_n) = (x, x, \dots)$ and $\chi_m^* = (x_n^*) = (\theta, \dots, \theta, x^*, \theta, \dots)$ with arbitrary $x, x^* \in X$. Clearly, $\lim_n x_n = x$ and $\lim_n x_n^* = \theta$ hold for χ_x and for χ_m^*, respectively.

From the above statements and in view of (7), Theorem A, and Remark A.1 we get for each χ_x that $\lim_n y_n = Bx + \sum_k B_k(x - x) = Bx$. Employing this result in (19), we get $\lim_n z_n = \alpha x + (1-\alpha)Bx$. Now it follows from (22) that $\alpha x + (1-\alpha)Bx = x$, or $Bx = x \quad (x \in X)$, signifying that $B = I$. Analogously, we get for every χ_m^* that $\lim_n y_n = B_m x^*$ and $\lim_n z_n = \alpha\theta + (1-\alpha)B_m x^*$, yielding $(1-\alpha)B_m x^* = \theta$, or $B_m x^* = \theta \quad (x^* \in X; m \in \mathbf{N})$, signifying that all $B_m = \theta$. So, the regularity of $\mathcal{B}$ is guaranteed by Corollary A.1.

Lemma 2. *Let $\mathcal{B} : \ell_X \to \ell_X$ and suppose that all assumptions for $\mathcal{B}$ and ζ are the same as in Theorem* 1. *If $\zeta \in \ell_X$ and if $|\frac{1-\alpha}{\alpha}| < \|\mathcal{B}\|^{-1}$, then*

$$\sum_n z_n = \sum_k x_k \tag{23}$$

if and only if $\mathcal{B}$ is absolutely regular.

Proof. The first part of our proof is analogous to the corresponding part of Lemma 1.

For the necessity, let (23) be valid for each $\chi \in \ell_X$. Then (23) is true also for every $\hat{\chi}_m = (\hat{x}_k)$ $(\hat{x} \in X)$ given in Lemma 1 as X_m^*. In this case it follows from (10) and Remark B.2 that $\mathcal{G}_k \hat{x}_m = \mathcal{G}_m \hat{x}$ if $k = m$ and $\mathcal{G}_k \hat{x}_m = \theta$ whenever $k \neq m$. Therefore and in view of (9) and (20) we get

$$\sum_n z_n = \alpha \sum_k \hat{x}_k + (1-\alpha) \sum_k \mathcal{G}_k \hat{x}_k = \alpha \hat{x} + (1-\alpha)\mathcal{G}_m \hat{x}.$$

From the last result, by using (23), we obtain $\alpha \hat{x} + (1-\alpha)\mathcal{G}_m \hat{x} = \hat{x}$ and then $\mathcal{G}_m \hat{x} = \hat{x}$ $(\hat{x} \in X; m \in \mathbf{N})$. Consequently, all these operators $\mathcal{G}_m$ satisfy the condition (13). Thus $\mathcal{B}$ is absolutely regular and (14) is valid (see Corollary B.1). The lemma is proved.

Various problems of the generalized summability theory necessitate examination of such M-theorems where the method $\mathcal{B}$ satisfies certain additional conditions which are typical of several well-known classical methods like the Cesàro, Riesz, Euler–Knopp methods, and also for generalized methods of the latters.

For such cases let us prove the following two theorems.

Theorem 2. *Let the operators* $\mathcal{B}_n$ *and* $\mathcal{B}$ *be defined by* (1) *and* (6) *with* $B_{nk} \in \mathcal{L}(X, X)$ $(n, k \in \mathbf{N})$.

Suppose the following conditions hold.

$$\|\mathcal{B}_n\| \leq 1 \;\; for \;\; \mathcal{B}_n : c_X \to X \qquad (n \in \mathbf{N}), \tag{24}$$

$$\mathcal{B}_n x = x \quad (x \in X; n \in \mathbf{N}), \tag{25}$$

$$\lim_n B_{nk} x = \theta \quad (x \in X; k \in \mathbf{N}). \tag{26}$$

If $\zeta = (z_n) \in c_X$ *is given by* (17) *and if* $\alpha > \frac{1}{2}$, *then* $\chi = (x_k) \in c_X$ *and* $\lim_k x_k = \lim_n z_n$.

Proof. It follows immediately from (1), $\|\mathcal{B}_n\| = \sup_{\|x\| \leq 1} \|\mathcal{B}_n \chi\|$, and (24) that the condition 3^0 of Theorem A holds. Because of that and by other premises for B_{nk} and $\mathcal{B}_n$ we can infer from Theorem A that $\mathcal{B}: \;\; c_X \to c_X$. Therefore $\|\mathcal{B}\| = 1$ (see [8], Theorem 3). The regularity of $\mathcal{B}$ follows now from Corollary A.1. Hence (see Lemma 1), $\chi \in c_X$ and (22) is valid, which completes the proof.

Analogously we can prove the following theorem using Theorem B and Corollary B.1 instead of Theorem A and Corollary A.1.

Theorem 3. *Let the operators* $\mathcal{B}$ *and* $\mathcal{G}_k$ *be defined by* (6) *and* (10) *with* $B_{nk} \in \mathcal{L}(X, X)$ $(n, k \in \mathbf{N})$.

Suppose the conditions (8) *and* (13) *hold. If* $\zeta = (z_n) \in \ell_X$, *where* ζ *is given by* (17), *and if* $\alpha > \frac{1}{2}$, *then* $\chi = (x_k) \in \ell_X$ *and* $\sum z_n = \sum x_k$.

3. MERCER'S THEOREMS FOR GENERALIZED RIESZ AND EULER–KNOPP METHODS

For any generalized summability method $\mathcal{B} = (B_{nk})$ with $B_{nk} \in \mathcal{L}(X, X)$, given by sequence-to-sequence transformation, there exists a corresponding method $\bar{\mathcal{B}} = (\bar{B}_{nk})$, given by series-to-series transformation, where

$$\bar{B}_{nk} = \sum_{\nu=k}^{n} \bar{\Delta} B_{n\nu}, \quad \bar{\Delta} B_{n\nu} = B_{n\nu} - B_{n-1,\nu} \quad (n, k \in \mathbf{N}). \tag{27}$$

These formulas, meant for triangular matrices, are completely analogous to the well-known formulas of scalar matrix methods (see, e.g., [9,10]).

In what follows the elements with negative indexes are everywhere taken to be zero.

Let $\Re = (\Re, P_n) = (R_{nk})$ be the generalized Riesz method given by sequence-to-sequence transformation and specified in [7,8] by

$$R_{nk} = \begin{cases} \mathcal{R}_n P_k & (k = 0, 1, \dots, n), \\ \theta & (k > n), \end{cases} \tag{28}$$

with $\mathcal{R}_n, P_k \in \mathcal{L}(X, X)$ and

$$\mathcal{R}_n \sum_{k=0}^{n} P_k x = x \qquad (x \in X; n \in \mathbf{N}). \tag{29}$$

Relying on (27)–(29), we can easily check that the corresponding method $\bar{\Re} = (\bar{\Re}, P_k) = (\bar{R}_{nk})$ is determined by

$$\bar{R}_{nk} = \begin{cases} (\mathcal{R}_{n-1} - \mathcal{R}_n) \sum\limits_{\nu=0}^{k-1} P_\nu & (k = 0, 1, \dots, n), \\ \theta & (k > n). \end{cases} \tag{30}$$

Now, using (9), (10), and the results of Theorem B with Remarks B.1, B.2, we prove that the operators $\mathcal{G}_k$ connected with $\bar{\Re} : \ \ell_X \to \ell_X$ satisfy the condition (13).

As for each $x \in X$, $k \in \mathbf{N}$ and because of (10)

$$\mathcal{G}_k x = \lim_m \sum_{n=k}^{m} (\Delta \mathcal{R}_{n-1}) \sum_{\nu=0}^{k-1} P_\nu x = (\mathcal{R}_{k-1} - \lim_m \mathcal{R}_m) \sum_{\nu=0}^{k-1} P_\nu x$$

and $\lim_m \mathcal{R}_m P_k x = \theta$ (see [8], Theorem 4), then $\mathcal{G}_k x = \mathcal{R}_{k-1} \sum_{\nu=0}^{k-1} P_\nu x$. Hence, (13) is true in view of (29).

Theorems 4 and 5 for the method $(\Re, P_n)$ extend Theorems 9 and 10 of [7]. Theorem 4 follows immediately from Theorem A and Corollary A.1 if we take (28) and (29) into account.

Theorem 4. *The method* $(\Re, P_n)$, *defined by* (6), (28), *and* (29) *with* $\mathcal{R}_n, P_k \in \mathcal{L}(X, X) \quad (n, k \in \mathbf{N})$, *is of* $c_X \to c_X$ *type if and only if*

$$\exists \lim_n \mathcal{R}_n x = \mathcal{R}^* x \qquad (x \in X) \tag{31}$$

and

$$\sup_{\|x_k\| \le 1} \|\mathcal{R}_n \sum_{k=0}^{n} P_k x_k\| = O(1). \tag{32}$$

The method $(\Re, P_n)$ *is regular if and only if the conditions* (31) *with* $\mathcal{R}^* = \theta$ *and* (32) *are valid.*

Theorem 5. *The method* $(\bar{\Re}, P_n)$, *defined by* (6), (29), *and* (30) *with* $\mathcal{R}_n, P_k \in \mathcal{L}(X, X) \quad (n, k \in \mathbf{N})$, *is of* $\ell_X \to \ell_X$ *type if and only if*

$$\sum_{n=k}^{\infty} \|(\Delta \mathcal{R}_{n-1}) \sum_{\nu=0}^{k-1} P_\nu x\| \le M \|x\| \qquad (x \in X; k \in \mathbf{N}), \tag{33}$$

where the constant M is independent from x *and* k.

The last method is absolutely regular and $\|\bar{\Re}\| = 1$.

Theorem 5 can be obtained as a direct application of Theorem B and Corollary B.1.

Next, with the help of various results of generalized summability methods, treated above, we obtain Mercer's theorems for generalized Riesz methods.

Theorem 6. *Let the method* $\Re = (\Re, P_n)$, *defined by* (6), (28), *and* (29) *with* $\mathcal{R}_n, P_k \in \mathcal{L}(X, X) \quad (n, k \in \mathbf{N})$, *satisfy the conditions* (31) *and* (32).

If $\zeta = (z_n) \in c_X$ *is given by* (17) *with* $\mathcal{B} = (\Re, P_n)$ *and if*

(a) $|\frac{1-\alpha}{\alpha}| < \|\Re\|^{-1}$ *for the case* $\|\Re\| > 1$

or if

(b) $\alpha > \frac{1}{2}$ *for the case* $\|\Re\| = 1$,

then $\chi \in c_X$;

(c) *and if, in addition to the assumptions of case* (b), *we suppose that* (31) *holds with* $\mathcal{R}^* = \theta$, *then* $\lim_k x_k = \lim_n z_n$.

Proof. The assertion $\Re : c_X \to c_X$ follows immediately from Theorem 4 because the validity of (31) and (32) is assumed. As (29) holds for each $(\Re, P_n)$, then, relying on Theorem 3 of [8], we have to discuss only two possible cases: $\|\Re\| > 1$ and $\|\Re\| = 1$.

Both statements (a) and (b) follow from Theorem 1 in view of Remark 1.1. By applying Theorems 2 and 4 we get the statement of case (c). This completes the proof.

Immediately from Theorems 3 and 5 we can infer Theorem 7.

Theorem 7. *Let for the method* $\bar{\mathfrak{R}} = (\bar{\mathfrak{R}}, P_n)$*, defined by* (6), (29), *and* (30) *with* $\mathcal{R}_n, P_k \in \mathcal{L}(X,X) \quad (n,k \in \mathbf{N})$*, the condition* (33) *hold.*

If $\zeta = (z_n) \in \ell_X$ *is given by* (17) *with* $\mathcal{B} = (\bar{\mathfrak{R}}, P_n)$ *and if* $\alpha > \frac{1}{2}$*, then* $\chi = (x_k) \in \ell_X$ *and* $\sum z_n = \sum x_k$.

The generalized Euler–Knopp method $\mathcal{E} = (\mathcal{E}, \Lambda) = (E_{nk})$, given by sequence-to-sequence transformation, is specified in [7,8] by

$$E_{nk} = \begin{cases} \binom{n}{k}\Lambda^k (I-\Lambda)^{n-k} & (k = 0, 1, \ldots, n), \\ \theta & (k > n), \end{cases} \tag{34}$$

where $\Lambda \in \mathcal{L}(X,X)$ and $\Lambda^0 = I$. In [7] it was proved that the equality (25), or, in our case $\mathcal{E}_n x = x \quad (x \in X; n \in \mathbf{N})$ with

$$\mathcal{E}_n \chi = \sum_{k=0}^{n} E_{nk} x_k \qquad (\chi \in s_X, n \in \mathbf{N}), \tag{35}$$

is valid for every $(\mathcal{E}, \Lambda)$. It is also known from [7] that $\mathcal{E}$ is conservative or regular if and only if

$$\|\Lambda\| + \|I - \Lambda\| = 1 \tag{36}$$

or (36) and

$$\|I - \Lambda\| < 1 \tag{37}$$

hold, respectively.

Let $\bar{\mathcal{E}} = (\bar{\mathcal{E}}, \Lambda) = (\bar{E}_{nk})$ denote the Euler–Knopp method given by series-to-series transformation. Sometimes we use also $\bar{E}_{nk}(\Lambda)$ instead of $\bar{E}_{nk}$. Transforming the elements E_{nk} with the help of (27), we can write (exactly as it is realized for the classical method (E, λ) in [9]) the elements $\bar{E}_{nk}$ in the following form:

$$\bar{E}_{nk} = \begin{cases} \frac{k}{n}\binom{n}{k}\Lambda^k (I-\Lambda)^{n-k} & (k = 1, \ldots, n), \\ \delta_{n0} I & (k = 0), \\ \theta & (k > n), \end{cases} \tag{38}$$

where δ_{ij} is the Kronecker symbol.

Further we need two well-known formulas:

$$\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}, \quad \binom{n-1}{-1} = \delta_{n0} \quad (n, k \in \mathbf{N}). \tag{39}$$

Lemma 3. *The both methods* $(\bar{\mathcal{E}}, I) = (\bar{E}_{nk}(I))$ *and* $(\bar{\mathcal{E}}, \theta) = (\bar{E}_{nk}(\theta))$ *are of* $\ell_X \to \ell_X$ *type, but only* $(\bar{\mathcal{E}}, I)$ *is absolutely regular.*

The validity of these assertions follows immediately from (38).

Lemma 4. *If for* $(\bar{\mathcal{E}}, \Lambda) = (\bar{E}_{nk})$ *with* $\Lambda \in \mathcal{L}(X, X)$ *the condition* (37) *is fulfilled, then the operators* $\mathcal{G}_k : \quad X \to X \quad (k \in \mathbf{N})$ *defined by* (10) *satisfy the condition* (13), *i.e.,*

$$\mathcal{G}_k x = \sum_{n=k}^{\infty} \bar{E}_{nk} x = x \qquad (x \in X; k \in \mathbf{N}). \tag{40}$$

Proof. It is known [1,2] that for each $A \in \mathcal{L}(X, X)$, $x \in X$, and $k, m \in \mathbf{N}$ the relations

$$A^k x = A(A^{k-1}x), \; A^k(A^m x) = A^{k+m}x = A^m(A^k x) \tag{41}$$

hold. Also, (see, e.g., [1,2]) the operator $I - A$ is invertible if $\|A\| < 1$. In this case $(I - A)^{-1} \in \mathcal{L}(X, X)$ and $(I - A)^{-1} = \sum_{i=0}^{\infty} A^i$. In addition to the last two facts and by (41), there exist the inverse operators $(I - A)^{-k} \in \mathcal{L}(X, X) \quad (k \in \mathbf{N})$ for which

$$(I - A)^{-k} = \sum_{i=0}^{\infty} \binom{k+i-1}{i} A^i \qquad (k \in \mathbf{N}). \tag{42}$$

Starting from (10) and (38), changing the index of summation by $i = n - k$, and taking account of (39) and (42) with $A = I - \Lambda$ and $\|I - \Lambda\| < 1$, we get for all $x \in X$ that

$$\sum_{n=k}^{\infty} \bar{E}_{nk} x = \Lambda^k \sum_{i=0}^{\infty} \binom{k-1+i}{i} (I - \Lambda)^i x = \Lambda^k [I - (I - \Lambda)]^{-k} x = x.$$

Lemma is proved.

The next theorem which gives the necessary and sufficient condition for $\bar{\mathcal{E}} = (\bar{\mathcal{E}}, \Lambda)$ to be of $\ell_X \to \ell_X$ type is in a sense more general than the analogous result obtained in [7].

Theorem 8. *The method* $\bar{\mathcal{E}} = (\bar{\mathcal{E}}, \Lambda)$ *defined by* (6) *and* (38) *with* $\Lambda \in \mathcal{L}(X, X)$ *is of* $\ell_X \to \ell_X$ *type if and only if* (36) *holds. This method is absolutely regular if and only if* $\Lambda \neq \theta$. *At that* $\|\bar{\mathcal{E}}\| = 1$.

Proof. In view of Lemma 3 we omit the case $\Lambda = \theta$ from the following discussion.

By Theorem B we have to prove that for $\mathcal{B} = \bar{\mathcal{E}}$ the condition (8) is valid if and only if (36) holds. Starting from the left-hand side of (8), using (38), (39), and changing the indexes by $n-k=i$, in view of (42) we see that the boundedness of $\sum_{n=k}^{\infty}\|\bar{E}_{nk}x\|$ will be guaranteed if and only if (36) holds. Actually, we get for all $x\in X$:

$$\sum_{n=k}^{\infty}\|\bar{E}_{nk}x\|\le\|\Lambda\|^k\sum_{i=0}^{\infty}\binom{k-1+i}{i}\|I-\Lambda\|^i\|x\|=\left(\frac{\|\Lambda\|}{1-\|I-\Lambda\|}\right)^k\|x\|\le\|x\|,$$

where we took into account that $\frac{\|\Lambda\|}{1-\|I-\Lambda\|}\le 1\Leftrightarrow\|\Lambda\|+\|I-\Lambda\|\le 1\Leftrightarrow$ $\|\Lambda\|+\|I-\Lambda\|=1$, yielding $\|I-\Lambda\|<1$, as $\Lambda\ne\theta$.

By the final result and because of Lemma 4 the condition (40) is fulfilled. The absolute regularity of $\bar{\mathcal{E}}$ and $\|\bar{\mathcal{E}}\|=1$ follow now from Corollary B.1.

Finishing this part, we shall apply the results of Sections 2 and 3 to obtain M-theorems for $(\mathcal{E},\Lambda)$ methods.

Theorem 9. *Let the method $\mathcal{E}=(\mathcal{E},\Lambda)$ defined by* (6), (34), *and* (35) *with $\Lambda\in\mathcal{L}(X,X)$ satisfy the condition* (36) *or both* (36) *and* (37).

If $\zeta=(z_n)\in c_X$ is given by (17) *with $\mathcal{B}=\mathcal{E}$ and if $\alpha>\frac{1}{2}$, then $\chi=(x_k)\in c_X$ or the equality* $\lim_k x_k=\lim_n z_n$ *hold, respectively.*

Proof. By the conditions (36) or $\{(36),(37)\}$ and in view of Theorems 6 or 5 from [7], the method $\mathcal{E}$ is conservative or regular, respectively. As in both cases $\|\mathcal{E}\|=1$ (see [8], Corollary 3.2), then let $\alpha>\frac{1}{2}$. Hence, the provable assertions follow from Corollary 3.2 of [8] and Theorem 2.

The next result follows from Theorem 3, Theorem 8, and Lemma 4.

Theorem 10. *Let the method $\bar{\mathcal{E}}=(\bar{\mathcal{E}},\Lambda)$ be defined by* (6) *and* (38) *with $\Lambda\in\mathcal{L}(X,X)$ satisfying the condition* (36).

If $\zeta=(z_n)\in\ell_X$ is given by (17) *with $\mathcal{B}=\bar{\mathcal{E}}=(\bar{E}_{nk})$ and if $\alpha>\frac{1}{2}$, then $\chi\in\ell_X$. If, in addition to previous assumptions, we suppose that $\Lambda\ne\theta$, then $\chi\in\ell_X$ and $\sum z_n=\sum x_k$.*

4. CONCLUDING REMARKS

In 1907 Mercer [11] showed for real sequences that if $y_n=(n+1)^{-1}\sum_{k=0}^{n}x_k$ and if $\alpha x_{n+1}+(1-\alpha)y_n\to x^*$, where $n\to\infty$ and $x^*\ne\infty$, then both x_{n+1} and y_n tend to x^*, provided that $\alpha>0$.

This theorem has been extended in various directions and numerous authors have studied its different modifications (see, e.g., [12–16]). Theorems of this kind, mainly in the scalar case, are still being examined.

Numerous M-theorems, associated with classical summability methods (among them several results of the works cited above), can be inferred from our generalized M-theorems.

Let now $B = (b_{nk})$ be a triangular matrix method with $b_{nk} \in \mathbf{K} \quad (n, k \in \mathbf{N})$. We can treat this method also in an operator form. To this end, instead of B, we use the method $\mathcal{B} = (\mathcal{B}_{nk})$ with

$$\mathcal{B}_{nk} = b_{nk} I, \ \mathcal{B}_n \chi = \sum_{k=0}^{n} b_{nk} I x_k, \ \mathcal{B}\chi = (\mathcal{B}_n \chi) \quad (n, k \in \mathbf{N}), \tag{43}$$

where $\chi = (x_k) \in s_X$. For this special case of the general method $\mathcal{B}$ the following formulas hold (see also [8], Summaries I, II):

$$\|\mathcal{B}\| = \sup_n \|\mathcal{B}_n\|, \ \|\mathcal{B}_n\| = \sum_{k=0}^{n} |b_{nk}| \quad (n \in \mathbf{N}) \tag{44}$$

for a general instance of $B = (b_{nk})$ and

$$\|\mathcal{B}\| = 1, \ \|\mathcal{B}_n\| = 1 \qquad (n \in \mathbf{N}) \tag{45}$$

for non-negative methods satisfying the condition

$$\sum_{k=0}^{n} b_{nk} = 1 \qquad (n \in \mathbf{N}). \tag{46}$$

In the case $X = \mathbf{K}$ the notations s, m, c, and ℓ will be used instead of s_X, m_X, c_X, and ℓ_X.

Employing (43)–(45), from Theorem 1 and Lemma 1 we can infer an M-theorem for $B = (b_{nk})$ and for $\zeta = (z_n) \in c_X$ with

$$z_n = \alpha x_n + (1 - \alpha) \sum_{k=0}^{n} b_{nk} x_k \qquad (n \in \mathbf{N}). \tag{47}$$

The result will be analogical to that obtained in [15].

For all non-negative and regular methods satisfying (46) we can deduce some M-theorems which occur in [15,16].

We denote, further, by $\mathcal{R} = (\mathcal{R}, p_n) = (r_{nk})$ and $E = E_\lambda = (E, q) = (e_{nk})$ with $p_n \in \mathbf{K}$, $q = \lambda^{-1} - 1$, and $\lambda \in \mathbf{R}$ the classical Riesz and Euler–Knopp methods, respectively. As we know, these methods, given by sequence-to-sequence transformation, and the methods $\bar{\mathcal{R}} = (\bar{r}_{nk})$ and $\bar{E} = (\bar{e}_{nk})$, given by series-to-series transformation, are defined by

$$r_{nk} = \mathcal{P}_n^{-1} p_k, \ \mathcal{P}_n = \sum_{k=0}^{n} p_k \neq 0 \ \text{ and } \ e_{nk} = \tbinom{n}{k} \lambda^k (1 - \lambda)^{n-k}, \tag{48}$$

and

$$\bar{r}_{nk} = \frac{\mathcal{P}_{k-1}p_k}{\mathcal{P}_{n-1}\mathcal{P}_n} \qquad \text{and} \qquad \bar{e}_{nk} = \frac{k}{n}e_{nk}, \tag{49}$$

respectively. At that (46) is true for both methods E and $\mathcal{R}$ (about E see [7–9]; for $\mathcal{R}$ it is clear) and therefore every regular method E is non-negative (see [8,9]).

The M-theorem for such $(\mathcal{R}, p_n)$ was first proved by Okada [16]. As a supplement to the last remark, let us observe the M-theorems in the classical form and for E_λ. Recall (see, e.g., [9,10]) that $\mathcal{E}_\lambda :\ c \to c$ or E_λ is regular if and only if $0 \le \lambda \le 1$ or $0 < \lambda \le 1$, respectively. Even more, $\|E\| = 1$ in view of (45) and (46).

The next result follows immediately from Theorem 9.

Corollary 9.1. *Let $\zeta = (z_n)$ be given by* (47)*, with $b_{nk} = e_{nk}$ defined by* (48)*, and let $0 \le \lambda \le 1$ or $0 < \lambda \le 1$.*

If $\zeta \in c$ and if $\alpha > \frac{1}{2}$, then $\chi = (x_k) \in c$ or $\lim_k x_k = \lim_n z_n$ hold, respectively.

An analogue of Mercer's original theorem for the case of absolute summability was first proved by Bosanquet [12] and later in a generalized form by Walsh in 1942. Afterwards M-theorems of $\ell \to \ell$ type were proved for different summability methods of scalars. For the case $(\mathcal{R}, p_n)$ such a theorem was proved by Hayashi [13]. For general triangular scalar methods $B = (b_{nk})$ this was done by Love [15], but Parameswaran proved in 1957 for this case an analogue of M-theorems studied by Agnew in 1954. Several of the mentioned results can be inferred from our M-theorems. For instance, the next Corollaries 7.1 and 10.1 follow immediately from Theorem 7 and Theorem 10, respectively.

Corollary 7.1. *Suppose that for $\bar{\mathcal{R}} = (\bar{r}_{nk})$ defined by* (49) *the condition $\sum_{n=k}^{\infty} |\bar{r}_{nk}| = O(1)$ holds and let $\zeta = (z_n)$ be given by* (47) *with $b_{nk} = \bar{r}_{nk}$.*

If $\zeta \in \ell$ and if $\alpha > \frac{1}{2}$, then $\chi = (x_k) \in \ell$ and $\sum x_k = \sum z_n$.

Corollary 10.1. *Suppose that for $\bar{E}_\lambda = (\bar{e}_{nk})$ defined by* (49) *the condition $0 \le \lambda \le 1$ holds and let $\zeta = (z_n)$ be given by* (47) *with $b_{nk} = \bar{e}_{nk}$.*

If $\zeta \in \ell$ and if $\alpha > \frac{1}{2}$, then $\chi = (x_k) \in \ell$. If, additionally to the previous propositions $\lambda \ne 0$, then $\sum x_k = \sum z_n$.

Note. In some cases our conditions about α differ from the corresponding conditions in the results of papers cited above. This is mainly caused by different methods used for investigations of that kind.

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MERCERI TEOREEMID SEOSES ÜLDISTATUD SUMMEERIMISMENETLUSTEGA BANACHI RUUMIDES

Tamara SÕRMUS

On üldistatud klassikalisest summeeruvusteooriast tuntud Merceri teoreemid (M-teoreemid) üldistatud summeerimismenetlustele $\mathcal{B} = (B_{nk})$ ja jadaruumidele Banachi ruumides X. Kõik operaatorid $B_{nk} : X \to X$ on pidevad ja lineaarsed. On tõestatud seitse M-teoreemi üldistatud kolmnurksete menetluste ja üldistatud Euleri–Knoppi ning Rieszi menetluste kohta, mis on koonduvust või absoluutset koonduvust säilitavad. Töö tulemusi on rakendatud arvmaatriksitega määratud üldiste või klassikaliste menetluste puhul.