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# GENERAL FORM OF $\beta$-MATRICES OF THE FIRST-ORDER WAVE EQUATIONS IN 16-DIMENSIONAL REPRESENTATION

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**Abstract.** General expressions of $\beta$-matrices of the first-order wave equations in the 16-dimensional representation in the direct-product (DP), Gelfand (G), and Kemmer–Duffin–Petiau (KDP) bases are computed. In the general case, 16 arbitrary parameters arise. Depending on these parameters, the $\beta$-matrices satisfy the KDP, the Dirac, and some new algebras and the corresponding equations may describe both the bosons and fermions. It appears that the reduction of the 16-dimensional representation to the direct sum of the 1-, 5-, and 10-dimensional irreducible representations is possible only in a special case, and in general, $\det\beta \neq 0$. Unitary transformations connecting the quantities of the DP basis with those of G and KDP bases, are expressed. Finally, the general expressions of Hermitianizing matrices in these bases are given.

**Key words:** first-order wave equations, 16-dimensional representation, Kemmer–Duffin–Petiau algebra, Dirac algebra, Hermitianizing matrices.

## 1. INTRODUCTION

The relativistic theory of free particles of finite mass and noninfinite spin has been well investigated [1–4] but there arise problems concerning the interactions between the particles. Usually, the system of the first-order equations is considered, because every higher-order equation is reducible to the system of the first-order equations. The well-known examples are the Dirac and Kemmer–Duffin–Petiau (KDP) equations [1–5]. All these equations contain some $N \times N$ matrices which determine the spin-state structure of the model. Here we are interested in the KDP-like equations, especially the case $N = 16$. We restrict ourselves to the theory of free particles, but some general expressions are computed also for the further investigations of models with interactions. Although several works give a general theory of the above-mentioned matrices [1,4], most of the authors confine themselves only to special cases [6–9], and possibly, some important features may have been lost. Below we present only the basic elements of the Lorentz-invariant wave equations. In the next three sections we find the general expressions of the above-mentioned matrices with respect to the three different bases: direct-product (DP), Gelfand (G), and KDP ones. The last section is dedicated to Hermitianizing matrices in these bases. Here we are not interested in the physical meaning of these results; this is the subject of the next papers.

It is well known [1–4] that the system of equations for a free field of arbitrary spin

$$\left(i\beta^{\mu}\partial_{\mu} - m\right)\psi(x) = 0 \tag{1}$$

(where $\beta_{\mu}$ is a set of four $N \times N$-dimensional matrices, independent of $x$) is invariant under the homogeneous Lorentz group if

$$T(\Lambda) : \psi(x) \to \psi^{'}(x^{'}) = T(\Lambda)\psi(x)\,,$$

$$T^{-1}(\Lambda)\beta_{\mu}T(\Lambda) = \Lambda_{\mu\nu}\beta^{\nu} \quad \Leftrightarrow \quad [\beta_{\mu,}, S_{\rho\sigma}] = i\left(\eta_{\mu\rho}\beta_{\sigma} - \eta_{\mu\sigma}\beta_{\rho}\right). \tag{2}$$

Here the transformation $T$ stands for the finite-dimensional representation of the homogeneous Lorentz group:

$$T : SO_{1,3} \ni \Lambda(\omega) \to T(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right).$$

Denoting $R_i = -\frac{1}{2}\varepsilon_{ijk}S^{jk}$, $S_i = S^{0i}$ ($i, j, k = 1, 2, 3$), one gets

$$\begin{aligned} &[[\beta_0,\ S_3], S_3] + \beta_0 = 0, \\ &[\beta_0,\ R_i] = 0 \end{aligned} \tag{3a}$$

and

$$\beta_i = -i[\beta_0, S_i]. \tag{3b}$$

Therefore, it is sufficient to find only $\beta_0$, other $\beta_i$ are derivable using boost-transformations $S_i$. So, for the well-known Dirac equation (the representation $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$) the $\gamma_\mu$ matrices satisfy

$$S_{\rho\sigma}=\frac{1}{4}(\gamma_\rho\gamma_\sigma-\gamma_\sigma\gamma_\rho)\,,\quad \gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu}\,, \tag{4}$$

and the field $\psi$ describes a spin-1/2 particle. The 16-dimensional KDP representation with the properties

$$S_{\rho\sigma}=\beta_\rho\beta_\sigma-\beta_\sigma\beta_\rho\,,\qquad \beta_\rho\beta_\mu\beta_\sigma+\beta_\sigma\beta_\mu\beta_\rho=g_{\mu\rho}\beta_\sigma+g_{\mu\sigma}\beta_\rho \tag{5}$$

consists of three irreducible fields, the first of which is trivial (one-dimensional), while the other two are spin 0 (five-dimensional) and spin 1 (ten-dimensional) fields. Attempts have been made to describe both, bosons and fermions, using the KDP equations [6–9], but unfortunately, using only the special cases of the $\beta$-matrices. Although the KDP algebra (5) is possible in the case of an arbitrary dimension of space-time [5], here we restrict ourselves to the common dimension 4.

## 2. DIRECT-PRODUCT BASIS

In the *direct-product* (DP) *basis* $((\frac{1}{2},0)\oplus(0,\frac{1}{2}))\otimes((\frac{1}{2},0)\oplus(0,\frac{1}{2}))$ all matrices may be expressed via the direct products of the Dirac $\gamma$-matrices, so the Lorentz generators are

$$S^{\mu\nu}=\frac{1}{2}(\gamma^\mu\gamma^\nu\otimes I+I\otimes\gamma^\mu\gamma^\nu) \tag{6}$$

and the common choise of $\beta$-matrices is

$$\beta^\mu=\frac{1}{2}(\gamma^\mu\otimes I+I\otimes\gamma^\mu)\,. \tag{7}$$

The DP basis is most favoured in the physical literature, but the spin states are mixed up here. In another basis (KDP), the choise (7) describes two particles with same mass *m*, but with a different spin, 0 or 1.

Direct calculations give that the most general expression of $\beta^0$ which satisfies the relativistic invariance conditions (3a) in this basis is

$$\beta^{0}{}_{DP} =$$

$$\begin{bmatrix}
0 & 0 & z_1 & 0 & 0 & 0 & 0 & 0 & w_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & z_2 & 0 & 0 & z_1-z_2 & 0 & 0 & w_2 & 0 & 0 & w_1-w_2 & 0 & 0 & 0 \\
z_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_5 & 0 & 0 & 0 & 0 & 0 \\
0 & z_4 & 0 & 0 & z_3-z_4 & 0 & 0 & 0 & 0 & 0 & 0 & w_6 & 0 & 0 & w_5-w_6 & 0 \\
0 & 0 & 0 & z_1-z_2 & 0 & 0 & z_2 & 0 & 0 & w_1-w_2 & 0 & 0 & w_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & z_1 & 0 & 0 & 0 & 0 & 0 & w_1 & 0 & 0 \\
0 & z_3-z_4 & 0 & 0 & z_4 & 0 & 0 & 0 & 0 & 0 & 0 & w_5-w_6 & 0 & 0 & w_6 & 0 \\
0 & 0 & 0 & 0 & 0 & z_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_5 \\
w_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_5 & 0 & 0 & 0 & 0 & 0 \\
0 & w_4 & 0 & 0 & w_3-w_4 & 0 & 0 & 0 & 0 & 0 & 0 & z_6 & 0 & 0 & z_5-z_6 & 0 \\
0 & 0 & w_7 & 0 & 0 & 0 & 0 & 0 & z_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & w_8 & 0 & 0 & w_7-w_8 & 0 & 0 & z_8 & 0 & 0 & z_7-z_8 & 0 & 0 & 0 \\
0 & w_3-w_4 & 0 & 0 & w_4 & 0 & 0 & 0 & 0 & 0 & 0 & z_5-z_6 & 0 & 0 & z_6 & 0 \\
0 & 0 & 0 & 0 & 0 & w_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_5 \\
0 & 0 & 0 & w_7-w_8 & 0 & 0 & w_8 & 0 & 0 & z_7-z_8 & 0 & 0 & z_8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & w_7 & 0 & 0 & 0 & 0 & 0 & z_7 & 0 & 0
\end{bmatrix}, \tag{8}$$

where $z_1,\dots, z_8, w_1,\dots, w_8$ are arbitrary 16 parameters. The determinant of this matrix is

$$\begin{aligned}\det \beta^0_{DP} = (z_3 z_5 - w_3 w_5)^3 (w_1 w_7 - z_1 z_7)^3 (z_3 z_5 - w_3 w_5 - 2 z_4 z_5 + 2 w_4 w_5 \\ +4 z_4 z_6 - 4 w_4 w_6 - 2 z_3 z_6 + 2 w_3 w_6)(w_1 w_7 - z_1 z_7 - 2 w_1 w_8 \\ +2 z_1 z_8 + 4 w_2 w_8 - 4 z_2 z_8 - 2 w_2 w_7 + 2 z_2 z_7).\end{aligned} \tag{9}$$

In the common case (7) $\det\beta = 0$ and there are no inverse matrices for $\beta$'s. But in the general case it is possible that $\det\beta \neq 0$.

Since every $4\times 4$-matrix may be decomposed with respect to the Dirac-basis $\Gamma_s = \{ I, \gamma_\mu, \gamma_5, \gamma_5\gamma_\mu, \sigma_{\mu\nu} \}$, it is possible to present general $\beta_\mu$ as in (7):

$$\beta_\mu = \beta_\mu{}^1 + \beta_\mu{}^2 ,$$

where

$$\begin{aligned}\beta_\mu{}^1 &= a_1(\gamma_\mu \otimes I) + b_1(\gamma_5\gamma_\mu \otimes \gamma_5) + c_1\varepsilon_{\mu\nu\alpha\beta}(\gamma^\nu \otimes \sigma^{\alpha\beta}) + d_1(\gamma^\nu \otimes \sigma_{\mu\nu}) \\ &\quad + l_1(\gamma_5\gamma_\mu \otimes I) + f_1(\gamma_\mu \otimes \gamma_5) + k_1\varepsilon_{\mu\nu\alpha\beta}(\gamma_5\gamma^\nu \otimes \sigma^{\alpha\beta}) + e_1(\gamma_5\gamma^\nu \otimes \sigma_{\mu\nu}), \\ \beta_\mu{}^2 &= a_2(I \otimes \gamma_\mu) + b_2(\gamma_5 \otimes \gamma_5\gamma_\mu) + c_2\varepsilon_{\mu\nu\alpha\beta}(\sigma^{\alpha\beta} \otimes \gamma^\nu) + d_2(\sigma_{\mu\nu} \otimes \gamma^\nu) \\ &\quad + l_2(I \otimes \gamma_5\gamma_\mu) + f_2(\gamma_5 \otimes \gamma_\mu) + k_2\varepsilon_{\mu\nu\alpha\beta}(\sigma^{\alpha\beta} \otimes \gamma_5\gamma^\nu) + e_2(\sigma_{\mu\nu} \otimes \gamma_5\gamma^\nu).\end{aligned} \tag{10}$$

Here $a_1, a_2, \dots, e_1, e_2$ are arbitrary 16 constants,

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right],$$

and $\varepsilon_{\mu\nu\alpha\beta}$ is a completely antisymmetric unit tensor.

We use the spinor representation

$$\gamma^0 = \sigma^1 \otimes I = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \gamma^k = -i\sigma^2 \otimes \sigma^k = \begin{bmatrix} 0 & -\sigma^k \\ \sigma^k & 0 \end{bmatrix},$$

where $I$ is $2\times 2$ unit matrix and $\sigma^k$ are the Pauli matrices.

The relations between the two systems of parameters, (8) and (10), are

$$\begin{aligned}
z_1 &= (a_2 + b_2 - c_2 - id_2) + (l_2 + f_2 - k_2 - ie_2), \\
z_2 &= (a_2 + b_2 + c_2 + id_2) + (l_2 + f_2 + k_2 + ie_2), \\
z_3 &= (a_2 - b_2 + c_2 + id_2) - (l_2 - f_2 + k_2 + ie_2), \\
z_4 &= (a_2 - b_2 - c_2 - id_2) - (l_2 - f_2 - k_2 - ie_2), \\
z_5 &= (a_2 - b_2 - c_2 + id_2) + (l_2 - f_2 - k_2 + ie_2), \\
z_6 &= (a_2 - b_2 + c_2 - id_2) + (l_2 - f_2 + k_2 - ie_2), \\
z_7 &= (a_2 + b_2 + c_2 - id_2) - (l_2 + f_2 + k_2 - ie_2), \\
z_8 &= (a_2 + b_2 - c_2 + id_2) - (l_2 + f_2 - k_2 + ie_2);
\end{aligned} \tag{11a}$$

$$\begin{aligned}
w_1 &= (a_1 + b_1 - c_1 - id_1) + (l_1 + f_1 - k_1 - ie_1), \\
w_2 &= (a_1 + b_1 + c_1 + id_1) + (l_1 + f_1 + k_1 + ie_1), \\
w_3 &= (a_1 - b_1 + c_1 + id_1) - (l_1 - f_1 + k_1 + ie_1), \\
w_4 &= (a_1 - b_1 - c_1 - id_1) - (l_1 - f_1 - k_1 - ie_1), \\
w_5 &= (a_1 - b_1 - c_1 + id_1) + (l_1 - f_1 - k_1 + ie_1), \\
w_6 &= (a_1 - b_1 + c_1 - id_1) + (l_1 - f_1 + k_1 - ie_1), \\
w_7 &= (a_1 + b_1 + c_1 - id_1) - (l_1 + f_1 + k_1 - ie_1), \\
w_8 &= (a_1 + b_1 - c_1 + id_1) - (l_1 + f_1 - k_1 + ie_1).
\end{aligned} \tag{11b}$$

The members with coefficients $a_i, b_i, c_i, d_i$ correspond to a vector, the members with $l_i, f_i, k_i, e_i$ to a pseudovector. On the other hand, if $\beta^\mu$ depends on $a_i, d_i, f_i, k_i$, then $\beta^0$ is Hermitian ($\beta^k$ is anti-Hermitian); if $\beta^\mu$ depends on $l_i, e_i, b_i, c_i$, then $\beta^0$ is anti-Hermitian ($\beta^k$ is Hermitian). In the special, symmetrical case $a_1 = a_2 \equiv a, \ldots, e_1 = e_2 \equiv e$ and therefore $w_1 = z_1, \ldots, w_8 = z_8$. By choosing the subcase $a = \frac{1}{2}, b = c = d = l = f = k = e = 0$, one gets the $\beta^\mu$ as in (7). Generally, the KDP algebra (5) is satisfied by $\beta^\mu(a, b, l, f)$, for which

$$\begin{aligned}
a^2 + f^2 - b^2 - l^2 &= \frac{1}{4}, \\
af - bl &= 0.
\end{aligned} \tag{12}$$

The nonsymmetrical parts of $\beta$-matrices $\beta(a_k, b_k, l_k, f_k)$, ($k = 1$ or $2$), satisfy the Dirac algebra (4) if

$$a_k^2 + f_k^2 - b_k^2 - l_k^2 = 1,$$
$$a_k f_k - b_k l_k = 0. \tag{13}$$

For the same expressions of $\beta(a_k, b_k, l_k, f_k)$, ($k = 1$ or $2$), $\beta$-matrices satisfy the *new algebra* [10]

$$\beta_\rho(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu)\beta_\sigma = 2g_{\mu\nu}\beta_\rho\beta_\sigma, \tag{14}$$

but in this case the specific condition

$$a_k^2 + f_k^2 - b_k^2 - l_k^2 = 0,$$
$$a_k f_k - b_k l_k = 0 \tag{15}$$

is valid. It is clear that the constraints of the new algebra (14) are weaker than those of the Dirac algebra.

## 3. GELFAND BASIS

This basis is used by Gelfand et al. [4] for building the general theory of the first-order relativistically invariant equations. The spin states are separated clearly in this basis. Here we use the *Gelfand basis* (G basis) in the following ordering:

$$(0, 0; 0) \oplus (0, 0; 0) \oplus (\tfrac{1}{2}, \tfrac{1}{2}; 0) \oplus (\tfrac{1}{2}, \tfrac{1}{2}; 0) \oplus (1, 0; 1) \oplus (0, 1; 1) \oplus (\tfrac{1}{2}, \tfrac{1}{2}; 1) \oplus (\tfrac{1}{2}, \tfrac{1}{2}; 1),$$

where $(k, l; j)$ denotes the reduction of the irreducible representation $(k, l)$ of $SO_{1,3}$ to the irreducible representation of $SO_3$. In this basis

$$S_j = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i(K_jV_0) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i(K_jV_0) \\ 0 & 0 & 0 & 0 & -im^j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -im^j & 0 & 0 \\ 0 & 0 & -i(K_jV_0)^+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i(K_jV_0)^+ & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{16}$$

$$R_j = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m^j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m^j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m^j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m^j \end{bmatrix}. \tag{17}$$

Here

$$m^1 = -\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, m^2 = -\frac{i}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, m^3 = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{18}$$

are the generators of the representation $D^1$ of the rotation group $SO_3$, and

$$K_1 = (1,0,0),\ K_2 = (0,1,0),\ K_3 = (0,0,1) \text{ and } V_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 \\ i & 0 & i \\ 0 & -\sqrt{2} & 0 \end{bmatrix} \tag{19}$$

are the Hurley matrices [11].

Now the parameters $y_i$ of $\beta_0$, corresponding to spin 0, and the parameters $x_k$, corresponding to spin 1, are separated and embedded into a $4 \times 4$-block and a $12 \times 12$-block, respectively:

$$\beta_G^0 = \begin{bmatrix} 0 & 0 & y_5 & y_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_7 & y_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_1 & y_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_3 & y_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & 0 & 0 & x_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & 0 & 0 & x_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & 0 & 0 & x_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_7 & 0 & 0 & x_8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_7 & 0 & 0 & x_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_7 & 0 & 0 & x_8 \\ 0 & 0 & 0 & 0 & x_1 & 0 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3 & 0 & 0 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_3 & 0 & 0 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_3 & 0 & 0 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{20}$$

The determinant is

$$\det \beta_G^0 = (x_1 x_4 - x_2 x_3)^3 (x_6 x_7 - x_5 x_8)^3 (y_1 y_4 - y_2 y_3)(y_6 y_7 - y_5 y_8). \tag{21}$$

The unitary transformation $U$, which connects the quantities of the DP basis with those of the G basis $R_G = U R_{DP} U^+$, has the form

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & p & 0 & 0 & -p & 0 & 0 & 0 & 0 & 0 & 0 & -r & 0 & 0 & r & 0 \\ 0 & r & 0 & 0 & -r & 0 & 0 & 0 & 0 & 0 & 0 & p & 0 & 0 & -p & 0 \\ 0 & 0 & 0 & s & 0 & 0 & -s & 0 & 0 & -q & 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & -q & 0 & 0 & s & 0 & 0 & -s & 0 & 0 & 0 \\ \sqrt{2}m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2}m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n & 0 & 0 & n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}n \\ 0 & 0 & \sqrt{2}s & 0 & 0 & 0 & 0 & 0 & \sqrt{2}q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s & 0 & 0 & s & 0 & 0 & q & 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}s & 0 & 0 & 0 & 0 & 0 & \sqrt{2}q & 0 & 0 \\ 0 & 0 & \sqrt{2}q & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}s & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & q & 0 & 0 & -s & 0 & 0 & -s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}q & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}s & 0 & 0 \end{bmatrix}. \tag{22}$$

Direct calculations give $\det U = -m^3 n^3 (s^2 + q^2)^4 (p^2 + r^2)$ and from the unitarity $U^+ U = I$ we conclude that $m^2 = n^2 = 1$, $s^2 + q^2 = 1$, $p^2 + r^2 = 1$. It is possible to take $m = 1$, $n = -1$, and $s = \cos\theta$, $q = \sin\theta$, $p = \cos\varphi$, $r = \sin\varphi$, so that there are only two independent parameters $\theta$ and $\varphi$. The connections between the parameters of the G basis and the DP basis can be easily written as follows:

$$\begin{aligned} x_1 &= m(qw_3 + sz_3), \\ x_2 &= n(qz_5 + sw_5), \\ x_3 &= m(qz_3 - sw_3), \\ x_4 &= n(qw_5 - sz_5), \\ x_5 &= m(qw_1 + sz_1), \\ x_6 &= m(qz_1 - sw_1), \\ x_7 &= n(qz_7 + sw_7), \\ x_8 &= n(qw_7 - sz_7), \\ y_1 &= ps(2z_4 - z_3) - pq(2w_4 - w_3) - rs(2w_6 - w_5) + rq(2z_6 - z_5), \\ y_2 &= ps(2w_6 - w_5) - pq(2z_6 - z_5) + rs(2z_4 - z_3) - rq(2w_4 - w_3), \\ y_3 &= ps(2w_4 - w_3) + pq(2z_4 - z_3) - rs(2z_6 - z_5) - rq(2w_6 - w_5), \\ y_4 &= ps(2z_6 - z_5) + pq(2w_6 - w_5) + rs(2w_4 - w_3) + rq(2z_4 - z_3), \\ y_5 &= ps(2z_2 - z_1) - pq(2w_2 - w_1) - rs(2w_8 - w_7) + rq(2z_8 - z_7), \\ y_6 &= ps(2w_2 - w_1) + pq(2z_2 - z_1) - rs(2z_8 - z_7) - rq(2w_8 - w_7), \\ y_7 &= ps(2w_8 - w_7) - pq(2z_8 - z_7) + rs(2z_2 - z_1) - rq(2w_2 - w_1), \\ y_8 &= ps(2z_8 - z_7) + pq(2w_8 - w_7) + rs(2w_2 - w_1) + rq(2z_2 - z_1). \end{aligned} \tag{23}$$

The $SO_3$-invariant $R^2 = R_1^2 + R_2^2 + R_3^2$ is not diagonal in the DP basis, but is diagonal in the G basis, having on the main diagonal 4 values 0 and 12 values $j(j+1) = 2$ corresponding to the spin 0 and spin 1 parts, respectively.

## 4. KEMMER–DUFFIN–PETIAU BASIS

In many physical applications [7–9] the *KDP basis* is used, ordered as

$$(0,0) \oplus ((\tfrac{1}{2},\tfrac{1}{2}) \oplus (0,0)) \oplus ((1,0) \oplus (0,1) \oplus (\tfrac{1}{2},\tfrac{1}{2})).$$

Thus, the 16-dimensional $\beta$-matrices reduce to the direct sum of 1-, 5-, and 10-dimensional $\beta$-matrices in the common case (7). Note that in this basis the group-theoretical properties of Eqs. (1) turn to be explicit and the separation of different physical states takes place. In the KDP basis the Lorentz generators have the form

$$S_j = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & iK_j & 0 & 0 & 0 & 0 & 0 \\ 0 & iK_j^+ & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -if^j & 0 & 0 \\ 0 & 0 & 0 & 0 & if^j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & iK_j^+ \\ 0 & 0 & 0 & 0 & 0 & 0 & iK_j & 0 \end{bmatrix}, \tag{24}$$

$$R_j = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -if^j & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -if^j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -if^j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -if^j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -if^j \end{bmatrix}, \tag{25}$$

where $\left[f^i, f^j\right] = -\varepsilon_k^{ij} f^k$, $f^k = iV_0 m^k V_0^+$, and $V_0, K_j, m^k$ are given in (18) and (19).

The unitary transformation, which connects the quantities of the DP basis with those of the KDP basis $R_{KDP} = VR_{DP}V^{+}$, is

$$V = \frac{1}{2}\begin{bmatrix} 0 & v & 0 & 0 & -v & 0 & 0 & 0 & 0 & 0 & 0 & -v & 0 & 0 & v & 0 \\ 0 & 0 & 0 & it & 0 & 0 & -it & 0 & 0 & it & 0 & 0 & -it & 0 & 0 & 0 \\ 0 & 0 & -it & 0 & 0 & 0 & 0 & it & it & 0 & 0 & 0 & 0 & -it & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & t & -t & 0 & 0 & 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & it & 0 & 0 & it & 0 & 0 & -it & 0 & 0 & -it & 0 & 0 & 0 \\ 0 & t & 0 & 0 & -t & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & -t & 0 \\ u & 0 & 0 & 0 & 0 & -u & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & -u \\ iu & 0 & 0 & 0 & 0 & iu & 0 & 0 & 0 & 0 & iu & 0 & 0 & 0 & 0 & iu \\ 0 & -u & 0 & 0 & -u & 0 & 0 & 0 & 0 & 0 & 0 & -u & 0 & 0 & -u & 0 \\ -iu & 0 & 0 & 0 & 0 & iu & 0 & 0 & 0 & 0 & iu & 0 & 0 & 0 & 0 & -iu \\ u & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & -u & 0 & 0 & -u \\ 0 & iu & 0 & 0 & iu & 0 & 0 & 0 & 0 & 0 & 0 & -iu & 0 & 0 & -iu & 0 \\ 0 & 0 & -iu & 0 & 0 & 0 & 0 & iu & -iu & 0 & 0 & 0 & 0 & iu & 0 & 0 \\ 0 & 0 & u & 0 & 0 & 0 & 0 & u & u & 0 & 0 & 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & iu & 0 & 0 & iu & 0 & 0 & iu & 0 & 0 & iu & 0 & 0 & 0 \\ 0 & 0 & 0 & iu & 0 & 0 & -iu & 0 & 0 & -iu & 0 & 0 & iu & 0 & 0 & 0 \end{bmatrix}, \tag{26}$$

where $\det V = iu^{10}t^{5}v$ and from $V^{+}V = I$ it follows that $u^{2} = t^{2} = v^{2} = 1$ (It is possible to take $u = t = v = 1$).

The most general $\beta^{0}$ which satisfies the relativistic invariance conditions (3a) is in the KDP basis

$$\beta^{0}{}_{KDP} = \begin{bmatrix} 0 & \xi_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_2 \\ \xi_3 & 0 & 0 & 0 & 0 & \xi_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \xi_5 & 0 & 0 & \xi_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_5 & 0 & 0 & \xi_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_5 & 0 & 0 & \xi_6 & 0 & 0 & 0 & 0 \\ 0 & \xi_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_8 \\ 0 & 0 & \xi_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_{10} & 0 \\ 0 & 0 & \xi_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \xi_{13} & 0 & 0 & \xi_{14} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_{13} & 0 & 0 & \xi_{14} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_{13} & 0 & 0 & \xi_{14} & 0 & 0 & 0 & 0 \\ \xi_{15} & 0 & 0 & 0 & 0 & \xi_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{27}$$

The determinant is

$$\det \beta^0_{KDP} = -(\xi_9\xi_{12} - \xi_{10}\xi_{11})^3(\xi_6\xi_{13} - \xi_5\xi_{14})^3(\xi_3\xi_{16} - \xi_4\xi_{15})(\xi_1\xi_8 - \xi_2\xi_7), \quad (28)$$

where, using parameters of the DP basis, parameters $\xi_i$ can be expressed as follows:

$$\begin{aligned}
\xi_1 &= \frac{1}{2}itv(z_1 - w_7 + w_1 - z_7 - 2z_2 + 2w_8 - 2w_2 + 2z_8),\\
\xi_2 &= \frac{1}{2}ivu(z_1 - w_7 - w_1 + z_7 - 2z_2 + 2w_8 + 2w_2 - 2z_8),\\
\xi_3 &= \frac{1}{2}itv(w_3 + z_5 - z_3 - w_3 + 2z_4 + 2w_4 - 2w_6 - 2z_6),\\
\xi_4 &= \frac{i}{2}(2z_4 + 2w_4 + 2w_6 + 2z_6 - z_3 - w_3 - w_5 - z_5),\\
\xi_5 &= \frac{1}{2}iut(w_3 - z_3 - w_5 + z_5),\\
\xi_6 &= \frac{1}{2}ut(z_3 - w_3 - w_5 + z_5),\\
\xi_7 &= \frac{i}{2}(z_1 + w_7 + w_1 + z_7 - 2z_2 - 2w_8 - 2w_2 - 2z_8),\\
\xi_8 &= \frac{1}{2}iut(z_1 + w_7 - w_1 - z_7 - 2z_2 - 2w_8 + 2w_2 + 2z_8), \qquad (29)\\
\xi_9 &= \frac{1}{2}iut(z_1 + w_7 - w_1 - z_7),\\
\xi_{10} &= \frac{i}{2}(z_1 + w_7 + w_1 + z_7),\\
\xi_{11} &= \frac{1}{2}ut(z_1 - w_7 - w_1 + z_7),\\
\xi_{12} &= \frac{1}{2}(z_1 - w_7 + w_1 - z_7),\\
\xi_{13} &= -\frac{i}{2}(z_3 + w_3 + w_5 + z_5),\\
\xi_{14} &= \frac{1}{2}(z_3 + w_3 - w_5 - z_5),\\
\xi_{15} &= \frac{1}{2}iuv(w_3 - z_3 + w_5 - z_5 + 2z_4 - 2w_4 - 2w_6 + 2z_6),\\
\xi_{16} &= \frac{1}{2}iut(w_3 - z_3 - w_5 + z_5 + 2z_4 - 2w_4 + 2w_6 - 2z_6).
\end{aligned}$$

In the concrete fashion (9), where $a_1 = a_2 \equiv a, \ldots, \quad e_1 = e_2 \equiv e$:

$$
\begin{aligned}
&\xi_1 = -2itv(l+f+3c+3ie),\\
&\xi_3 = -2itv(l-f+3c-3ie),\\
&\xi_4 = 2i(a-b+3k-3id),\\
&\xi_7 = -2i(a+b+3k+3id),\\
&\xi_{10} = 2i(a+b-k-id),\\
&\xi_{12} = 2(l+f-c-ie),\\
&\xi_{13} = -2i(a-b-k+id),\\
&\xi_{14} = -2(l-f-c+ie),\\
&\xi_2 = \xi_5 = \xi_6 = \xi_8 = \xi_9 = \xi_{11} = \xi_{15} = \xi_{16} = 0.
\end{aligned}
\tag{30}
$$

It appears that the reduction $16 = 1 \oplus 5 \oplus 10$ (or, $16 = 5 \oplus 1 \oplus 10$) of the $\beta$-matrices takes place only if $\beta^\mu$'s depend solely on $a, b, k, d$ (or, on $l, f, c, e$). In the case (30) only the reduction $16 = 6 \oplus 10$ is possible.

A general form of $\beta^\mu$, which satisfies the *weak conditions of discrete symmetries* in the representation $N = 10$, is given in [12]. The same calculations in the case $N = 5$ give nothing new. Since the case $N = 1$ is trivial, the independent parameters of $\beta$ in the KDP basis are

$$
\begin{aligned}
&\xi_1 = \xi_2 = \xi_3 = \xi_5 = \xi_6 = \xi_8 = \xi_9 = \xi_{11} = \xi_{15} = \xi_{16} = 0\\
&\xi_4 = -\xi_7 = i,\\
&\xi_{10} = \frac{i(q+i)x}{2}, \quad \xi_{12} = \frac{(i-q)x}{2}, \quad \xi_{13} = -\frac{q+i}{2qx}, \quad \xi_{14} = \frac{i(i-q)}{2qx},
\end{aligned}
\tag{31}
$$

where $q, x$ are arbitrary complex members. If we demand the validity of (30) and (31) together, then

$$
\begin{aligned}
&a = \frac{1}{2} - 3k,\\
&b = -3id,\\
&l = -3c,\\
&f = -3ie,\\
&c = \frac{(q-i)(qx^2-i)}{32qx},\\
&d = \frac{i(q+i)(qx^2+i)}{32qx},\\
&k = \frac{1}{8} - \frac{(q+i)(qx^2-i)}{32qx},\\
&e = \frac{(q-i)(qx^2+i)}{32iqx}.
\end{aligned}
\tag{32}
$$

The corresponding parameters in the DP basis are

$$
\begin{aligned}
z_1 &= w_1 = \frac{ix}{2}, \\
z_2 &= w_2 = \frac{1}{4}(1+ix), \\
z_3 &= w_3 = -\frac{i}{2x}, \\
z_4 &= w_4 = \frac{1}{4}\left(1-\frac{i}{x}\right), \\
z_5 &= w_5 = \frac{1}{2qx}, \\
z_6 &= w_6 = \frac{1}{4}\left(1+\frac{1}{qx}\right), \\
z_7 &= w_7 = \frac{qx}{2}, \\
z_8 &= w_8 = \frac{1}{4}(1+qx).
\end{aligned}
\tag{33}
$$

In the special case, if $q = i, x = -i$, we get the well-known expressions (7). There are only two free parameters $x, q$ (or, $z_1 \neq 0, z_5 \neq 0$), all others are expressed via these as follows:

$$
\begin{aligned}
z_2 &= \frac{1}{4} + \frac{z_1}{2}, \\
z_3 &= \frac{1}{4z_1}, \\
z_4 &= \frac{1}{4} + \frac{1}{8z_1}, \\
z_6 &= \frac{1}{4} + \frac{z_5}{2}, \\
z_7 &= \frac{1}{4z_5}, \\
z_8 &= \frac{1}{4} + \frac{1}{8z_5}.
\end{aligned}
\tag{34}
$$

One can easily verify that here $\beta^{\mu}$'s satisfy the KDP algebra condition.

The invariant bilinear form in this theory is defined as $\overline{\psi}\psi$, where $\overline{\psi} \equiv \psi^{+} H$. This means [11,12] that there exists a nonsingular Hermitian matrix $H$ such that the transformation $T(\Lambda)$ is $H$-unitary:

$$T^{+}(\Lambda)H = HT^{-1}(\Lambda) \quad \Leftrightarrow \quad HS_{\mu\nu} - S_{\mu\nu}^{+}H = 0 \tag{35}$$

or
$$HR_k - R_k^{+}H = 0, \;\; HS_k - S_k^{+}H = 0 \quad (k = 1, 2, 3)\,. \tag{35'}$$

Because of the nonunitarity of any finite-dimensional representation of $SO_{1,3}$ this form is always indefinite. In the common KDP theory [1–5] $H = 2\beta_0^2 - I$.

The most general *Hermitianizing matrix* satisfying Eqs. (35) has the form

$$H_G = \begin{bmatrix}
h_1 & h_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h_{10} & h_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -h_5 & -h_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -h_8 & -h_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & h_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & h_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & h_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & h_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_5 & 0 & 0 & h_7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_5 & 0 & 0 & h_7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_5 & 0 & 0 & h_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_8 & 0 & 0 & h_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_8 & 0 & 0 & h_6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_8 & 0 & 0 & h_6
\end{bmatrix}. \tag{36}$$

From the Hermiticity condition it follows that

$$h_4 = h_3^{*},\, h_{10} = h_9^{*},\, h_8 = h_7^{*},\, h_1^{*} = h_1,\, h_2^{*} = h_2,\, h_5^{*} = h_5,\, h_6^{*} = h_6.$$

The nondegenerancy is satisfied if

$$\det H_G = -h_3^3 h_4^3 (h_5 h_6 - h_7 h_8)^4 (h_1 h_2 - h_9 h_{10}) \neq 0. \tag{37}$$

The space inversion operator $I_r$, [12], may be presented in the same fashion by choosing

$$\begin{gathered} h_7 = h_8 = h_9 = h_{10} = 0,\, h_1 = h_2 = h_5 = h_6 = q_0, \\ h_4 = q_0 q,\, h_3 = q_0 / q \end{gathered} \tag{38}$$

and $\det I_r = -q_0^{10}$.

The Hermitianizing matrix $H$ in the DP basis has the form

$$
H_{DP} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\tilde{h}_9 & 0 & 0 & 0 & 0 & 0 \\
0 & \tilde{h}_1 & 0 & 0 & -\tilde{h}_1 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{h}_9+\tilde{h}_2 & 0 & 0 & \tilde{h}_9-\tilde{h}_2 & 0 \\
0 & 0 & \tilde{h}_5 & 0 & 0 & 0 & 0 & 0 & \tilde{h}_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \tilde{h}_5 & 0 & 0 & \tilde{h}_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\tilde{h}_1 & 0 & 0 & \tilde{h}_1 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{h}_9-\tilde{h}_2 & 0 & 0 & \tilde{h}_9+\tilde{h}_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\tilde{h}_9 \\
0 & 0 & 0 & \tilde{h}_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{h}_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{h}_5 & 0 & 0 & 0 & 0 & 0 & \tilde{h}_6 & 0 & 0 \\
0 & 0 & \tilde{h}_7 & 0 & 0 & 0 & 0 & 0 & \tilde{h}_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \tilde{h}_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{h}_8 & 0 & 0 & 0 \\
2\tilde{h}_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \tilde{h}_{10}+\tilde{h}_3 & 0 & 0 & \tilde{h}_{10}-\tilde{h}_3 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{h}_4 & 0 & 0 & -\tilde{h}_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \tilde{h}_7 & 0 & 0 & \tilde{h}_8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{h}_7 & 0 & 0 & 0 & 0 & 0 & \tilde{h}_8 & 0 & 0 \\
0 & \tilde{h}_{10}-\tilde{h}_3 & 0 & 0 & \tilde{h}_{10}+\tilde{h}_3 & 0 & 0 & 0 & 0 & 0 & 0 & -\tilde{h}_4 & 0 & 0 & \tilde{h}_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 2\tilde{h}_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \tag{39}
$$

where

$$
\begin{aligned}
\tilde{h}_1 &= \frac{1}{2}(p^2 h_1 + pr(h_{10}+h_9) + r^2 h_2), \\
\tilde{h}_2 &= \frac{1}{2}(p^2 h_9 + pr(h_2 - h_1) - r^2 h_{10}), \\
\tilde{h}_3 &= \frac{1}{2}(p^2 h_{10} + pr(h_2 - h_1) - r^2 h_9), \\
\tilde{h}_4 &= \frac{1}{2}(p^2 h_2 - pr(h_9 + h_{10}) + r^2 h_1), \\
\tilde{h}_5 &= q^2 h_6 + qs(h_7 + h_8) + s^2 h_5, \\
\tilde{h}_6 &= q^2 h_8 + qs(h_5 - h_6) - s^2 h_7, \\
\tilde{h}_7 &= q^2 h_7 + qs(h_5 - h_6) - s^2 h_8, \\
\tilde{h}_8 &= q^2 h_5 - qs(h_7 + h_8) + s^2 h_6, \\
\tilde{h}_9 &= \frac{1}{2} mn h_3, \\
\tilde{h}_{10} &= \frac{1}{2} mn h_4,
\end{aligned} \tag{40}
$$

and

$$
\det H_{DP} = -256 \tilde{h}_9^3 \tilde{h}_{10}^3 (\tilde{h}_6 \tilde{h}_7 - \tilde{h}_5 \tilde{h}_8)^4 (\tilde{h}_1 \tilde{h}_4 - \tilde{h}_2 \tilde{h}_3). \tag{41}
$$

In the usual case (7) $H_{DP} = \gamma_0 \otimes \gamma_0$, which is provided by the choise $\tilde{h}_1 = \tilde{h}_4 = \tilde{h}_5 = \tilde{h}_8 = 0$, $\tilde{h}_6 = \tilde{h}_7 = 1$, $\tilde{h}_2 = \tilde{h}_3 = \tilde{h}_9 = \tilde{h}_{10} = \frac{1}{2}$.

In the KDP basis

$$H_{KDP} = \begin{bmatrix}
\hat{h}_1 & 0 & 0 & 0 & 0 & \hat{h}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\hat{h}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\hat{h}_4 \\
0 & 0 & \hat{h}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{h}_4 & 0 & 0 & 0 \\
0 & 0 & 0 & \hat{h}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{h}_4 & 0 & 0 \\
0 & 0 & 0 & 0 & \hat{h}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{h}_4 & 0 \\
\hat{h}_5 & 0 & 0 & 0 & 0 & \hat{h}_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \hat{h}_7 & 0 & 0 & \hat{h}_8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{h}_7 & 0 & 0 & \hat{h}_8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{h}_7 & 0 & 0 & \hat{h}_8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \hat{h}_8 & 0 & 0 & -\hat{h}_7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{h}_8 & 0 & 0 & -\hat{h}_7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{h}_8 & 0 & 0 & -\hat{h}_7 & 0 & 0 & 0 & 0 \\
0 & 0 & \hat{h}_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{h}_{10} & 0 & 0 & 0 \\
0 & 0 & 0 & \hat{h}_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{h}_{10} & 0 & 0 \\
0 & 0 & 0 & 0 & \hat{h}_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{h}_{10} & 0 \\
0 & -\hat{h}_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\hat{h}_{10}
\end{bmatrix}, \tag{42}$$

where

$$\begin{aligned}
\hat{h}_1 &= \frac{1}{2}\left[p^2(h_1 + h_2 - h_9 - h_{10}) + r^2(h_1 + h_2 + h_9 + h_{10}) + 2pr(h_1 - h_2)\right], \\
\hat{h}_2 &= \frac{1}{2}vt\left[p^2(h_1 - h_2 + h_9 - h_{10}) - r^2(h_1 - h_2 - h_9 + h_{10}) + 2pr(h_9 + h_{10})\right], \\
\hat{h}_3 &= \frac{1}{2}\left[q^2(h_5 + h_6 - h_7 - h_8) + s^2(h_5 + h_6 + h_7 + h_8) - 2qs(h_5 - h_6)\right], \\
\hat{h}_4 &= \frac{1}{2}ut\left[-q^2(h_5 - h_6 + h_7 - h_8) + s^2(h_5 - h_6 - h_7 + h_8) + 2qs(h_7 + h_8)\right], \\
\hat{h}_5 &= \frac{1}{2}vt\left[p^2(h_1 - h_2 - h_9 + h_{10}) - r^2(h_1 - h_2 + h_9 - h_{10}) + 2pr(h_9 + h_{10})\right], \\
\hat{h}_6 &= \frac{1}{2}\left[p^2(h_1 + h_2 + h_9 + h_{10}) + r^2(h_1 + h_2 - h_9 - h_{10}) - 2pr(h_1 - h_2)\right], \\
\hat{h}_7 &= \frac{1}{2}mn(h_4 + h_3), \\
\hat{h}_8 &= \frac{1}{2}mn(h_4 - h_3), \\
\hat{h}_9 &= \frac{1}{2}ut\left[-q^2(h_5 - h_6 - h_7 + h_8) + s^2(h_5 - h_6 + h_7 - h_8) + 2qs(h_7 + h_8)\right], \\
\hat{h}_{10} &= \frac{1}{2}\left[q^2(h_5 + h_6 + h_7 + h_8) + s^2(h_5 + h_6 - h_7 - h_8) + 2qs(h_5 - h_6)\right],
\end{aligned} \tag{43}$$

and

$$\det H_{KDP} = -(\hat{h}_7^2 + \hat{h}_8^2)^3 (\hat{h}_4 \hat{h}_9 - \hat{h}_3 \hat{h}_{10})^4 (\hat{h}_1 \hat{h}_6 - \hat{h}_2 \hat{h}_5)\,. \tag{44}$$

Finally, let us note that these *H*-matrices play an important role in the investigation of the discrete symmetries of Eqs. (1). The *weak conditions of discrete symmetries* are presented in [[12]].

## 7. CONCLUSIONS

We presented some general expressions for $\beta$-matrices of the first-order wave equations in the 16-dimensional representation using three different bases. Hopefully these representations allow of the construction of a more realistic theory of elementary particles and their interactions. Of course, this model must contain some nonlinearity. Except the cases (12)–(14), we have not concretized the algebra of $\beta$-matrices. The general 16-component theory of the massive single spin-1 particle, based on the new algebra (14), will be published in the nearest future. We hope that the 16-dimensional representation will enable us to include successfully both the bosons and fermions within the framework of some realistic model.


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## ESIMEST JÄRKU LAINEVÕRRANDITE $\beta$-MAATRIKSITE ÜLDINE AVALDIS 16-DIMENSIOONILISES ESITUSES

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On arvutatud esimest järku lainevõrrandite $\beta$-maatriksite üldised avaldised 16-dimensioonilises esituses otsekorrutise (DP), Gelfandi (G) ja Kemmeri–Duffini–Petiau' (KDP) baasis. Üldjuhul tekib 16 suvalist parameetrit. Sõltuvalt nendest parameetritest rahuldavad $\beta$-maatriksid kas KDP-, Diraci või teatud uut algebrat ning vastavad võrrandid võivad kirjeldada nii bosoneid kui ka fermione. Ilmneb, et 16-dimensioonilise esituse reduktsioon 1-, 5- ja 10-dimensiooniliste esituste otsesummaks on võimalik ainult erijuhul ning harilikult $\det \beta \neq 0$. On toodud ka unitaarsed teisendused, mis seovad suurusi DP-baasis vastavate suurustega G- ja KDP-baasis. Samuti on avaldatud hermitiseerivate maatriksite üldkujud nendes baasides.