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GRAPHS AND LATTICE VARIETIES

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Abstract. The congruence lattices of graphs satisfying a given lattice identity are studied. A complete characterization of all finite graphs with the congruence lattice lying in a given lattice variety is presented.

Key words: graph, congruence lattice, lattice variety, lattice identity.

1. INTRODUCTION

It has been proved in [1] that the set of all congruence relations of a given graph G is a complete lattice. In the current paper the following problem is solved.

Problem. *Let I be a lattice identity. Find a characterization for all finite graphs G such that* $\mathrm{Con}\, G$ *satisfies I.*

To solve this problem, the *J-radical* defined in [1] as the meet of all co-atoms in $\mathrm{Con}\, G$ is useful. If $J(G) = 0$, then either $\mathrm{Con}\, G \cong \mathbf{\Pi}_n$ or $\mathrm{Con}\, G \cong \mathbf{\Pi}_2^n$, where $\mathbf{\Pi}_n$ is the lattice of all partitions of $n = \{0, ..., n-1\}$. For every J-semisimple graph G define a positive integer $\eta(G)$ such that $\eta(G) = n$ if G is a complete or edgeless graph with n vertices (i.e. if $\mathrm{Con}\, G \cong \mathbf{\Pi}_n$) and $\eta(G) = 2$ otherwise. Let $\mathcal{V}$ be a lattice variety such that $L \notin \mathcal{V}$ for at least one lattice L. There is a unique positive integer n such that $\mathbf{\Pi}_1, ..., \mathbf{\Pi}_n \in \mathcal{V}$, but $\mathbf{\Pi}_{n+1} \notin \mathcal{V}$. Denote this n as $\eta(\mathcal{V})$.

Theorem. *If $G/J(G) = \{G_1, ..., G_n\}$ and $\mathcal{V}$ is a lattice variety, then* $\mathrm{Con}\, G$ *lies in $\mathcal{V}$ if and only if $\eta(G/J(G)) \leq \eta(\mathcal{V})$ and* $\mathrm{Con}\, G_i \in \mathcal{V}$ *for $i = 1, ..., n$.*

$G/J(G)$ denotes the *factor-graph* of G by $J(G)$ and G_i are the congruence classes viewed as subgraphs of G. It follows from the theorem that for every lattice identity I there is an $O(|V|^2)$ algorithm that determines whether I holds in $\mathrm{Con}\, G$, where V is the vertex set of G.

2. BASICS

A pair $G = (V, E)$ is called a *graph* if E is an antireflexive binary relation on V. The elements of V and E are called *vertices* and *edges* of the graph G, respectively. A graph G is said to be *undirected* if the relation E is symmetric. Usually we use the short notation $xy \in E$ instead of the correct notation $(x, y) \in E$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. We say that a mapping $V_1 \xrightarrow{f} V_2$ is a *graph morphism* if the condition

$$f(x) \neq f(y) \Rightarrow [xy \in E_1 \leftrightarrow f(x)f(y) \in E_2] \tag{1}$$

holds for arbitrary vertices $x, y \in V_1$. Then we can write $G_1 \xrightarrow{f} G_2$. Such a morphism has been introduced in [2]. It is easy to verify that we get the usual structure of a category. In other words, the *identity* mapping $G \xrightarrow{1_G} G$ is always a morphism, and the *composition* $G_1 \xrightarrow{g\circ f} G_3$ of two morphisms $G_1 \xrightarrow{f} G_2$ and $G_2 \xrightarrow{g} G_3$ is a morphism as well.

An equivalence relation ρ on the vertex set V of the graph $G = (V, E)$ is called a *congruence relation* on G if the condition

$$x\rho x' \wedge y\rho y' \wedge \neg(x\rho y) \Rightarrow [xy \in E \leftrightarrow x'y' \in E]$$

holds for arbitrary vertices $x, x', y, y' \in V$. It is easy to see that the kernel $\operatorname{Ker} f$ of every morphism $G \xrightarrow{f} H$ is a congruence relation and conversely, every congruence relation ρ of G is a kernel of some morphism. This is true because there is a unique graph structure on the factor set V/ρ such that the natural projection $V \xrightarrow{\pi} V/\rho$ is a morphism. This graph is called a *factor graph* of G by ρ and is denoted as G/ρ.

A subset $M \subseteq V$ is called a *module* (by Spinrad [3]) of $G = (V, E)$ if $zx \in E \Rightarrow zy \in E$ and $xz \in E \Rightarrow yz \in E$ for arbitrary $x, y \in M$ and $z \notin M$. It is easy to prove that an equivalence relation ρ on the vertex set V is a congruence relation of G if and only if all the ρ-classes are modules of G.

Lemma 1. *The union $A \cup B$ of two intersecting modules A and B is a module. If A and B are overlapping modules (i.e. they are intersecting and $A - B \neq \emptyset \neq B - A$,) then $A - B$ is a module.*

The proof is trivial ([1,3]).

Let $\operatorname{Con} G$ denote the set of all congruence relations of the graph G. Note that $\operatorname{Con} G$ is *partially ordered* by the inclusion relation $\subseteq$. It is proved in [1] that $\operatorname{Con} G$ is a complete lattice for every graph $G = (V, E)$. Let $(\rho]$ and $[\rho)$ denote the principal ideal generated by ρ and its dual, respectively.

Theorem 1. *If $G = (V, E)$ is a graph and $\rho \in \operatorname{Con} G$, then*

$$[\rho) \cong \operatorname{Con} G/\rho.$$

Theorem 2. *If G is a graph, $\rho \in \operatorname{Con} G$ is an arbitrary congruence relation, and $G/\rho = \{G_j\}_{j \in \mathcal{J}}$, then*

$$(\rho] \cong \prod_{j \in \mathcal{J}} \operatorname{Con} G_j.$$

Proofs can be found in [1].

For any graph G the intersection of all co-atoms of $\operatorname{Con} G$ is called a *radical* of G and will be denoted as $J(G)$. A graph G is said to be *J-radical* if $r(G) = 1$ and G is said to be *J-semisimple* if $r(G) = 0$. Here 0 and 1 denote trivial congruence relations.

Theorem 3. *A graph G is J-radical iff there are no co-atoms in $\operatorname{Con} G$ and G is J-semisimple iff it satisfies at least one of the following conditions:*

- *G is simple,*
- *G is edgeless,*
- *G is complete,*
- *G is isomorphic to a linear ordering.*

The proof can be found in [1].

3. PARTITION LATTICES

Let A be an arbitrary set and $S \subseteq \mathbf{\Pi}(A)$ be a nonempty set of equivalence relations on A. A finite sequence

$$a_0, a_1, \ldots, a_\ell,$$

where $a_i \in A$, is called an *S-chain* if for every i there is an equivalence relation $\rho_i \in S$ such that $(a_i, a_{i+1}) \in \rho_i$. Two S-chains $a : a_0, \ldots, a_\ell$ and $b : b_0, \ldots, b_\ell$ are *equivalent* if for every i there is $\rho_i \in S$ such that $(a_i, a_{i+1}) \in \rho_i$ and $(b_i, b_{i+1}) \in \rho_i$.

It is well known that $(x, y) \in \sup S$ iff x and y can be connected with an S-chain. If $(x_1, y_1) \in \sup S$ and $(x_2, y_2) \in \sup S$, then the corresponding S-chains can be chosen in such a way that they are equivalent.

Let $G = (V, E)$ be a graph.

Theorem 4. $\operatorname{Con} G$ *is a complete sublattice of* $\mathbf{\Pi}(V)$.

Proof. It is sufficient to show that the least upper bound (in $\mathbf{\Pi}(V)$) of every nonempty set S of congruence relations is a congruence relation. Let $\rho = \sup S$, $x\rho x'$, $y\rho y'$, and $\neg(x\rho y)$. So, there are equivalent S-chains

$$x = x_0, x_1, \ldots, x_\ell = x',$$
$$y = y_0, y_1, \ldots, y_\ell = y'$$

such that $x_i \rho_i x_{i+1}$ and $y_i \rho_i y_{i+1}$ $\forall i < \ell$.

Let $(x, y) \in E$. We will show by induction that $(x_i, y_i) \in E$ for all $0 \leq i \leq \ell$. Indeed, the case $i = 0$ is trivial and if $(x_{i-1}, y_{i-1}) \in E$, then $x_{i-1} \rho_{i-1} x_i$, $y_{i-1} \rho_{i-1} y_i$. But $\neg(x_{i-1} \rho_{i-1} y_{i-1})$, because otherwise there would be an S-chain

$$x = x_0, x_1, ..., x_{i-1}, y_{i-1}, ..., y_0 = y.$$

As $\rho_{i-1} \in \mathrm{Con}(G)$, we get from the definition of congruence relation that $(x_i, y_i) \in E$. Therefore, $\rho = \sup S$ is a congruence relation. □

4. NEUTRAL AND STRONGLY NEUTRAL ELEMENTS

An element α of a lattice L is said to be *neutral* [4,5] if

$$(\alpha \wedge x) \vee (x \wedge y) \vee (y \wedge \alpha) = (\alpha \vee x) \wedge (x \vee y) \wedge (y \vee \alpha)$$

for all $x, y \in L$. The following theorem gives us two equivalent formulations of neutrality.

Theorem 5. *Let L be a lattice and let α be an element of L. The following conditions are equivalent:*

- *α is neutral;*
- *α is distributive, dually distributive, and $\alpha \wedge x = \alpha \wedge y$ and $\alpha \vee x = \alpha \vee y$ imply $x = y$ for any $x, y \in L$;*
- *the mapping*

$$\varphi\colon \begin{cases} L \longrightarrow (\alpha] \times [\alpha) \\ x \mapsto (x \wedge \alpha, x \vee \alpha) \end{cases}$$

is a lattice embedding.

The proof is given in [5].

Let V be an arbitrary set, L be a complete sublattice of $\mathbf{\Pi}(V)$ and $\rho \in L$. We say that ρ is *strongly neutral* in L if

$$\rho \vee \sigma = \sigma \cup \rho$$

for arbitrary $\sigma \in L$.

Lemma 2. *An equivalence relation $\rho \in L$ is strongly neutral iff*

$$v/\rho \subseteq v/\sigma \quad or \quad v/\sigma \subseteq v/\rho \tag{2}$$

for arbitrary $\sigma \in L$ and $v \in V$, where v/ρ denotes the ρ-class containing v.

Proof. Let us assume that ρ is strongly neutral, $\sigma \in L$, $v \in V$, and $v/\rho \nsubseteq v/\sigma$. We will show that $v/\sigma \subseteq v/\rho$. As v/ρ is not a subset of v/σ, there has to be an $u \in v/\rho$ such that $u \notin v/\sigma$. Let w be an arbitrary element of v/σ. As $(w, v) \in \sigma$ and $(v, u) \in \rho$, we have

$$(w, u) \in \rho \vee \sigma = \rho \cup \sigma$$

and thereby $(w, u) \in \rho$ because $(w, u) \notin \sigma$. Now we have $w \in u/\rho = v/\rho$. As w has been chosen arbitrarily, we conclude that $v/\sigma \subseteq v/\rho$.

Let us assume that the condition (2) holds. It is sufficient to prove that $\rho \cup \sigma$ is an equivalence relation. It is obvious that $\rho \cup \sigma$ is reflexive and symmetric. We will prove the transitivity. Let $(u, v), (v, w) \in \rho \cup \sigma$. If these pairs lie both in ρ or in σ, the transitivity is obvious. Let $(u, v) \in \sigma$ and $(v, w) \in \rho$. If $(u, w) \notin \rho$, then $u \notin w/\rho$ and therefore

$$v/\sigma = u/\sigma \nsubseteq w/\rho = v/\rho$$

and by condition (2) we have $w \in w/\rho = v/\rho \subseteq v/\sigma = u/\sigma$ showing that $(u, w) \in \sigma$. □

Lemma 3. *Every strongly neutral element is neutral.*

Proof. Suppose $\rho \in L$ is strongly neutral. Let us prove at first that the mappings $\sigma \mapsto \sigma \cap \rho$ and $\sigma \mapsto \sigma \vee \rho$ are endomorphisms of the lattice L. The second mapping is obviously a morphism because of the distributivity of the lattice of all subsets of A. Let $\gamma = \sigma \vee \delta$. We will prove the equality

$$\gamma \cap \rho = (\sigma \cap \rho) \vee (\delta \cap \rho).$$

Indeed, as $\sigma \subseteq \gamma$ and $\delta \subseteq \gamma$, it follows that $\sigma \cap \rho \subseteq \gamma \cap \rho$ and $\delta \cap \rho \subseteq \gamma \cap \rho$. Therefore $\gamma \cap \rho$ is an upper bound of the equivalence relations $\sigma \cap \rho$ and $\delta \cap \rho$. It remains to show that it is the least upper bound. Let

$$\sigma \cap \rho \subseteq \tau, \qquad \delta \cap \rho \subseteq \tau, \tag{3}$$

and $(u, v) \in \gamma \cap \rho$. Accordingly, $(u, v) \in \sigma \vee \delta$ and $(u, v) \in \rho$. Therefore there is a $\{\sigma, \delta\}$-chain

$$c: \; u = v_0, v_1, ..., v_\ell = v.$$

If $(v_i, v_{i+1}) \in \rho$ for each $i < \ell$, then obviously c is a $\{\sigma\cap\rho, \delta\cap\rho\}$-chain. Therefore, by the inclusions (3) and the transitivity of τ, we have $(u, v) \in \tau$.

Let i be the smallest index such that $(v_i, v_{i+1}) \notin \rho$. Then either $v_i/\sigma \nsubseteq v_i/\rho$ or $v_i/\delta \nsubseteq v_i/\rho$. Accordingly, by Lemma 2, either $v_i/\rho \subseteq v_i/\sigma$ or $v_i/\rho \subseteq v_i/\delta$, which gives that either $(u, v) \in \sigma$ or $(u, v) \in \delta$. This implies $(u, v) \in \tau$. Thus, ρ is a distributive and dually distributive element.

We assume now that $\sigma \vee \rho = \delta \vee \rho$ and $\sigma \cap \rho = \delta \cap \rho$ and show that $\sigma = \delta$. Let $(u, v) \in \sigma$. If $(u, v) \in \rho$, then $(u, v) \in \sigma \cap \rho = \delta \cap \rho \subseteq \delta$ and therefore $(u, v) \in \delta$. If $(u, v) \notin \rho$, then from $(u, v) \in \sigma \subseteq \sigma \vee \rho = \delta \vee \rho = \delta \cup \rho$ it follows that $(u, v) \in \delta$. Therefore $\sigma \subseteq \delta$. The proof of $\delta \subseteq \sigma$ is similar. Accordingly, ρ is a neutral element of L. □

Lemma 4. *If there is a co-atom ρ in* Con G *such that there are at least* 3 *vertices in G/ρ, then ρ is a unique co-atom of* Con G *and, furthermore, ρ is the least upper bound of all congruence relations different from* 1.

The proof is given in [1].

Theorem 6. *If $G/J(G)$ is edgeless [complete], then every $J(G)$-class G_ι is connected [complement-connected].*

Proof. Let $G/J(G)$ be edgeless and $G_\iota \in G/J(G)$ be not connected. Let G'_ι be an arbitrary connected component of G_ι. It is obvious that there are no edges between G'_ι and $G - G'_\iota$ and therefore $\{G'_\iota, G - G'_\iota\}$ is a congruence partition and the corresponding congruence relation ρ is a co-atom in Con G and $J(G) \nleq \rho$, which is a contradiction with the definition of $J(G)$.

If $G/J(G)$ is complete, the proof is similar. □

By A_n we mean the graph $(\{0, 1, ..., n-1\}, \{(i, j) \mid 0 \le i < j < n\})$ (a linear ordering with n elements).

Theorem 7. *If $G/J(G)$ is a linear ordering, then none of the $J(G)$-classes have a factor-graph isomorphic to A_2.*

Proof. Let $G/J(G) = (V_0, E_0)$ be a linear ordering and $H \in G/J(G)$. Suppose there is an epimorphism $H \xrightarrow{f} A_2$ and $\{H_1, H_2\}$ is a partition corresponding to Ker f; $(f(H_1), f(H_2)) \in E(A_2)$.

We say that $H' \in V_0$ *is less than* $H \in V_0$ if $H'H \in E_0$. Let $\mathbf{G}_1 \subseteq G/J(G)$ be the set of elements of $G/J(G)$ less than H and $\mathbf{G}_2$ be the set of elements greater than H. It is obvious that the partition

$$\{(\cup\mathbf{G}_1) \cup H_1, \quad (\cup\mathbf{G}_2) \cup H_2\}$$

is a co-atom of Con G which is not comparable with $J(G)$. This is a contradiction. □

Theorem 8. *The radical $J(G)$ is a strongly neutral element of* Con G.

Proof. The statement is trivially true if $|G| \leq 2$. Let us assume that $|G| \geq 3$. As the graph $G/J(G)$ is J-semisimple, we know that $G/J(G)$ is either simple, complete, edgeless or linear (Theorem 3).

The cases when $G/J(G)$ is edgeless or complete are dual, so it is sufficient to consider only one of them. Let $G/J(G)$ be edgeless, $v \in V$ be an arbitrary vertex, $\sigma \in \operatorname{Con} G$, and $v/\sigma \nsubseteq v/J(G)$. Consequently, there is a vertex $u \in v/\sigma - v/J(G)$. By Theorem 6 the induced subgraph $v/J(G)$ is connected. Thereby, for any vertex $w \in v/J(G)$, there is a chain of vertices $v = v_0, v_1, ..., v_\ell = w \in v/J(G)$ such that $v_i v_{i+1} \in E \cup E^{-1}, 0 \leq i \leq n-1$. We will prove by induction that all the vertices v_i lie in v/σ. Obviously, v lies in v/σ. Assume that $v_i \in v/\sigma$ and $v_{i+1} \notin v/\sigma$. Since there is an edge between v_i and v_{i+1} and u is contained in the module v/σ which does not contain v_{i+1}, there must be an edge between u and v_{i+1} as well. Since $G/J(G)$ is edgeless, this implies $u/J(G) = v_{i+1}/J(G)$, a contradiction.

Let $G/J(G)$ be linear. Let $H = v/J(G)$ and v/σ be overlapping modules. By Theorem 1 the intersection $H \cap v/\sigma$ and set difference $H - v/\sigma$ are modules of the induced subgraph H, and thereby we have a partition of H into modules. It is easy to show that the corresponding factor-graph is linear, which is a contradiction with Theorem 7.

If $G/J(G)$ is simple, then $J(G)$ is the unique co-atom of Con G. If $|G/J(G)| = 2$, then $G/J(G)$ is either complete, edgeless or linear. Thus we can assume without loss of generality that $| G/J(G) | \geq 3$. It follows directly from Lemma 4 that every congruence relation $\rho \in \operatorname{Con} G$ is comparable with $J(G)$ and therefore $J(G) \vee \rho = J(G) \cup \rho$ for every $\rho \in \operatorname{Con} G$. □

5. PARTITION NUMBER OF A VARIETY

We will show in this section that the finite partition lattices $\mathbf{\Pi}_\ell$ play an important role in studying the lattice identities holding in Con G.

Let $\mathcal{L}$ be the class of all lattices, $\mathcal{V} \subseteq \mathcal{L}$ be a lattice variety, and $L_1, ..., L_n$ be arbitrary lattices. It is obvious that their direct product

$$L = L_1 \times L_2 \times ... \times L_n$$

lies in $\mathcal{V}$ if and only if every L_i lies in $\mathcal{V}$. For example, L is modular [distributive] iff every L_i is modular [distributive].

As proved by Sachs in 1961 [6], every lattice identity that holds in every finite partition lattice must hold in every lattice. Thereby, for every lattice variety $\mathcal{V} \neq \mathcal{L}$, there is a unique natural number $\eta(\mathcal{V})$ such that $\mathbf{\Pi}_{\eta(\mathcal{V})} \notin \mathcal{V}$ and $\mathbf{\Pi}_m \in \mathcal{V}$ for every natural number $m < \eta(\mathcal{V})$. The natural number $\eta(\mathcal{V})$ is called a *partition number* of the variety $\mathcal{V}$.

For example, the class of all distributive lattices has the partition number 3, the class of all modular lattices has the partition number 4.

Lemma 5. *If $\mathcal{V}$ is a lattice variety and $\eta(\mathcal{V}) = 2$, then every lattice in $\mathcal{V}$ is trivial.*

Proof. If $L \in \mathcal{V}$ is a nontrivial lattice, then it has a two-element sublattice isomorphic to $\mathbf{\Pi}_2$. Therefore $\eta(\mathcal{V}) > 2$. □

Let $\mathcal{L}_\ell$ denote the variety generated by the lattice $\mathbf{\Pi}_{\ell-1}$. Obviously,

$$\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}_3 \subseteq ... \subseteq \mathcal{L}_\ell \subseteq ... \subseteq \mathcal{L}.$$

It follows from the Jónsson lemma [7,8] that all these inclusions are proper, i.e. for every natural number $n > 0$ there is a lattice identity that holds in $\mathbf{\Pi}_n$ but not in $\mathbf{\Pi}_{n+1}$. For the case $n = 1$, the suitable identity is $x = y$, for the case $n = 2$, it is distributivity and for the case $n = 3$, modularity. It turns out that the suitable identity for $n = 4$ is

$$\begin{aligned} x_0 \wedge (x_1 \vee x_2 \vee x_3 \vee x_4) = \quad & [x_0 \wedge (x_1 \vee x_2 \vee x_3)] \vee [x_0 \wedge (x_1 \vee x_2 \vee x_4)] \\ & \vee [x_0 \wedge (x_1 \vee x_3 \vee x_4)] \vee [x_0 \wedge (x_2 \vee x_3 \vee x_4)]. \end{aligned}$$

Obviously, the left-hand side is greater than or equal to the right-hand side. To show the opposite inequality, it is sufficient to mention that for arbitrary elements $a_0, a_1, ..., a_4$ of $\mathbf{\Pi}_4$ the equivalence relation $a_0 \wedge (a_1 \vee ... \vee a_4)$ is equal to one of

$$\begin{aligned} a_0 \wedge (a_1 \vee a_2 \vee a_3),\ a_0 \wedge (a_1 \vee a_2 \vee a_4), \\ a_0 \wedge (a_1 \vee a_3 \vee a_4),\ a_0 \wedge (a_2 \vee a_3 \vee a_4), \end{aligned}$$

because there are no chains of length 5 in $\mathbf{\Pi}_4$ and therefore the chain

$$0 \leq a_1 \leq a_1 \vee a_2 \leq a_1 \vee a_2 \vee a_3 \leq a_1 \vee a_2 \vee a_3 \vee a_4 \leq 1$$

must have two equal elements.

But this identity does not hold in $\mathbf{\Pi}_5$. Take, for example, $x_0 = (04)$, $x_1 = (01)$, $x_2 = (12)$, $x_3 = (23)$, and $x_4 = (34)$, where (ij) denotes the minimal equivalence relation containing the pair (i, j).

A proof of the general case is similar to the proof of the case $n = 4$. Let $X = \{x_0, x_1, ..., x_n\}$ be the set of variable letters. A suitable identity that separates $\mathbf{\Pi}_n$ and $\mathbf{\Pi}_{n+1}$ is

$$x_0 \wedge (x_1 \vee ... \vee x_n) = \bigvee_\iota [x_0 \wedge (x_{\iota(1)} \vee ... \vee x_{\iota(n-1)})].$$

where the join in the right-hand side is calculated over all possible injections

$$\{1, ..., n-1\} \xrightarrow{\iota} \{1, ..., n\}.$$

6. IDENTITIES IN $\operatorname{Con} G$

Let $\mathcal{L}$ be the class of all lattices, $\mathcal{L}_{\mathcal{G}}$ be the class of congruence lattices of all finite graphs. We say that two lattice varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ are *equivalent* and write $\mathcal{V}_1 \sim \mathcal{V}_2$ iff $\mathcal{V}_1 \cap \mathcal{L}_{\mathcal{G}} = \mathcal{V}_2 \cap \mathcal{L}_{\mathcal{G}}$.

Lemma 6. *Let G be a graph, $\mathcal{V}$ be a lattice variety, and $G/J(G) = \{G_i\}_{i\in\mathcal{I}}$. The congruence lattice $\operatorname{Con} G$ lies in $\mathcal{V}$ iff all the lattices $\operatorname{Con} G_i$ and the lattice $\operatorname{Con}(G/J(G))$ lie in $\mathcal{V}$.*

Proof. Assume $\operatorname{Con} G \in \mathcal{V}$. Now $\operatorname{Con} G_i \in \mathcal{V}$, because there are lattice embeddings $\operatorname{Con} G_i \longrightarrow (J(G)] \leq G$, and $\operatorname{Con} G/J(G) \in \mathcal{V}$, because $\operatorname{Con} G/J(G) \cong [J(G)) \leq \operatorname{Con} G$.

And, conversely, if every $\operatorname{Con} G_i$ and $\operatorname{Con} G/J(G)$ lie in $\mathcal{V}$, then by Theorem 8 there is a lattice embedding

$$\operatorname{Con} G \longrightarrow (J(G)] \times [J(G)) \cong \operatorname{Con}(G/J(G)) \times \prod_{i\in\mathcal{I}} \operatorname{Con} G_i$$

and therefore $\operatorname{Con} G \in \mathcal{V}$. □

Let $\mathcal{L}_J$ be the class of all congruence lattices of finite J-semisimple graphs. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be lattice varieties. We say that $\mathcal{V}_1$ and $\mathcal{V}_2$ are J-equivalent and write $\mathcal{V}_1 \sim_J \mathcal{V}_2$ iff $\mathcal{V}_1 \cap \mathcal{L}_J = \mathcal{V}_2 \cap \mathcal{L}_J$.

Lemma 7. *Two lattice varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ are J-equivalent if and only if $\eta(\mathcal{V}_1) = \eta(\mathcal{V}_2)$.*

Proof. Theorem 3 gives us a complete characterization of the class $\mathcal{L}_J$. It is obvious that if any of the direct powers $\mathbf{\Pi}_2^k$ lies in $\mathcal{V}_1$ or in $\mathcal{V}_2$, then all $\mathbf{\Pi}_2^\ell$, $\ell = 1, 2, 3, \ldots$, lie in $\mathcal{V}_1$ or in $\mathcal{V}_2$, respectively. Therefore, $\mathcal{V}_1$ and $\mathcal{V}_2$ are J-equivalent if and only if they contain the same partition lattices $\mathbf{\Pi}_1, \mathbf{\Pi}_2, \ldots, \mathbf{\Pi}_\ell, \ldots$, i.e. iff $\eta(\mathcal{V}_1) = \eta(\mathcal{V}_2)$. □

Theorem 9. *Two varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ are equivalent if and only if their partition numbers coincide, i.e.*

$$\mathcal{V}_1 \sim \mathcal{V}_2 \Leftrightarrow \eta(\mathcal{V}_1) = \eta(\mathcal{V}_2).$$

Proof. If $\mathcal{V}_1 \cap \mathcal{L}_{\mathcal{G}} = \mathcal{V}_2 \cap \mathcal{L}_{\mathcal{G}}$, then $\eta(\mathcal{V}_1) = \eta(\mathcal{V}_2)$. Indeed, if K_n is a complete graph of n vertices, then $\operatorname{Con} K_n \cong \mathbf{\Pi}_n$ and therefore $\{\mathbf{\Pi}_1, \mathbf{\Pi}_2, \ldots\} \subseteq \mathcal{L}_{\mathcal{G}}$.

Let us prove the opposite implication. Let G be a finite graph. Let $\eta(\mathcal{V}_1) = \eta(\mathcal{V}_2)$. Then, by Lemma 7, $\mathcal{V}_1 \cap \mathcal{L}_J = \mathcal{V}_2 \cap \mathcal{L}_J$. If G is J-semisimple, then obviously

$$\operatorname{Con} G \in \mathcal{V}_1 \Leftrightarrow \operatorname{Con} G \in \mathcal{V}_2. \tag{4}$$

Assume that G is not J-semisimple and (4) is valid for all finite graphs smaller than G. Let $G/J(G) = \{G_i\}_{i \in \mathcal{I}}$. By Lemma 6 $\operatorname{Con} G$ lies in $\mathcal{V}_1$ iff every $\operatorname{Con} G_i$ and $\operatorname{Con} G/J(G)$ lie in $\mathcal{V}_1$. But G_i and $G/J(G)$ are smaller than G and therefore by (4) we get that $\operatorname{Con} G$ is in $\mathcal{V}_1$ iff $\operatorname{Con} G_i, i \in \mathcal{I}$, and $\operatorname{Con} G/J(G)$ lie in $\mathcal{V}_2$, and by Lemma 6

$$\operatorname{Con} G \in \mathcal{V}_1 \Leftrightarrow \operatorname{Con} G \in \mathcal{V}_2.$$

□

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Ahto BULDAS

On uuritud graafide kongruentside võresid, milles kehtib fikseeritud võresamasus. On esitatatud täielik kirjeldus kõigi selliste graafide kohta, mille kongruentside võre rahuldab kindlat võresamasust.