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"DYNAMICAL" REPRESENTATION OF THE POINCARÉ ALGEBRA AND RARITA-SCHWINGER EQUATION

Rein SAAR[a], Ilmar OTS[a], and Leiger MÄTTAS[b]

[a] Eesti Teaduste Akadeemia Füüsika Instituut (Institute of Physics, Estonian Academy of Sciences), Riia 142, EE-2400 Tartu, Eesti (Estonia)
[b] Eesti Põllumajandusülikool (Estonian Agricultural University), Riia 12, EE-2400 Tartu, Eesti (Estonia)

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Abstract. Rarita-Schwinger spin-3/2 equation in the presence of a special external field invariantly introduced into the Poincaré algebra is given. It is shown that in this case the Rarita-Schwinger equation is free from algebraic inconsistencies.

Key words: Rarita-Schwinger equation, Poincaré algebra, "dynamical" representation.

1. INTRODUCTION

The theory of relativistic wave equations is based on the representations of the Poincaré group. However, in the presence of an external electromagnetic field introduced minimally, the Poincaré invariance is violated and the theory of higher-spin ($s \geq 1$) fields suffers serious difficulties (acausal propagation, indefinite commutators, algebraic inconsistencies). In papers [1, 2] an attempt has been made to preserve the Poincaré invariance in the presence of interaction. For this purpose the external field has been introduced invariantly into the Poincaré algebra. It has been shown that it can be done for the special form of an external field. The algebra which has been built up from the generators of the Poincaré algebra and from a specially chosen external field and which obeys the commutation relations of the Poincaré algebra, has been called "dynamical". Now, analogously to the free field theory, the new wave equations with respect to the "dynamical" representation of the Poincaré algebra can be constructed. In such a theory the particle behaves in spite of the presence of interaction, like a free particle. As a consequence, one can hope that some of the troubles existing in the ordinary theory can be avoided here.

In papers [1, 2] the Kemmer-Duffin spin-1 case in the "dynamical" representation has been investigated to some extent. In this paper, we investigate the Rarita-Schwinger spin-3/2 equation with respect to the "dynamical" Poincaré representation. We show that such a theory is free from algebraic inconsistencies.

2. ORDINARY RARITA-SCHWINGER THEORY

By using the Poincaré algebra

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= i\,(g_{\mu\sigma} M_{\nu\rho} + g_{\nu\rho} M_{\mu\sigma} - g_{\mu\rho} M_{\nu\sigma} - g_{\nu\sigma} M_{\mu\rho}),\\
[M_{\mu\nu}, P_{\sigma}] &= i\,(g_{\nu\sigma} P_{\mu} - g_{\mu\sigma} P_{\nu}),\\
[P_{\mu}, P_{\nu}] &= 0,
\end{aligned}
\tag{1}
$$

where

$$
\begin{aligned}
P_{\mu} &= i\partial_{\mu},\\
M_{\mu\nu} &= L_{\mu\nu} + S_{\mu\nu},\\
L_{\mu\nu} &= x_{\mu} P_{\nu} - x_{\nu} P_{\mu},
\end{aligned}
\tag{2}
$$

one can construct the free spin-3/2 particle Rarita-Schwinger equation:

$$(P_{\mu}\gamma^{\mu} - m)\,\psi_{\sigma}(x) = 0, \tag{3a}$$

$$\gamma_{\mu}\psi^{\mu}(x) = 0, \tag{3b}$$

$$P_{\mu}\psi^{\mu}(x) = 0. \tag{3c}$$

Usually, (3a) is called equation and the two others, subsidiary conditions. Not all equations in (3) are independent. Indeed, by multiplying Eq. (3a) from the left by γ^{σ} and by using subsidiary condition (3b) one gets condition (3c).

The minimal electromagnetic interaction in the ordinary theory is introduced by the change

$$P_{\mu} \to D_{\mu} = P_{\mu} - eA_{\mu}. \tag{4}$$

Then the system of equations (3) becomes

$$(D_{\mu}\gamma^{\mu} - m)\,\tilde{\psi}_{\sigma} = 0, \tag{5a}$$

$$\gamma_{\mu}\tilde{\psi}^{\mu} = 0, \tag{5b}$$

$$D_{\mu}\tilde{\psi}^{\mu} = 0. \tag{5c}$$

To show that Eqs. (5) are inconsistent let us multiply Eq. (5a) by D_{σ} and use the commutator

$$[D_\mu, D_\nu] = -ieF_{\mu\nu} \tag{6}$$

to get the algebraic relation

$$e\gamma_\mu \tilde{F}^{\mu\nu} \psi_\nu = 0, \tag{7}$$

which is satisfied only if

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0,$$

or

$$\tilde{\psi}_\nu = 0.$$

To avoid the algebraic inconsistence, one tries to use the equations without subsidiary conditions (the Bhabha type equation) [3–5]

$$(P_\mu \beta^\mu - m)\,\psi(p) = 0, \tag{8}$$

where

$$(\beta^\sigma)^\mu{}_\nu = g^\mu_\nu \gamma^\sigma + y_1 g^\sigma_\nu \gamma^\mu + y_2 g^{\sigma\mu}\gamma_\nu + y_3 \gamma^\sigma \gamma^\mu \gamma_\nu, \tag{9}$$

which is equivalent to Eqs. (3) if the real parameters satisfy conditions:

$$\begin{aligned} y_1 + y_2 - 2y_3 &= 0, \\ \frac{1}{3} + y_1 + y_2 + y_1(y_1 + 2y_2) &= 0. \end{aligned} \tag{10}$$

However, after introducing the minimal electromagneting interaction into (8), the acausality problems arise.

In what follows we use another way to avoid the algebraic contradiction.

3. RARITA-SCHWINGER EQUATION WITH RESPECT TO "DYNAMICAL" REPRESENTATION OF THE POINCARÉ GROUP

In [1, 2] it has been shown that the easiest way to get the "dynamical" representation of the Poincaré algebra $\tilde{p}_{1,3}$, is to use the nonsingular operator V that depends on the external field. If such an operator exists, it transforms the ordinary representation of the Poincaré algebra $p_{1,3}$ into the "dynamical" one in the following way:

$$p_{1,3} \to \tilde{p}_{1,3} = Vp_{1,3}V^{-1} = p_{1,3} + [V, p_{1,3}]\,V^{-1}. \tag{11}$$

There is no general method to prove the existence of V. However, V exists for the external field A_μ, being an arbitrary function of $kx = k_\mu x^\mu$.

$$A_\mu(x) = A_\mu(kx) \tag{12}$$

with the Lorentz gauge

$$k_\mu A^\mu = 0, \tag{13}$$

where k_μ is a lightlike 4-vector ($k^2 = 0$). As it is shown in [1, 2], one gets the simplest Poincaré invariant interaction by choosing

$$V = \exp i\,(\hbar + bG_{\mu\nu}S^{\mu\nu})\,, \tag{14}$$

where

$$\hbar = \frac{e}{kP}\int\left(\frac{e}{2}A^2 - AP\right)d(kx)\,,$$
$$b = -\frac{e}{2kP},\quad G_{\mu\nu} = k_\mu A_\nu - k_\nu A_\mu \tag{15}$$

with $kP = k_\mu P^\mu$, $AP = A_\mu P^\mu$, $A^2 = A_\mu A^\mu$.

For spin-3/2

$$S_{\mu\nu} = -ie_{\mu\nu} \otimes 1 + 1 \otimes s_{\mu\nu}, \tag{16}$$

where $(e_{\mu\nu})^\rho_{\ \sigma} = -g^\rho_\mu g_{\nu\sigma} + g_{\mu\sigma}g^\rho_\nu$ and $s_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$.

The operator V introducing the Poincaré invariant interaction can be given as

$$V = V_0\,(V_P \otimes V_D)\,, \tag{17}$$

with

$$V_0 = \exp\left\{\frac{ie}{kP}\int\left(\frac{e}{2}A^2 - AP\right)d(kx)\right\}$$

as scalar (Klein-Gordon) part,

$$V_P = e^{-2bG} = \exp\left\{-\frac{e}{2kP}(k_\mu A_\nu - k_\nu A_\mu)\,e^{\mu\nu}\right\}$$

as spin-1 (Proca) part
and

$$V_D = e^{ibG_{\mu\nu}s^{\mu\nu}} = \exp\left\{\frac{e}{4kP}(k_\mu A_\nu - k_\nu A_\mu)\gamma^\mu\gamma^\nu\right\}$$

as bispinor (Dirac) part.

Now, by using the operator V one can transform the free Poincaré algebra (2) into the "free" Poincaré algebra with respect to the "dynamical" representation:

$$P_\mu \to \pi_\mu = P_\mu - k_\mu\frac{e}{kP}\left(AP - \frac{e}{2}A^2 + \frac{1}{2}F_{\rho\sigma}S^{\rho\sigma}\right),$$

$$S_{\mu\nu} \to \sigma_{\mu\nu} = S_{\mu\nu} - \frac{e}{kP}\left\{\frac{e}{2kP}(g_{\mu\rho}k_\nu - g_{\nu\rho}k_\mu)\,k_\sigma + g_{\mu\rho}G_{\nu\sigma} - \right.$$

$$\left. - g_{\nu\rho}G_{\mu\sigma} - \frac{e}{kP}G_{\mu\nu}k_\rho A_\sigma\right\}S^{\rho\sigma}, \tag{18}$$

$$P^2 \to \pi^2 = D^2 - eF_{\mu\nu}S^{\mu\nu}.$$

In order to get Eqs. (3) in the "dynamical" representation, we must cast these into the matrix form [6]:

$$\begin{aligned}(P_\mu(1\otimes\gamma^\mu)-m)\,\psi(p) &= 0,\\ (E_{\mu\nu}\otimes\gamma^\nu)\,\psi(p) &= 0,\\ P^\mu(E_{\rho\mu}\otimes 1)\,\psi(p) &= 0,\end{aligned} \tag{19}$$

where $(E_{\mu\nu})^{\rho}_{\ \sigma} = g^{\rho}_{\mu} g_{\nu\sigma}$.

By taking V as given in Eq. (17) and writing in terms of free fields, Eq. (3) becomes explicitly as follows:

$$\{(\not{D}-m)\,g_{\mu\nu} - \frac{ie}{kP}\not{k}F_{\mu\nu}\}\,\Psi^\nu = 0, \tag{20a}$$

$$\gamma_\mu\Psi^\mu = 0, \tag{20b}$$

$$\{D_\mu - \frac{ie}{4kP}(F_{\rho\sigma}\gamma^\rho\gamma^\sigma)\,k_\mu\}\,\Psi^\mu = 0. \tag{20c}$$

The second-order equation has the form

$$\{(\not{D}^2-m^2)\,g_{\mu\rho} - 2ieF_{\mu\rho}\}\,\Psi^\rho = 0. \tag{21}$$

One can see from these equations that the simplest Poincaré invariant interaction differs from the minimal one: besides the external field contained in D_μ, there are terms with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Like in the free field case, not all Eqs. (20) are independent. By contracting Eq. (20a) with γ_μ one gets Eq. (20c) and the system of Eqs. (20) containing the "dynamical" interaction, is not inconsistent algebraically.

By applying the "dynamical" interaction transformation V to free Bhabha-type Eqs. (8), (9), one finds the spin-3/2 Bhabha-type equation in the presence of dynamical interaction:

$$\{g_{\mu\nu}(\not{D}-m) - \frac{ie}{kP}\not{k}F_{\mu\nu} + y_1\gamma_\mu L_\nu + (y_2+2y_3)\,L_\mu\gamma_\nu - $$

$$-y_3\gamma_\mu\not{D}\gamma_\nu\}\,\Psi^\nu = 0, \tag{22}$$

where $L_\mu = D_\mu - k_\mu\frac{ie}{4kP}(1\otimes F_{\rho\sigma}\gamma^\rho\gamma^\sigma)$, $\not{D} = D_\mu\gamma^\mu$.

However, it is extremely difficult to prove that this equation is equivalent to Eqs. (20) if Y_i satisfy Eq. (10), or to investigate its causality properties due to the fact that $[D_\mu, D_\nu] \neq 0$.

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POINCARÉ ALGEBRA "DÜNAAMILINE" ESITUS JA RARITA-SCHWINGERI VÕRRAND

Rein SAAR, Ilmar OTS, Leiger MÄTTAS

Rarita-Schwingeri vôrrand spinnile 3/2 on antud spetsiaalse välise välja korral, mis on viidud invariantselt Poincaré algebrasse. On näidatud, et sel juhul on Rarita-Schwingeri vôrrand vaba algebralistest vastuoludest.

"ДИНАМИЧЕСКОЕ" ПРЕДСТАВЛЕНИЕ АЛГЕБРЫ ПУАНКАРЕ И УРАВНЕНИЕ РАРИТЫ-ШВИНГЕРА

Рейн САAР, Ильмар ОТС, Лейгер МЯТТАС

Дано уравнение Рариты-Швингера для спина 3/2 в присутствии специального внешнего поля, инвариантно введенного в алгебру Пуанкаре. Показано, что в этом случае уравнение Рариты-Швингера свободно от алгебраических противоречий.