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SOME REMARKS ON RELATIVISTIC WAVE EQUATIONS

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Abstract. A modified conception of the theory of the first-order relativistic wave equations is given and in the framework of the modified theory the Kemmer–Duffin spin-1 equation is considered.

Key words: relativistic wave equations, Kemmer-Duffin equation.

1. INTRODUCTION

The fundamental role of the Poincaré group in the quantum field theory was discovered by E. P. Wigner [1]. From the point of view in which free particle states are given by unitary irreducible representations of the Poincaré group, Wigner considered the problem of representations of the group and revealed its structure. As it appears, the invariants of this group have a direct physical meaning – they determine the mass and the spin of the corresponding physical state.

On the other hand, the quantum field theory is based on the homogeneous Lorentz group, since the field operators are the Lorentz quantities. Therefore, the Poincaré representations are realized via Lorentz representations. Such realizations have generally more components than needed for a given physical state. In the free field case these superfluous components are eliminated by additional conditions not ensuing from the Poincaré group. In this sense, the relativistic wave equations can be used to define a unique mass and a unique spin.

In this paper, we modify the well-known general principles of the theory of the first-order relativistic wave equations and consider the Kemmer-Duffin spin-1 equation in this modified theory. Contrary to the ordinary procedure, we demand the validity of the discrete symmetric after the unique spin and mass conditions have been applied. Further, we do not demand the symmetry conditions to be valid algebraically, but only when applied on the solutions of equations. In this way these conditions do not restrict the parameters connected with the superfluous spins contained in the representation. In other words, using the procedure, we do not subject the superfluous components to the same symmetry properties, as the ones connected with a given spin. In fact, it seems more essential that the symmetry properties are applied only to the given spin components [2].

The main advantage of our approach is the increase of the number of arbitrary parameters in β-matrices, which offers more possibilities of treating interactions and quantization. We hope that our approach enables to avoid some well-known difficulties which exist in the ordinary theory.

The paper consists of two parts. In the first part, the modified axiomatics of the building of the first-order relativistic wave equations is given. In the second part, the Kemmer-Duffin spin-1 wave equation in the framework of the modified theory is considered.

2. GENERAL THEORY

The requirement of relativistic invariance means that to each physical system corresponds a unitary representation U of the Poincaré group $P_{1,3}$. We take the transformation property of the wave-function ψ as

$$(U(a,\Lambda)\bar{\bar{\Psi}})(p) = e^{ipa}T(\Lambda)\bar{\bar{\Psi}}(\Lambda^{-1}p), \qquad (1)$$

where $(a,\Lambda)\in P_{1,3}$ and $T(\Lambda)$ is a N-dimensional matrix representation of the Lorentz group ($\Lambda \in SO_{1,3}$). The finite-dimensional representation of the Lorentz group is generally irreducible and its spin content is determined by the decomposition

$$T_{|SO_3} = \sum_{j\in J} n_j D^{(j)}, \qquad (2)$$

where $D^{(j)}$ denotes the irreducible representation of the rotation group SO_3 and n_j, its multiplicity. Besides the invariance with respect to the restricted Poincaré group, the theory has to be invariant with respect to discrete symmetries of which only the space inversion is considered here. To make the theory space inversion invariant one must take only such representations of Lorentz group for which

$$I_r T(\Lambda) = T(\eta\Lambda\eta) I_r, \qquad (3i)$$

I_r – nonsingular.
This choice yields the transformation

$$(U(\eta)U(a,\Lambda)\psi)(\bar{p}) = e^{-ipa} I_r T(\Lambda)\psi(\Lambda^{-1}p). \qquad (3ii)$$

Here I_r is the space inversion operator corresponding to the reflection in the Minkovski space

$$I_r : p^{\mu} \to \bar{p}^{\mu} = \eta^{\mu\mu} p^{\mu},$$

with $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ as the metric tensor. It follows from Eq. (3) that the operator I_r belongs to the commutant of the representation $T_{|SO_3}$, and due to the Weyl's theorem

$$I_r = \sum_{j \in J} I_r(j) . \tag{4}$$

There is yet one restriction called H-unitary condition which is needed for the Lagrangian theories. Hence, the Lorentz representation T is assumed to be H-unitary:

$$T^{+}(\Lambda) H = H T(\Lambda^{-1}), \tag{5}$$

$H^{+} = H$, H – nonsingular.

Due to this relation the Weyl' theorem states that

$$H = \sum_{j \in J} H(j) . \tag{6}$$

To make the invariant system of functions with transformation law (1) into a single-particle theory, one needs some additional restrictions not proposed by the Poincaré group. These restrictions – the unique mass and the unique spin conditions – are put on the operators contained in wave equations. Since any n-th-order differential equation is equivalent to the first-order system of equations of the form [3–9]

$$(p_{\mu} \beta^{\mu} - \kappa)\, \psi(p) = 0, \tag{7}$$

where β^{μ} are N-dimensional matrices and κ is a non-zero constant, only the restrictions on β-matrices can be considered. Relativistic covariance restricts the structure of β-matrices, since under a homogeneous Lorentz transformation ψ transforms according to Eq. (1). Thus we must have

$$T^{-1}(\Lambda)\, \beta_{\mu}\, T(\Lambda) = \Lambda_{\mu\nu} \beta^{\nu} \tag{8}$$

if Eq. (7) is to be covariant.

Written in terms of the generators $S_{\rho\sigma}$ of the Lorentz representation T, this relation becomes

$$[\beta_{\mu}, S_{\rho\sigma}] = i(\eta_{\mu\rho} \beta_{\sigma} - \eta_{\mu\sigma} \beta_{\rho}), \tag{9}$$

or, in more detail,

$$[\beta_0, S_{ik}] = 0, \tag{10i}$$

$$[[\beta_0, S_{03}], S_{03}] = -\beta_0, \tag{10ii}$$

$$\beta_k = -i\, [\beta_0, S_{0k}] \tag{10iii}$$

($\mu, \rho, \ldots = 0, 1, 2, 3$; $i, k, \ldots = 1, 2, 3$). Thus, β_0 is the solution of Eqs. (10i) and (10ii) and β_k are determined by Eq. (10iii). Incidentally, Weyl's theorem states that

$$\beta_0 = \sum_{j \in J} \beta_0(j) \cdot \tag{11}$$

Now, if we demand that the solutions of Eq. (7) describe a particle with unique mass, i.e. if we assume

$$(p^2 - \kappa^2)\,\psi_A(p) = 0, \quad A = 1, 2, \ldots, N, \tag{12}$$

then β_μ must satisfy the condition [9]

$$(p_\mu \beta^\mu)^{n+1} = p^2 (p_\mu \beta^\mu)^{n-1} . \tag{13}$$

Thus, the spectrum of $\beta_0, \sigma(\beta_0)$, is 0, ±1 and Eq. (7) describes the particle with the rest mass m, $m^2 = p^2 = \kappa^2$. However, this mass m can occur several times.

The separation of one desired spin s is possible thanks to the decomposition (11). Taking into account the spectrum $\sigma(\beta_0)$, we conclude that Eq. (8) contains the spin s if

$$\beta_0^{n_s+1}(s) = \beta_0^{n_s-1}(s) \neq 0, \tag{14i}$$

and does not contain the spin $j \in J$ if

$$\beta_0^{n_j-1}(j) = 0 \tag{14ii}$$

for some n_s, $n_j \leq n$, i.e. if the submatrix $\beta_0(j)$ is nilpotent.

Usually the discrete symmetry conditions are put on β-matrices algebraically, and before the single mass and the single spin have been separated. It is more natural to follow a radically different procedure. That is to impose the discrete symmetry properties by demanding its validity only on the solutions of Eq. (7) with conditions (13) and (14). Thus for space inversion invariance we shall demand that

$$\bar{p}^\mu \beta_\mu I_r \psi(p) = I_r p^\mu \beta_\mu \psi(p) , \tag{15}$$

where the parity operator I_r is required to be H-unitary, i.e.

$$I_r^+ H = H I_r^{-1} .$$

Since we can apply decomposition (11), we get that

$$\beta_0(j)\, I_r(j) = I_r(j)\, \beta_0(j) \tag{16}$$

only for these $j \in J$ for which condition (14i) is fulfilled. For other spins the space inversion invariance does not give rise to any restriction.

Similarly, the hermiticity condition may be introduced by the condition

$$p^{\mu}\beta_{\mu}^{+}H\Psi(p) = Hp^{\mu}\beta_{\mu}\Psi(p). \tag{17}$$

As a result of decompositions (6) and (11), we get

$$\beta_0^{+}(j)H(j) = H(j)\beta_0(j) \tag{18}$$

only for these $j \in J$ for which condition (14i) is fulfilled. On the contrary, on the blocks $\beta_0(j)$, which obey condition (14ii), hermiticity condition (17) does not generate any restriction.

The physical meaning of the last two conditions proceeds in part from decomposition (11). Due to conditions (13) and (14) Eq. (7) yields only solutions with unique mass and spin and therefore conditions (15) and (17) mean the hermiticity and space inversion invariance of only the particle selected by (13) and (14). On the contrary, the usual algebraic conditions of discrete symmetries mean invariance under these symmetries of all the particles contained in the representation. Otherwise we do not subject the spin-s particle to the discrete symmetry properties of the spin-j particles, $s \neq j \in J$. In principle, a spin-s particle cannot be looked upon as composed by lower spin-j particles $(s \neq j)$ according to the Clebsh-Gordon procedure. A higher spin particle is in itself a phenomenon of nature. This does not mean that the nilpotent parts of β_0 are useless or superfluous: the particle is characterized by the whole β_0-matrix, and the parameters of nilpotent parts play an important role in the presence of interactions.

3. KEMMER-DUFFIN EQUATION

It is natural to wonder how the above-mentioned axioms determine the β-matrices of the relativistic wave equations. For this purpose, let us consider the Kemmer-Duffin spin-1 equation. To construct it, we depart from the representation $(1,0) \oplus (1/2, 1/2) \oplus (0,1)$. In the Gel'fand basis [4] the generators have the form

$$S^{j} = -\frac{1}{2}\varepsilon^{j}_{kl}S^{kl} = \begin{bmatrix} m^{j} & & & \\ & m^{j} & & \\ & & m^{j} & \\ & & & 0 \end{bmatrix},$$

$$S^{0j} = \begin{bmatrix} -im^{j} & & & \\ & im^{j} & & \\ & & 0 & -(k_jV)^{+} \\ & & -k_jV & 0 \end{bmatrix},$$

where

$$m^1 = -\frac{\sqrt{2}}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad m^2 = -\frac{i\sqrt{2}}{2}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad m^3 = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

are the generators of the representation $D^{(1)}$ of the rotation group SO_3. The K-matrices have been defined by Hurley [10],

$$K_1 = (1\ 0\ 0),\ K_2 = (0\ 1\ 0),\ K_3 = (0\ 0\ 1)$$

and

$$V = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 0 & -1 \\ i & 0 & i \\ 0 & -\sqrt{2} & 0 \end{bmatrix}.$$

The discrete symmetry operators, the hermitianizing matrix H and the space inversion operator I_r are expressed as

$$H = h_0\begin{bmatrix} 0 & h & & \\ h^* & 0 & & \\ & & 1 & 0 \\ & & 0 & -1 \end{bmatrix}, \quad h_0^* = h_0 \neq 0 \tag{19}$$

$$I_r = q_0\begin{bmatrix} 0 & \frac{1}{q} & & \\ q & 0 & & \\ & & 1 & 0 \\ & & 0 & -1 \end{bmatrix}, \quad q_0 q_0^* = 1, \tag{20}$$

$$(hq)^* = hq.$$

The general form of β_0, which satisfies condition of relativistic invariance (9), is [11]

$$\beta_0 = \begin{bmatrix} 0 & 0 & x_1 & \\ 0 & 0 & x_2 & \\ x_3 & x_4 & 0 & \\ & & & 0 \end{bmatrix}. \tag{21}$$

The characteristic polynomial of β_0 has the form

$$\det(\beta_0 - \lambda) = \lambda^4 (\lambda^2 - z)^3 \tag{22}$$

and the minimal polynomial is

$$\beta_0 (\beta_0^2 - z) = 0, \tag{23}$$

$$z = x_1 x_3 + x_2 x_4.$$

If we take $z = 1$, unique mass condition (13) is satisfied and the minimal polynomial guarantees the unique spin without any additional restrictions on the parameters x_j.

Now, let us consider discrete symmetries. Space inversion invariance condition (15) generates the relation

$$x_2 = q x_1. \tag{24}$$

The strong algebraic condition

$$\beta_0 I_r = I_r \beta_0$$

yields the restrictions

$$x_2 = q x_1, \quad x_3 = q x_4.$$

Hermiticity condition (17) generates the relations

$$x_3 = h^* x_2{}^*, \quad x_4 = h x^*{}_1 \tag{25}$$

which coincide with the algebraic condition

$$\beta_0^+ H = H \beta_0.$$

Thus, we may draw a conclusion that discrete symmetries (15) and (17) together coincide with the corresponding algebraic conditions. It is natural, because in the present case the lower-spin part of β_0 is trivial. As is shown in [2] for the spin-3/2 particle, the difference between ordinary algebraic conditions and modified conditions is essential.

Therefore, we have

$$\beta^0 = \frac{\sqrt{2}}{2} \begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & qx & 0 \\ \frac{1}{x} & \frac{1}{qx} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \beta^i = \frac{\sqrt{2}}{2} \begin{bmatrix} 0 & 0 & x m^j & -x (K_j V)^+ \\ 0 & 0 & -q x m^j & -q x (K_j V)^+ \\ -\frac{1}{x} m^j & \frac{1}{qx} m^j & 0 & 0 \\ \frac{1}{x} K_j V & \frac{1}{qx} K_j V & 0 & 0 \end{bmatrix}, \tag{26}$$

$$(hq)^* = hq > 0, \quad qhxx^* = 1.$$

These explicit forms of β-matrices imply the validity of the Kemmer-Duffin algebra relation

$$\beta^\mu \beta^\rho \beta^\nu + \beta^\nu \beta^\rho \beta^\mu = \eta^{\mu\rho} \beta^\nu + \eta^{\nu\rho} \beta^\mu,$$

and therefore there exists the Klein-Gordon divisor. As to the Proca equation, for its equivalence to first-order Eq. (7) with β-matrices given by (26), it is necessary to take $q = i$. In this case $h^* = -h$, and the parameters h and x are connected, as $|h||x|^2 = 1$. Therefore, the class of the first-order wave equations based on the Kemmer-Duffin algebra is larger than the Proca second-order wave equation.

For completeness, let us give β-matrices (26) in the Kemmer-Duffin basis. By somewhat tedious calculation one obtains that

$$\beta^0_{KD} = \frac{1}{2}\begin{bmatrix} 0 & 0 & -i(q+i)x & 0 \\ 0 & 0 & (q-i)x & 0 \\ \dfrac{q+i}{qx} & \dfrac{i(q-i)}{qx} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\beta^j_{KD} = \frac{1}{2}\begin{bmatrix} 0 & 0 & (q-i)xe^j & -i(q+i)xK^+_j \\ 0 & 0 & i(q+i)xe^j & (q-i)xK^+_j \\ \dfrac{i(q-i)}{qx}e^j & -\dfrac{q+i}{qx}e^j & 0 & 0 \\ -\dfrac{q+i}{qx}K_j & -\dfrac{q-i}{qx}K_j & 0 & 0 \end{bmatrix},$$

where $(e^j)^p_q = \varepsilon^{ip}_q$ and ε^{ipq} is the antisymmetric tensor, $\varepsilon^{123} = 1$. The usual choice of parameters is $q = 1$, $x = -i$.

4. CONCLUSIONS AND DISCUSSIONS

The aim of this modified theory is to enlarge the number of parameters in β-matrices. In this way we hope to avoid the difficulties that appear in the ordinary theory (acausal modes of propagation, indefinite anticommutation relations). In fact, the acausal propagation of wave appears even of the external field and is coupled in the Poincaré-invariant way [12]. Indeed, the Courant's characteristics method was used for the estimation of causality of the Kemmer-Duffin spin-1 equation with "dynamical" coupling and fixed parameters $q = i$, $x = -i$. The answer was – noncausality in spite of the fact that the second-order Proca equation with the same coupling gives rise to causal propagation. A natural question arises now whether one can obtain the causality by the variation of the parameters q and x.

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MÕNINGAID MÄRKUSI RELATIVISTLIKE LAINEVÕRRANDITE KOHTA

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On antud esimest järku relativistlike lainevôrrandite modifitseeritud käsitlus. Selle modifitseeritud teooria raames on vaadatud Kemmeri-Duffini vôrrandit spinnile 1.

НЕКОТОРЫЕ ЗАМЕЧАНИЯ О РЕЛЯТИВИСТСКИХ ВОЛНОВЫХ УРАВНЕНИЯХ

Рейн СААР, Ильмар ОТС, Ираида ЕРШОВА

Дано модифицированное представление о релятивистских волновых уравнениях первого порядка. В рамках модифицированной теории рассмотрено уравнение Кеммера-Дэффина для спина 1.