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ON THE ROBUST STABILITY OF POLYTOPES

Ülo NURGES

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Abstract. A family of stable (Schur) polynomials is generated, starting from a single stable polynomial (or polytope) by means of a transform from the unit circle into itself. The transform is linear in respect of polynomial coefficients, but nonlinear in respect of a free parameter. By repeated use of the transform and the edge theorem, a sequence of stable polytopes is produced.

Key words: robust stability, discrete-time systems, polytopes.

1. INTRODUCTION

The stability of linear control systems with structured parameter perturbations is a problem of current interest. In the continuous-time case, and with independently varying polynomial coefficients, Kharitonov's theorem [1] provide necessary and sufficient robust stability conditions. Kharitonov's theorem fails to hold for discrete-time systems of fourth and higher order [2].

The edge theorem [3] handles both the continuous-time and discrete-time systems. It is particularly convenient for polytopic perturbations, that is, for linearly interdependent polynomial coefficients. Unfortunately, and contrary to Kharitonov's theorem, the edge theorem suffers from a dimensionality curse. For those reasons the search has continued for alternative robust stability conditions and tests.

Many interesting results have been obtained for discrete-time systems by the use of a bilinear transformation and the so-called "barycentric coordinates" starting from the continuous-time case [2, 4]. Special kind of parameter perturbations have been found out which do not destroy the necessity of Kharitonov's conditions [4].

In this paper a different approach is proposed. The question is: do there exist special perturbations which do not destroy the stability of a stable system? The discrete-time case will be studied. For the sake of simplicity only one free parameter will be dealt with. The key idea is to use such a linear-fractional transformation with a free parameter on the roots of a polynomial, which maps the unit circle into itself [5].

A polynomial is called the Schur (stable) polynomial if all its roots lie inside the unit circle. Our aim is to generate a family of Schur polynomials starting from a single stable polynomial (or polytope). This family of polynomials is not an interval polynomial, because the transform is nonlinear in respect of the free parameter. Fortunately, the transform is linear in respect of the polynomial coefficients. Hence we can generate different kind of stable polyhedrons by the use of the edge theorem.

The paper is composed as follows. In Section 2, the transform with a free parameter is recalled which does not alter the stability of discrete polynomials [5]. In Section 3, the stability conditions for polytopes with a free parameter are obtained by the use of the edge theorem. The transform with a free parameter works as a generating rule. By repeated use of it we can produce a sequence of stable polytopes starting from a stable one.

2. STABLE POLYNOMIALS WITH A FREE PARAMETER

Let us consider a polynomial

$$a(z) = \sum_{i=0}^{n} a_i z^i, \qquad a_i \in R,$$

with roots λ_i in the unit circle, $|\lambda_i| < 1$, $i = 1,\dots,n$. We are looking for another polynomial

$$b(z;\zeta, a) = \sum_{i=0}^{n} b_i(\zeta, a)\, z^i \tag{1}$$

with coefficients $b_i(\zeta, a) \in R$ depending linearly on the coefficients a_j, $j = 0,\dots,n$ and nonlinearly on a single parameter ζ

$$b_i(\zeta, a) = \sum_{j=0}^{n} f_{ij}(\zeta)\, a_j. \tag{2}$$

Our aim is to choose such functions $f_{ij}(\zeta)$ that the polynomial $b(z;\zeta,a)$ will be stable if the polynomial $a(z)$ is stable and if the parameter ζ is placed in an interval $\zeta \in [\underline{\zeta}, \bar{\zeta}]$.

It can be easily shown [5] that the polynomial (1) with coefficients (2) has roots

$$\mu_s(\zeta) = \frac{\lambda_s - \zeta}{1 - \zeta\lambda_s}, \quad s = 1, \dots, n, \tag{3}$$

if

$$f_{ij}(\zeta) = \sum_{k=0}^{j} \binom{n-j}{i-k} \binom{j}{k} \zeta^{i+j-2k}, \tag{4}$$

where $\binom{j}{k}$ are binomial coefficients. The linear-fractional mapping (3) transforms the unit circle into itself and $b_i \in R$ if $\zeta \in (-1, 1)$.

P r o p o s i t i o n 1. [5] *The polynomial* (1) *with real coefficients* $b_i(\zeta, a)$ *defined by* (2) *and* (4) *will be Schur stable if the polynomial a(z) is Schur stable and if the free parameter* ζ *lies in the interval* $\zeta \in (-1, 1)$.

Let us denote the mapping $S: R^n \times R \to R^n$ from the coefficient space of $a(z)$ with a real free parameter ζ into the coefficient space of $b(z)$ as $S(a, \zeta) = b(z; \zeta, a)$. Here and in the following we shall use transform (2) with $f_{ij}(\zeta)$ from (4). For $\zeta = 0$, $S(a,0) = b(z; 0, a) = a(z)$.

3. STABLE POLYTOPES WITH A FREE PARAMETER

Let us now consider a polytope

$$P(a, b) = \gamma a(z) + (1-\gamma) b(z), \quad \gamma \in [0, 1]$$

of polynomials $a(z)$ and $b(z)$. The polytope $P(a,b)$ is said to be stable if the polynomials $p(z; a, b, \gamma) \in P(a, b)$ are stable for all $\gamma \in [0, 1]$.

Applying the transform (2) to the polynomials $a(z)$ and $b(z)$, we obtain a polytope

$$P[S(a, \zeta), S(b, \zeta)] = \gamma S(a, \zeta) + (1-\gamma) S(b, \zeta), \quad \gamma \in [0, 1].$$

P r o p o s i t i o n 2. *The polytope* $P[S(a,\zeta), S(b,\zeta)]$ will *be stable for any* ζ *from the interval* $\zeta \in (-1, 1)$ *if the polytope* $P(a,b)$ is stable.

P r o o f: The proof is straightforward because the transform (2) is linear with respect to the coefficients a_j, $j = 0,\ldots,n$. For some fixed ζ_f and γ_f, $\zeta_f \in (-1, 1)$, $\gamma_f \in [0, 1]$ we obtain the polynomial

$$p[z; S(a, \zeta_f), S(b, \zeta_f), \gamma_f] = \gamma_f \sum_{i=0}^{n} \sum_{j=0}^{n} f_{ij}(\zeta_f) a_j z^i +$$

$$+(1-\gamma_f) \sum_{n=0}^{n} \sum_{j=0}^{n} f_{ij}(\zeta_f) b_j z^i = \sum_{i=0}^{n} \sum_{j=0}^{n} f_{ij}(\zeta_f) [\gamma_f a_j + (1-\gamma_f) b_j] z^i =$$

$$= S[p(z; a, b, \gamma_f), \zeta_f].$$

The polynomial $p(z;a,b,\gamma_f)$ is stable by assumption. By proposition 1 the polynomial $S[p(z;a,b,\gamma_f),\zeta_f]$ is then also stable. Hence, the polynomial $p[z; S(a, \zeta_f), S(b, \zeta_f), \gamma_f] \in P[S(a, \zeta), S(b, \zeta)]$ is stable for all $\zeta_f \in (-1, 1)$ and $\gamma_f \in [0, 1]$. Δ

Let us now consider a polytope

$$P[a, S(a, \zeta)] = \gamma a(z) + (1-\gamma) a(z; \zeta),$$

where the polynomials $a(z)$ and $a(z;\zeta) = S[a(z),\zeta]$ are related by transform

(2). Taking into account the properties of this transform, we can formulate the following proposition.

P r o p o s i t i o n 3. *Suppose that the polytope $P[a,S(a,\zeta)]$ is stable for some ζ, $\zeta \in (-1,1)$. Then*
1) *the polytope $P[S(a,\zeta), S(a,\zeta_1)]$ will be stable, where $\zeta_1 = 2\zeta/(1+\zeta^2)$;*
2) *the polytope $P[a,S(a,-\zeta)]$ will be stable;*
3) *the polytopes $P[S(a,\zeta_k),S(a,\zeta_{k+1})]$ will be stable, where*

$$\zeta_{k+1} = \frac{2\zeta_k}{1+\zeta_k^2},\ k = 0, 1, 2, \dots \quad (5)$$

P r o o f: Let us denote $b(z) = S[a(z),\zeta]$. Then, by proposition 2, the polytope $P[S(a,\zeta),S(b,\zeta)]$ will be stable if the polytope $P[a,S(b,\zeta)]$ is stable. To prove the first assertion we have to show that $P[S(a,\zeta),S(a,\zeta_1)] = P[S(a,\zeta), S(b,\zeta)]$.
By the twofold use of transform (3), we obtain

$$\mu_i(\zeta,\zeta_f) = \frac{\mu_i(\zeta)-\zeta_f}{1-\zeta_f\mu_i(\zeta)} = \frac{\lambda_i-\tilde{\zeta}}{1-\tilde{\zeta}\lambda_i},$$

or $S\{S[a(z),\zeta],\zeta_f\} = S[a(z),\tilde{\zeta}]$, where

$$\tilde{\zeta} = \frac{\zeta+\zeta_f}{1+\zeta\zeta_f}.$$

For $\zeta_f = \zeta$ we obtain $\tilde{\zeta} = \frac{2\zeta}{1+\zeta^2} = \zeta_1$ and we have proved the first assertion.

Let now $\zeta_f = -\zeta$. Then $S\{S[a(z),\zeta],-\zeta\} = S[a(z),0] = a(z)$. From the stability of the polytope $P[a,S(a,\zeta)]$ follows, by the proposition 2, that the polytope $P\{S(a,-\zeta), S[S(a,\zeta),-\zeta]\} = P[S(a,-\zeta),a]$ is stable. Hence the second assertion holds.

By repeated use of the first assertion and proposition 2, we can easily prove the third assertion. Δ

Using the edge theorem [3], we can generalize the assertions of proposition 3. Let us denote a polytope of m polynomials as follows

$$P(a_1,\dots,a_m) = \gamma_1 a_1(z) + \dots + \gamma_m a_m(z), \qquad \sum_{k=1}^{m} \gamma_k = 1,$$

and the repeated use of transform (2)

$$S^{k+1}(a,\zeta) = S(S^k(a,\zeta),\zeta), \qquad k = 1, 2, \dots .$$

P r o p o s i t i o n 4. *Suppose that the polytopes $P[a,S(a,\zeta)]$ and $P[a,S^2(a,\zeta)]$ are stable for some ζ, $\zeta \in (-1,1)$. Then*
1) *the polytope $P[a,S(a,\zeta),S^2(a,\zeta)]$ is stable,*
2) *the polytope $P[a,S(a,\zeta),S(a,-\zeta)]$ is stable,*
3) *the polytopes $P[S^k(a,\zeta),S^{k+1}(a,\zeta),S^{k+2}(a,\zeta)]$ are stable.*

P r o o f: From the stability of the polytope $P[a,S(a,\zeta)]$ it follows, by propositions 3, that the polytope $P[S(a,\zeta),S^2(a,\zeta)]$ is stable. Now all the exposed edges of the triangle (polytope) $P[a,S(a,\zeta),S^2(a,\zeta)]$ are stable. Therefore, by the edge theorem the whole triangle is robustly stable, i.e. the polytope $P\left[a, S(a,\zeta), S^2(a,\zeta)\right]$ is stable.

In a similar way we can prove the assertions 2) and 3), using, accordingly, the assertions 2) and 3) of proposition 3. Δ

Increasing the number of edges we can obtain similar statements for polytopes of dimension k. For example, if the polytopes $P[a,S(a,\zeta)], P[a,S^2(a,\zeta)], \dots, P[a,S^k(a,\zeta)]$ are stable, then the polytopes (polyhedrons) $P[S^m(a,\zeta), \dots, S^{m+k}(a,\zeta)]$, $m = 1, 2, \dots$ are stable.

By the use of propositions 2 – 4 we can prove the stability of various polygons and polyhedrons starting from the stability assumption of the polytopes $P(a_1,a_2)$, $P[a_1,S(a_1,\zeta)]$, $P[a_2,S(a_2,\zeta)]$.

The case of special interest arises if $a_2(z) \equiv 0$. The origin is a fixed-point of the transform (2) for any $\zeta \in (-1,1)$, i.e. $S^k(0,\zeta) \equiv 0$, $k = 1, 2, \dots$ As a corollary of propositions 2 and 3, we obtain then the following.

P r o p o s i t i o n 5. *Suppose that the polytopes $P(a,0)$ and $P[a,S(a,\zeta)]$ are stable for some ζ, $\zeta \in (-1,1)$. Then*

1) *the polytope $P[0,a,S(a,\zeta)]$ is stable,*
2) *the polytope $P[0,a,S(a,-\zeta)]$ is stable,*
3) *the polytopes $P[0,S^k(a,\zeta),S^{k+1}(a,\zeta)]$, $k = 1, 2, \dots$ are stable.*

E x a m p l e. Let us start with a (Schur) stable polynomial

$$a(z) = z^2 + 0.7z + 0.1,$$

and let $\zeta = 0.2$. Then we have, according to (1), $\zeta_1 = 0.38$, $\zeta_2 = 0.69$, $\zeta_3 = 0.94$, and according to (2) – (4),

$$b(z,\zeta) = 1.14z^2 + 1.17z + 0.28,$$

$$b(z,\zeta_1) = 1.28z^2 + 1.68z + 0.51,$$

$$b(z,\zeta_2) = 1.53z^2 + 2.58z + 1.06,$$

$$b(z,\zeta_3) = 1.75z^2 + 3.25z + 1.6,$$

$$b(z,-\zeta) = 0.86z^2 + 0.29z.$$

By proposition 1 the polynomials $b(z,\zeta)$, $b(z,\zeta_1)$, $b(z,\zeta_2)$, $b(z,\zeta_3)$ and $b(z,-\zeta)$ are (Schur) stable.

Consider now the polytope

$$P[a,b(\zeta)] = (1.14 - 0.14\gamma)z^2 + (1.17 - 0.47\gamma)z + 0.28 - 0.17\gamma, \quad \gamma \in [0,1].$$

The polytope $P[a,b(\zeta)]$ will be stable if the matrix $H(a,b) = H(a)H^{-1}(b)$ has no real eigenvalues $\lambda(H)$ in $(-\infty, 0)$ [6], where

$$H(a) = \begin{bmatrix} a_n & a_{n-1} & \dots & a_3 & a_2 - a_0 \\ 0 & a_n & \dots & a_4 - a_0 & a_3 - a_1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & -a_0 & \dots & a_n - a_{n-4} & a_{n-1} - a_{n-3} \\ -a_0 & -a_1 & \dots & -a_{n-3} & a_n - a_{n-2} \end{bmatrix}.$$

We have

$$H(a) = \begin{bmatrix} 1 & 0.9 \\ -0.1 & 0.9 \end{bmatrix}, \quad H[b(\zeta)] = \begin{bmatrix} 1.14 & 0.86 \\ -0.28 & 0.86 \end{bmatrix},$$

$$H[a,b(\zeta)] = \begin{bmatrix} 0.91 & 0.14 \\ 0.14 & 0.91 \end{bmatrix}, \quad \begin{matrix} \lambda_1(H) = 1.05, \\ \lambda_2(H) = 0.78. \end{matrix}$$

Hence the polytope $P[a,b(\zeta)]$ is stable. By proposition 3, the polytopes

$$P[b(\zeta), b(\zeta_1)] = (1.28 - 0.14\gamma)z^2 + (1.68 - 0.51\gamma)z + 0.51 - 0.23\gamma,$$

$$P[b(\zeta_1), b(\zeta_2)] = (1.53 - 0.25\gamma)z^2 + (2.58 - 0.90\gamma)z + 1.06 - 0.55\gamma,$$

$$P[b(\zeta_2), b(\zeta_3)] = (1.75 - 0.22\gamma)z^2 + (3.25 - 0.67\gamma)z + 1.60 - 0.54\gamma,$$

$$P[a, b(-\zeta)] = (1.00 - 0.14\gamma)z^2 + (0.70 - 0.41\gamma)z + 0.28(1 - \gamma),$$

$\gamma \in [0, 1]$ are also stable.

For the polytope

$$P[a, b(\zeta_1)] = (1.28 - 0.28\gamma)z^2 + (1.68 - 0.98\gamma)z + 0.51 - 0.41\gamma,$$

we have

$$H[a, b(\zeta_1)] = \begin{bmatrix} 0.89 & 0.28 \\ 0.28 & 0.89 \end{bmatrix}, \quad \begin{matrix} \lambda_1(H) = 0.61, \\ \lambda_2(H) = 1.17. \end{matrix}$$

Hence the polytope $P[a,b(\zeta_1)]$ is stable. Since the polytope $P[a,b(\zeta)]$ is also stable, then, by proposition 4, the polytopes

$$P[a, b(\zeta), b(\zeta_1)] = (\gamma_1 + 1.14\gamma_2 + 1.28\gamma_3)z^2 + (0.7\gamma_1 + 1.17\gamma_2 + 1.68\gamma_3)z + 0.1\gamma_1 + 0.28\gamma_2 + 0.51\gamma_3,$$

$$P[a, b(\zeta), b(-\zeta)] = (\gamma_1 + 1.14\gamma_2 + 0.86\gamma_3)z^2 + (0.7\gamma_1 + 1.17\gamma_2 + 0.29\gamma_3)z + 0.1\gamma_1 + 0.28\gamma_2,$$

$$P[b(\zeta), b(\zeta_1), b(\zeta_2)] = (1.14\gamma_1 + 1.28\gamma_2 + 1.53\gamma_3) z^2 + (1.17\gamma_1 + 1.68\gamma_2 + 2.58\gamma_3) z + 0.28\gamma_1 + 0.51\gamma_2 + 1.06\gamma_3,$$

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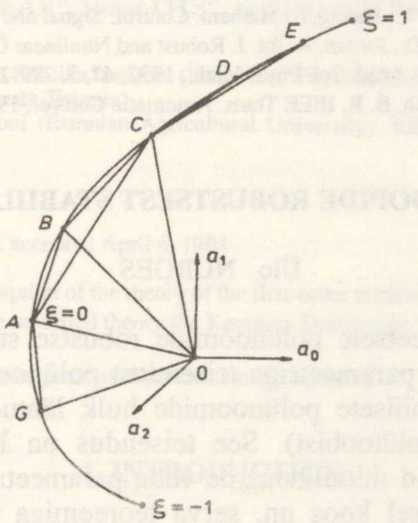
$\gamma_1 + \gamma_2 + \gamma_3 = 1$, are stable.



Stable polynomials and polytopes generated by the linear-fractional transform with a free parameter ζ.

Figure illustrates the example. In coefficient space the polynomial $a(z)$ is represented by the point A. Since A is stable, the curve $GABCDE$ is stable by proposition 1. The polytope $P[a,b(\zeta)]$ is represented by the line segment AB. By proposition 3, the line segments BC, CD, DE, AG are stable if AB is stable. The polytope $P[a,b(\zeta),b(\zeta_1)]$ is represented by the triangle ABC. By proposition 4, the triangles ABC, AGB, BCD, ... are stable if the line segments AB and AC are stable. By proposition 5, the triangles AOB, AOG, BOC, COD, ... are stable if line segments AO and AB are stable.

4. CONCLUSIONS

Some sufficient stability conditions for the polytopes of discrete polynomials are obtained by means of a transform from the unit circle into itself. This transform works as a generating rule. By the repeated use of it and the edge theorem, a sequence of stable polytopes can be produced starting from a stable one.

In this paper only one parameter ζ is allowed to vary independently. This restriction is not principial – more complicated transforms with many free parameters exist which map the unit circle into itself. In the next paper we will consider the problem with the multiparametric stable transform.

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POLÜTOOPIDE ROBUSTSEST STABIILSUSEST

Ülo NURGES

On uuritud diskreetsete polünoomide robustse stabiilsuse probleemi. Kasutades ühe vaba parameetriga teisendust polünoomi juurte ühikringis on genereeritud stabiilsete polünoomide hulk lähtudes ühest stabiilsest polünoomist (vôi polütoobist). See teisendus on lineaarne polünoomi kordajate suhtes, kuid mittelineaarne vaba parameetri suhtes. Teisenduse korduval rakendamisel koos nn. serva teoreemiga on leitud stabiilsete polütoopide perekond.

О РОБАСТНОЙ УСТОЙЧИВОСТИ ПОЛИТОПОВ

Юло НУРГЕС

Изучается проблема робастной устойчивости дискретных многочленов. С помощью преобразования в единичном круге корней многочлена с одним свободным параметром генерируется множество устойчивых многочленов исходя из одного устойчивого многочлена (или политопа). Это преобразование линейное относительно коэффициентов многочлена, и нелинейное относительно свободного параметра. С помощью так наз. теоремы ребра и многократного преобразования найдено семейство устойчивых политопов.