Proc. Estonian Acad. Sci. Phys. Math., 1994, **43**, 2, 64 – 78
https://doi.org/10.3176/phys.math.1994.2.02

DISCRETE-TIME MODELS OF A NONLINEAR CONTINUOUS-TIME SYSTEM

Ülle KOTTA

Eesti Teaduste Akadeemia Küberneetika Instituut (Institute of Cybernetics, Estonian Academy of Sciences), Akadeemia tee 21, EE-0026 Tallinn, Eesti (Estonia)

Presented by Ü. Jaaksoo

Received January 20, 1994; accepted April 6, 1994

Abstract. The paper studies the problem of sampling the multi-input analytic nonlinear continuous-time (CT) system. It derives the exact as well as the τth-order approximate (with respect to sampling period T) discrete-time step-invariant model of CT system.

Key words: nonlinear system, sampled-data system, digital control.

1. INTRODUCTION

There are two basic approaches for designing a discrete-time (DT) control law for continuous-time (CT) plant. The first approach which is equivalent to the classical Euler method and is known to be lacking accuracy, is to design a control law in the CT domain and then to discretize it. One of the advantages of this approach is the wide variety of existing design techniques which are available in the CT domain. Of course, the approach is efficient only when the sampling period T is sufficiently small with respect to the plant (process) dynamics. Even in the case of small T, this approach is not a good choice in case of an arbitrary design technique in the CT domain. When precise requirements are imposed on the input-output behaviour, as in case of input-output decoupling, input-output linearization and model matching, the control objectives cannot be achieved even at given time instants. The obtained accuracy may be unsatisfactory and the error will rise as the sampling period increases. This means that the outputs of the plant under continuous and discretized control schemes may become quite different in time. The method which is usually employed is that of neglecting the error due to sampling by increasing the sampling frequency. However, increasing the sampling frequency is not always possible or desirable.

The second approach is to obtain first a (exact or approximate) DT model of the CT plant, and then to design a controller in the DT domain.

This is, in principle at least, preferable since it deals directly with the issue of time sampling. Provided we use an exact DT model, the latter approach guarantees exact achieving the control objectives at given time instants. This approach is especially suitable for the cases when rapid sampling is impossible and/or precise requirements are imposed on the input-output behaviour. Of course, in the case of latter approach one should not forget about the possible loss of design flexibility caused by the fact that only particular type of input is allowed. Usually one uses the step input, i.e. the control input is required to be constant between the sampling times. It may happen that certain properties of a continuous-time system will be not preserved under sampling. This yields that if the applicability of a given design method depends upon these properties of the system, then a DT model of a CT plant is not suitable for controller design via the given method.

The exact step invariant DT models have been earlier obtained for single-input (SI) bilinear system [1], for SI linear-analytic system [2], for multi-input (MI) linear-analytic system [3], and for general analytic SI nonlinear system [4, 5]. In [6] the programme written with the aid of computer algebra system REDUCE for computing DT model of MI linear-analytic system has been presented. Our objective in this paper is to derive the exact as well as the τth-order approximate step invariant DT model of multi-input analytic nonlinear CT system, i.e. to generalize the results of [1–5] to a larger class of systems. While our solution is similar to the one given by Monaco and Normand-Cyrot [4, 5] in the sense that the derivation of DT model is based on the representation of the solution of the continuous-time system in terms of a formal Lie exponential series [7], and that our formulas and theirs give the same DT models when applied to any concrete SI system, it has several important differences. In particular, we compute recursively the τth-order approximate DT model on the bases of (τ-1)th-order model. Moreover, the DT models in [4, 5] have been obtained as a direct consequence of certain combinatorial formula which allow to study several properties of sampled-data systems such as controllability and observability. In our solution we have tried to avoid combinatorics, shuffle products and *ad* operators, since these are not important if we want just to compute the sampled-data model.

2. EXACT STEP-INVARIANT DISCRETE-TIME MODEL OF A NONLINEAR CONTINUOUS-TIME PLANT

Consider a CT nonlinear plant, described by the differential equations

$$\dot{x} = f(x, u), \tag{1}$$

where the state $x \in R^n$, the control input $u \in R^m$, and $f: R^{n+m} \to R^n$ is an analytic function.

The exact step-invariant DT model of CT system is defined as the one whose response to a step input (i.e. the type of control usually available under digital control)

$$u(kT+t) = u(kT)\,, \quad 0 \le t < T, \tag{2}$$

is identical to that of the CT system at discrete instants of time. The step invariant model is one of the most popular for implementing DT control systems. This is because the step invariant model corresponds to using the zero-order-hold and the sampler, both of which are readily available, and because it is one of the most simple models in state-space formulation. The other DT models include impulse-invariant model, mapping models and the matched z-transform model [8].

Denote the ith component of vector-valued function $f(x,u)$ by $f_i(x,u)$, and define the Lie differential operator $L_{f(\bullet,u)}$ associated with the function $f(\bullet,u)$ as follows

$$L_{f(\bullet,u)} = \sum_{i=1}^{n} f_i(\bullet,u)\frac{\partial}{\partial x_i}.$$

In case of r-multiple composition of differential operators, the following notation is used

$$L^{r}_{f(\bullet,u)} = L_{f(\bullet,u)}(L^{r-1}_{f(\bullet,u)})\,, \quad r \ge 2\,,$$

and $L^{0}_{f(\bullet,u)}$ is defined as identity operator Id. Note that the differential operator $L^{r}_{f(\bullet,u)}$, $r \ge 0$, acts on the (scalar – or vector-valued) function defined on R^n.

Like [2-5], our derivation of DT model is based on the representation of the solution of the differential Eq. (1) in terms of a formal Lie exponential series [7]

$$x(kT+t) = \sum_{r\ge 0}^{k} \frac{t^r}{r!} L^{r}_{f(\bullet,u(kT))}\, x|_{x(kT)}\,, \quad 0 \le t < T, \tag{3}$$

where $|_{x(kT)}$ denotes the evaluation at $x(kT)$.

As we are intrested in the states x only at sampling instant $kT+T$, we obtain from (3) for $t = T$

$$x(kT+T) = \sum_{r\ge 0} \frac{T^r}{r!} L^{r}_{f(\bullet,u(kT))}\, x|_{x(kT)} := F(x(kT), u(kT))\,. \tag{4}$$

Neglecting in (4) the terms of order $O(T^2)$, the Euler discretization scheme is obtained

$$x(kT+T) = x(kT) + Tf(x(kT),u(kT))\,.$$

Discretizing the CT plant more accurately than Euler method does, we need to take into account higher order terms in T.

Note that the analytic function $f(\bullet,u)$ can be expanded in a Taylor series around a point $u = u^0 = (u_1{}^0,\dots,u_m{}^0)^{\mathrm{T}}$

$$f(x,u) = f_0(x) + \sum_{i_1=1}^{m} f_{i_1}(x)\,v_{i_1} + \sum_{i_1,i_2=1}^{m} f_{i_1 i_2}(x)\,v_{i_1} v_{i_2} + \dots$$

$$\ldots + \sum\nolimits_{i_1, \ldots, i_s = 1}^{m} f_{i_1 \ldots i_s} v_{i_1} \ldots v_{i_s} + \ldots , \tag{5}$$

where

$$v_i = u_i - u_i^0, \quad i = 1, \ldots, m,$$

$$f_0(x) = f(x, u^0), \qquad f_{i_1 \ldots i_s}(x) = \frac{1}{s!} \frac{\partial^s f(x, u)}{\partial u_{i_1} \ldots \partial u_{i_s}} \bigg|_{u = u^0}, \qquad s \geq 1.$$

Taking into account (5), $L_{f(\bullet,u)}$ can be represented in the following form

$$L_{f(\bullet, u)} = L_{f_0} + \sum\nolimits_{i_1 = 1}^{m} v_{i_1} L_{f_{i_1}} + \sum\nolimits_{i_1, i_2 = 1}^{m} v_{i_1} v_{i_2} L_{f_{i_1 i_2}} + \ldots$$

$$\ldots + \sum\nolimits_{i_1, \ldots, i_s = 1}^{m} v_{i_1} \ldots v_{i_s} L_{f_{i_1 \ldots i_s}} + \ldots . \tag{6}$$

Moreover, straightforward but tedious computation shows that

$$L^r_{f(\bullet, u)} = L^r_{f_0} + \sum_{i_1 = 1}^{m} v_{i_1} \sum_{\substack{c_1 + c_2 = r - 1 \\ c_1 \geq 0, c_2 \geq 0}} L^{c_1}_{f_0} L_{f_{i_1}} L^{c_2}_{f_0} + \sum_{i_1, i_2 = 1}^{m} v_{i_1} v_{i_2} \times$$

$$\times \left(\sum_{\substack{c_1 + c_2 = r - 1 \\ c_1 \geq 0, c_2 \geq 0}} L^{c_1}_{f_0} L_{f_{i_1 i_2}} L^{c_2}_{f_0} + \sum_{\substack{c_1 + \ldots + c_3 = r - 2 \\ c_1 \geq 0, \ldots, c_3 \geq 0}} L^{c_1}_{f_0} L_{f_{i_1}} L^{c_2}_{f_0} L_{f_{i_2}} L^{c_3}_{f_0} \right) +$$

$$+ \sum_{i_1, i_2, i_3 = 1}^{m} v_{i_1} v_{i_2} v_{i_3} \Bigg[\sum_{\substack{c_1 + c_2 = r - 1 \\ c_1 \geq 0, c_2 \geq 0}} L^{c_1}_{f_0} L_{f_{i_1 i_2 i_3}} L^{c_2}_{f_0} +$$

$$+ \sum_{\substack{c_1 + \ldots + c_3 = r - 2 \\ c_1 \geq 0, \ldots, c_3 \geq 0}} \left(L^{c_1}_{f_0} L_{f_{i_1 i_2}} L^{c_2}_{f_0} L_{f_{i_3}} L^{c_3}_{f_0} + L^{c_1}_{f_0} L_{f_{i_1}} L^{c_2}_{f_0} L_{f_{i_2 i_3}} L^{c_3}_{f_0} \right) +$$

$$+ \sum_{\substack{c_1 + \ldots + c_4 = r - 3 \\ c_1 \geq 0, \ldots, c_4 \geq 0}} L^{c_1}_{f_0} L_{f_{i_1}} L^{c_2}_{f_0} L_{f_{i_2}} L^{c_3}_{f_0} L_{f_{i_3}} L^{c_4}_{f_0} \Bigg] + \ldots$$

$$\ldots + \sum_{i_1,\ldots,i_s=1}^{m} v_{i_1}\ldots v_{i_s}\Big[\sum_{\substack{c_1+c_2=r-1\\ c_1\ge 0,\, c_2\ge 0}} L_{f_0}^{c_1} L_{f_{i_1\ldots i_s}} L_{f_0}^{c_2} \; +$$

$$+ \sum_{\substack{c_1+\ldots+c_3=r-2\\ c_1\ge 0,\ldots,c_3\ge 0}} \; \sum_{j_1=1}^{s-1} L_{f_0}^{c_1} L_{f_{i_1\ldots i_{j_1}}} L_{f_0}^{c_2} L_{f_{i_{j_1+1}\ldots i_s}} L_{f_0}^{c_3} \; +$$

$$+ \sum_{\substack{c_1+\ldots+c_4=r-3\\ c_1\ge 0,\ldots,c_4\ge 0}} \; \sum_{j_1=1}^{s-2} \; \sum_{j_2=j_1+1}^{s-1} L_{f_0}^{c_1} L_{f_{i_1\ldots i_{j_1}}} L_{f_0}^{c_2} L_{f_{i_{j_1+1}\ldots i_{j_2}}} \times$$

$$\times \; L_{f_0}^{c_3} L_{f_{i_{j_2+1}\ldots i_s}} L_{f_0}^{c_4} + \ldots + \sum_{\substack{c_1+\ldots+c_{s+1}=r-s\\ c_1\ge 0,\ldots,c_{s+1}\ge 0}} L_{f_0}^{c_1} L_{f_{i_1}} L_{f_0}^{c_2} L_{f_{i_2}} \ldots L_{f_{i_s}} L_{f_0}^{c_{s+1}} \Big] . \quad (7)$$

Now it is clear that $F(x(kT),u(kT))$ takes the form

$$F(x(kT),u(kT)) = F_0(x(kT)) + \sum\nolimits_{i_1=1}^{m} F_{i_1}(x(kT))\, v_{i_1}(kT) \; +$$
$$+ \sum\nolimits_{i_1,i_2=1}^{m} F_{i_1 i_2}(x(kT))\, v_{i_1}(kT)\, v_{i_2}(kT) + \ldots$$
$$\ldots + \sum\nolimits_{i_1,\ldots,i_s=1}^{m} F_{i_1\ldots i_s}(x(kT))\, v_{i_1}(kT)\ldots v_{i_s}(kT) + \ldots \quad (8)$$

Of course, (8) may only be defined for T sufficiently small unless the vector field $f(\bullet,u)$ in (1) is complete [9].

Taking into account (4) – (7), the vector-valued functions $F_0(x)$ and $F_{i_1\ldots i_k}(x)\,, k\ge 1,\, i_1,\ldots,i_k \in \{1,\ldots,m\}$ are defined as follows

$$F_0(x) = \sum\nolimits_{r=0}^{\infty} \frac{T^r}{r!} L_{f_0}^{r} x \; ,$$

$$F_{i_1\ldots i_p} x = \sum_{k=1}^{\infty}\sum_{r=0}^{\infty} \frac{T^{r+k}}{(r+k)!} \sum_{\substack{c_1+\ldots+c_{k+1}=r\\ c_1\ge 0,\ldots,c_{k+1}\ge 0}} \left(\sum_{v=1}^{k-1} \; \sum_{j_v=j_{v-1}+1}^{p-(k-v)} L_{f_0}^{c_1} L_{f_{i_{j_0+1}\ldots i_{j_1}}} L_{f_0}^{c_2} L_{f_{i_{j_1+1}\ldots i_{j_2}}} L_{f_0}^{c_3} \ldots \right) L_{f_0}^{c_k} L_{f_{i_{j_{k-1}+1}\ldots i_{j_k}}} L_{f_0}^{c_{k+1}} x \, , \quad (9)$$

where $j_0 := 0,\;\; j_k := p$.

Special case of single-input systems

Consider the special case of nonlinear plant (1) where $u \in R$, i.e. where $m = 1$. Then in Eqs. (5), (6) for every j, $i_j = 1$. Let us introduce the shorter notations

$$F_{i_1 \dots i_p} = F^p, \quad f_{i_1 \dots i_k} = f^k .$$

In these notations

$$f_{i_{j_{s-1}+1} \dots i_{j_s}} = f^{\,j_s - j_{s-1}} .$$

Moreover, denote for $s = 1, \dots, p$,

$$i_s = j_s - j_{s-1} . \tag{10}$$

By Eqs. (10) and (9), for $s = 1, \dots, p$, $i_s \geq 1$, and $i_1 + i_2 + \dots + i_k = j_k = p$. So, from Eqs. (8) and (9) we finally obtain

$$x(kT+T) = F_0(x(kT)) + F^1(x(kT))\,v(kT) + F^2(x(kT))\,v^2(kT) + \dots$$
$$\dots + F^p(x(kT))\,v^p(kT) + \dots ,$$

where for $p \geq 1$

$$F^p(x) = \sum_{k=1}^{\infty} \sum_{r=0}^{\infty} \frac{T^{r+k}}{(r+k)!} \sum L_{f_0}^{c_1} L_{f^{j_1}} L_{f_0}^{c_2} L_{f^{j_2}} \dots L_{f^{j_k}} L_{f_0}^{c_{k+1}} x \tag{11}$$

and the last summation is taken over all sets of integers $c_1 \geq 0, \dots, c_{k+1} \geq 0$, $j_1 \geq 1, \dots, j_k \geq 1$ such that the equalities $c_1 + \dots + c_{k+1} = r$ and $j_1 + \dots + j_k = p$ hold.

Special case of linear-analytic systems

Consider the case when the nonlinear plant is described by equations of the following form

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x)\, u_i . \tag{12}$$

In the case of model (12) $L_{f_{i_1 \dots i_s}} = 0$, if only $s \geq 2$ and, consequently, in formulas (9) only the term, corresponding to $k = p$ is different from zero. So, Eq. (9) yield (see also [3])

$$F_{i_1 \dots i_p}(x) = \sum_{r=0}^{\infty} \frac{T^{r+p}}{(r+p)!} \sum_{\substack{c_1 + \dots + c_{p+1} = r \\ c_1 \geq 0, \dots, c_{p+1} \geq 0}} L_{f_0}^{c_1} L_{f_{i_1}} L_{f_0}^{c_2} L_{f_{i_2}} \dots L_{f_{i_p}} L_{f_0}^{c_{p+1}} x . \tag{13}$$

3. APPROXIMATE STEP-INVARIANT DISCRETE-TIME MODEL OF A NONLINEAR CONTINUOUS-TIME PLANT

The exact step-invariant DT model (4), (8), (9) of a nonlinear CT plant (1) is defined in terms of the infinite series both with respect to the sampling period T and the control input v. So, in the general case, the exact step-invariant DT model is usually not computable. In reality, to compute the model, one must confine oneself with finite number of terms in this series. In that way we reach the notion of approximate DT models. Computing approximate DT models corresponds to the truncation of the infinite series with respect to the sampling period T as well as to the control input v, at the fixed orders τ and λ, respectively, which define the orders of approximation of the sampled system.

The (τ,λ)th-order approximate step-invariant DT model of a CT system (1) is defined as the one whose response to a step input (2) will agree with that of the CT system at given instants of time $t = kT$, $k \geq 0$, up to an error of the order $O(T^{\tau+1}, v^{\lambda+1})$.

An approximate DT model may be computed considering in Eqs. (4), (9), (11) the terms until the power τ in T and until the power λ in v :

$$x(kT+T) = F_0^a(x(kT)) + \sum_{i_1=1}^{m} F_{i_1}^a(x(kT))\, v_{i_1}(kT) + \ldots$$

$$\ldots + \sum_{i_1,\ldots,i_\lambda=1}^{m} F_{i_1\ldots i_\lambda}^a(x(kT))\, v_{i_1}(kT)\ldots v_{i_\lambda}(kT),$$

$$F_0^a(x) = \sum_{r=0}^{\tau} \frac{T^r}{r!} L_{f_0}^r x,$$

$$F_{i_1\ldots i_p}^a(x) = \sum_{k=1}^{p}\sum_{r=0}^{\tau-k} \frac{T^{r+k}}{(r+k)!} \sum_{\substack{c_1+\ldots+c_{k+1}=r \\ c_1\geq 0,\ldots,c_{k+1}\geq 0}} \left[\sum_{\nu=1}^{k-1}\sum_{j_\nu=j_{\nu-1}+1}^{p-(k-\nu)} L_{f_0}^{c_1} L_{f_{i_{j_0+1}\ldots i_{j_1}}} \times\right.$$

$$\left.\times L_{f_0}^{c_2} L_{f_{i_{j_1+1}\ldots i_{j_2}}} \ldots L_{f_0}^{c_3} L_{f_{i_{j_{k-2}+1}\ldots i_{j_{k-1}}}}\right] L_{f_0}^{c_k} L_{f_{i_{j_{k-1}+1}}} \ldots L_{f_0}^{c_k} L_{f_{i_{j_{k-1}+1}\ldots i_{j_k}}} L_{f_0}^{c_{k+1}} x, \quad 1 \leq p \leq \lambda,$$

where $j_0 := 0$, $j_k := p$.

The first-order approximate DT model with respect to the sampling period T is equivalent to the classical Euler discretization scheme.

4. RECURSIVE COMPUTATION OF VECTOR-VALUED FUNCTIONS IN THE EXPRESSION OF DISCRETE-TIME MODEL

In this section we shall show how to compute recursively the τth-order approximate DT model on the bases of the (τ-1)th-order model, starting with the first-order model which is equivalent to the classical Euler discretization scheme. The recursive formulas which we obtain are much more appropriate for computing DT model than the nonrecursive Eqs. (9), (11), (13).

Consider Eq. (4) and denote

$$L^r_{f(\bullet,\, u(kT))}\, x\big|_{x(kT)} := F_r(x(kT), u(kT))\,. \tag{14}$$

By (5) – (7), $F_r(x(kT),u(kT))$ takes the form

$$\begin{aligned} F_r(x(kT), u(kT)) &= F_{0,r}(x(kT)) + \sum_{i_1=1}^{m} F_{i_1,r}(x(kT))\, v_{i_1}(kT) + \\ &+ \sum_{i_1,i_2=1}^{m} F_{i_1 i_2,r}(x(kT))\, v_{i_1}(kT)\, v_{i_2}(kT) + \ldots \\ &\ldots + \sum_{i_1,\ldots,i_p=1}^{m} F_{i_1\ldots i_p,r}(x(kT))\, v_{i_1}(kT)\ldots v_{i_p}(kT) + \ldots \end{aligned} \tag{15}$$

Since

$$\begin{aligned} F_s(x(kT), u(kT)) &\overset{\text{by (14)}}{=} L_{f(\bullet,\, u(kT))} F_{s-1}(x(kT), u(kT)) \overset{\text{by (6),(15)}}{=} \\ &= \{L_{f_0} + \sum_{i_1=1}^{m} L_{f_{i_1}} v_{i_1}(kT) + \sum_{i_1,i_2=1}^{m} L_{f_{i_1 i_2}} v_{i_1}(kT)\, v_{i_2}(kT) + \ldots \\ &\ldots + \sum_{i_1,\ldots i_p=1}^{m} L_{f_{i_1\ldots i_p}} v_{i_1}(kT)\ldots v_{i_p}(kT) + \ldots\} \times \{F_{0,s-1}(x(kT)) + \\ &+ \sum_{i_1=1}^{m} F_{i_1,s-1}(x(kT))\, v_{i_1}(kT) + \sum_{i_1,i_2=1}^{m} F_{i_1 i_2,s-1}(x(kT))\, v_{i_1}(kT)\, v_{i_2}(kT) + \ldots \\ &\ldots + \sum_{i_1,\ldots i_p=1}^{m} F_{i_1\ldots i_p,s-1}(x(kT))\, v_{i_1}(kT)\ldots v_{i_p}(kT) + \ldots\}\,, \end{aligned}$$

we easily find that $F_{0,s}$ and $F_{i_1 \dots i_p, s}$, for $s \geq 2$, can be computed recursively

$$F_{0,s} = \Gamma_{0,1} F_{0,s-1} \tag{16}$$

$$F_{i_1,s} = \Gamma_{0,1} F_{i_1,s-1} + \Gamma_{i_1,1} F_{0,s-1},$$

and for $p \geq 2$

$$F_{i_1 \dots i_p, s} = \Gamma_{0,1} F_{i_1 \dots i_p, s-1} + \Gamma_{i_1,1} F_{i_2 \dots i_p, s-1} +$$

$$+ \Gamma_{i_1 i_2, 1} F_{i_3 \dots i_p, s-1} + \dots + \Gamma_{i_1 \dots i_{p-1}, 1} F_{i_p, s-1} + \Gamma_{i_1 \dots i_p, 1} F_{0, s-1} \tag{17}$$

with initial conditions

$$F_{0,1} = L_{f_0} x, \qquad F_{i_1 \dots i_p, 1} = L_{f_{i_1 \dots i_p}} x \tag{18}$$

and with

$$\Gamma_{0,1} = L_{f_0}, \qquad \Gamma_{i_1 \dots i_p, 1} = L_{f_{i_1 \dots i_p}}.$$

Special case of single-input systems

In case of single-input plant we obtain from (16), (17)

$$F_s^p = \Gamma_{0,1} F_{s-1}^p + \Gamma_{1,1} F_{s-1}^{p-1} + \Gamma_{2,1} F_{s-1}^{p-2} + \dots + \Gamma_{p-1,1} F_{s-1}^1 + \Gamma_{p,1} F_{s-1}^0, \tag{19}$$

where

$$\Gamma_{1,1} = L_{f^1}, \qquad \Gamma_{2,1} = L_{f^2}, \dots, \Gamma_{p,1} = L_{f^p}.$$

Special case of linear-analytic systems

In case of linear-analytic plant (12), we obtain from (16), (17)

$$F_{i_1 \dots i_p, s} = \Gamma_{0,1} F_{i_1 \dots i_p, s-1} + \Gamma_{i_1,1} F_{i_2 \dots i_p, s-1} \tag{20}$$

since $\Gamma_{i_1 \dots i_s, 1} = 0$ if $s \geq 2$.

5. EXAMPLES

E x a m p l e 1. The first example describes the dynamics of fish population [10] p. 99

$$\dot{x} = Kx(M - x) - u.$$

Here x is a measure of the size of the fisheries resource (for example the total weight of fish which can be harvested), and u denotes the harvesting rate.

In this model $f_0(x) = Kx(M-x)$, $f^1(x) = -1$, $f^k(x) = 0$, $k \geq 0$. So, $F_1^0(x) = Kx(M-x)$, $F_1^1(x) = -1, F_1^k(x) = 0$, $k \geq 2$ and by Eqs. (16) and (19) we obtain

$$F_2^0(x) = \Gamma_{0,1} F_1^0(x) = K^2 x(M-x)(M-2x),$$
$$F_2^1(x) = \Gamma_{0,1} F_1^1(x) + \Gamma_{1,1} F_1^0(x) = -K(M-2x),$$
$$F_2^k(x) = \Gamma_{0,1} F_1^k(x) + \Gamma_{1,1} F_1^{k-1}(x) = 0, \qquad k \geq 2.$$

The second-order discrete-time model is described by the equations

$$x(kT+T) = x(kT) + T\{Kx(kT)[M-x(kT)] - u(kT)\} +$$
$$+ \frac{T^2}{2}\{K^2 x(kT)[M-x(kT)][M-2x(kT)] - K[M-2x(kT)]u(kT)\}.$$

E x a m p l e 2. The second example describes the dynamics of asynchroneus induction motor [11]

$$\dot{x}_1 = -\frac{x_1}{\tau} + u_2 x_2,$$

$$\dot{x}_2 = -\frac{x_2}{\tau} + \frac{K}{\tau} u_1 - x_1 u_2,$$
$$\dot{x}_3 = -K\alpha x_3^2 + Kg x_1 u_1.$$

In this model

$$f_0(x) = \begin{bmatrix} -x_1/\tau \\ -x_2/\tau \\ -K\alpha x_3^2 \end{bmatrix}, \qquad f_1(x) = \begin{bmatrix} 0 \\ K/\tau \\ Kgx_1 \end{bmatrix}, \qquad f_2(x) = \begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix},$$

$$f_{i_1 \dots i_s}(x) = 0, \quad \text{for} \quad s \geq 2, \; i_1, \dots, i_s \in \{1, 2\}.$$

Since $F_{0,1}(x) = f_0(x)$, $F_{1,1}(x) = f_1(x)$, $F_{2,1}(x) = f_2(x)$, $F_{i_1 \dots i_s, 1}(x) = 0$, for $s \geq 2$, we obtain by Eqs. (16), (20):

$$F_{0,2} = \Gamma_{0,1} F_{0,1} = L_{f_0} f_0(x) = \left[x_1/\tau^2, x_2/\tau^2, 2K^2\alpha^2 x_3^3\right]^{\mathrm{T}},$$
$$F_{1,2} = \Gamma_{0,1} F_{1,1} + \Gamma_{1,1} F_{0,1} = L_{f_0} f_1 + L_{f_1} f_0 =$$
$$= [0, -K/\tau^2, -2K^2\alpha g x_1 x_3 - Kgx_1/\tau]^{\mathrm{T}},$$

$$F_{2,2} = \Gamma_{0,1} F_{2,1} + \Gamma_{2,1} F_{0,1} = L_{f_0} f_2 + L_{f_2} f_0 = [-2x_2/\tau, 2x_1/\tau, 0]^{\mathrm{T}},$$

$$F_{11,2} = \Gamma_{0,1} F_{11,1} + \Gamma_{1,1} F_{1,1} = L_{f_1} f_1 = 0,$$

$$F_{12,2} = \Gamma_{0,1}F_{12,1} + \Gamma_{1,1}F_{2,1} = L_{f_1}f_2 = [K/\tau, 0, 0]^T$$

$$F_{21,2} = \Gamma_{0,1}F_{21,1} + \Gamma_{2,1}F_{1,1} = L_{f_2}f_1 = [0, 0, Kgx_2]^T,$$

$$F_{22,2} = \Gamma_{0,1}F_{22,1} + \Gamma_{2,1}F_{2,1} = L_{f_2}f_2 = [-x_1, -x_2, 0]^T.$$

Of course for arbitrary $i_1, i_2, i_3 \in \{1, 2\}$ and $k \geq 1$, $F_{i_1 i_2 i_3, k} = 0$.

The second-order discrete-time model is described by the equations

$$x_1(kT+T) = x_1(kT) + T\left\{-\frac{x_1(kT)}{\tau} + x_2(kT)u_2(kT)\right\} +$$

$$+\frac{T^2}{2}\left\{\frac{x_1(kT)}{\tau^2} - \frac{2x_2(kT)}{\tau}u_2(kT) + \frac{K}{\tau}u_1(kT)u_2(kT) - x_1(kT)u_2^2(kT)\right\},$$

$$x_2(kT+T) = x_2(kT) + T\left\{-\frac{x_2(kT)}{\tau} + \frac{K}{\tau}u_1(kT) - x_1(kT)u_2(kT)\right\} +$$

$$\frac{T^2}{2}\left\{\frac{x_2(kT)}{\tau^2} - \frac{K}{\tau^2}u_1(kT) + \frac{2x_1(kT)}{\tau}u_2(kT) - x_2(kT)u_2^2(kT)\right\}$$

$$x_3(kT+T) = x_3(kT) + T\{-K\alpha x_3^2(kT) + Kgx_1(kT)u_1(kT)\} +$$

$$+\frac{T^2}{2}\{2K^2\alpha^2x_3^3(kT) - \left[\frac{Kg}{\tau}x_1(kT) + 2K^2\alpha g x_3(kT)x_1(kT)\right]u_1(kT) +$$

$$+ Kgx_2(kT)u_1(kT)u_2(kT)\}.$$

E x a m p l e 3. The third example describes the motion of a rocket near the earth [10], p. 152

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = x_1x_4^2 - \frac{gR^2}{x_1^2} + \frac{\delta}{m}\cos u,$$

$$\dot{x}_3 = x_4,$$

$$\dot{x}_4 = -\frac{2x_2x_4}{x_1} + \frac{\delta}{m}\sin u.$$

Expanding $\cos u$ and $\sin u$ in a Taylor series around a point $u = 0$,

$$\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \ldots (-1)^{p-1}\frac{u^{2p-2}}{(2p-2)!} + \ldots,$$

$$\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \ldots (-1)^{p-1}\frac{u^{2p-1}}{(2p-1)!} + \ldots,$$

we obtain

$$f_0(x) = \begin{bmatrix} x_2 \\ x_1 x_4^2 - \frac{gR^2}{x_1^2} + \frac{\delta}{m} \\ x_4 \\ -\frac{2x_2 x_4}{x_1} \end{bmatrix}, \quad f^{2k-1}(x) = \frac{\delta}{m} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{(-1)^{k-1}}{(2k-1)!} \end{bmatrix},$$

$$f^{2k}(x) = \frac{\delta}{m} \begin{bmatrix} 0 \\ \frac{(-1)^k}{(2k)!} \\ 0 \\ 0 \end{bmatrix}, \quad k \geq 1.$$

Since $F_1^0 = f_0, F_1^k = f^k, k \geq 1$, we obtain by Eqs. (16) and (19)

$$F_2^0 = \Gamma_{0,1} F_1^0 = L_{f_0} f_0 = \begin{bmatrix} x_1 x_4^2 - \frac{gR^2}{x_1^2} + \frac{\delta}{m} \\ -3x_2 x_4^2 + 2gR^2 \frac{x_2}{x_1^3} \\ -\frac{2x_2 x_4}{x_1} \\ \frac{6x_2^2 x_4}{x_1^2} - 2x_4^3 + \frac{2gR^2 x_4}{x_1^3} - \frac{2\delta x_4}{m x_1} \end{bmatrix},$$

$$F_2^{2k-1} = \Gamma_{2k-1,1} F_1^0 = L_{f^{2k-1}} f_0 = \frac{(-1)^{k-1}\delta}{m\,(2k-1)!} [0, 2x_1 x_4, 1, -2x_2/x_1]^{\mathrm{T}},$$

$$F_2^{2k} = \Gamma_{2k,1} F_1^0 = L_{f^{2k}} f_0 = \frac{(-1)^k \delta}{m\,(2k)!} [1, 0, 0, -x_4/x_1]^{\mathrm{T}}, \quad k \geq 1.$$

The (2,3)th-order approximate discrete-time model is described by the equations

$$x_1(kT+T) = x_1(kT) + Tx_2(kT) + \frac{T^2}{2}\left\{x_1(kT)x_4^2(kT) - \frac{gR^2}{x_1^2(kT)} + \frac{\delta}{m} - \right.$$

$$\left. -\frac{\delta}{2m}u^2(kT)\right\},$$

$$x_2(kT+T) = x_2(kT) + T\left\{x_1(kT)x_4^2(kT) - \frac{gR^2}{x_1^2(kT)} + \frac{\delta}{m} - \frac{\delta}{2m}u^2(kT)\right\} +$$

$$+ \frac{T^2}{2}\left\{-3x_2(kT)x_4^2(kT) + 2gR^2\frac{x_2(kT)}{x_1^3(kT)} + \frac{\delta}{m}2x_1(kT)x_4(kT)u(kT) - \right.$$

$$\left. -\frac{\delta}{m3!}2x_1(kT)x_4(kT)u^3(kT)\right\},$$

$$x_3(kT+T) = x_3(kT) + Tx_4(kT) + \frac{T^2}{2}\left\{-\frac{2x_2(kT)x_4(kT)}{x_1(kT)} + \frac{\delta}{m}u(kT) - \right.$$

$$\left. -\frac{\delta}{3!m}u^3(kT)\right\},$$

$$x_4(kT+T) = x_4(kT) + T\left\{-\frac{2x_2(kT)x_4(kT)}{x_1(kT)} + \frac{\delta}{m}u(kT) - \frac{\delta}{3!m}u^3(kT)\right\} +$$

$$+ \frac{T^2}{2}\left\{\frac{6x_2^2(kT)x_4(kT)}{x_1^2(kT)} - 2x_4^3(kT) + \frac{2gR^2x_4(kT)}{x_1^3(kT)} - \frac{2\delta x_4(kT)}{mx_1(kT)} - \right.$$

$$\left. -\frac{2\delta}{m}\frac{x_2(kT)}{x_1(kT)}u(kT) + \frac{\delta}{2m}\frac{x_4(kT)}{x_1(kT)}u^2(kT) + \frac{2\delta}{3!m}\frac{x_2(kT)}{x_1(kT)}u^3(kT)\right\}.$$

6. CONCLUSIONS

In this paper we have studied the problem of sampling the multi-input analytic nonlinear CT system. We have derived the exact as well as the τth-order approximate (with respect to sampling period T) DT step-invariant model of CT system. The exact step-invariant DT model of CT system is defined as the one whose response to a step input (i.e. the type of control usually available under digital control) is identical to that of the CT system at sampling times. The τth-order approximate DT model is defined as the one whose response to a step input at sampling times will agree with that of the CT system up to an error of the order $O(T^{\tau+1})$. The agree with that of the CT system up to an error of the order $O(T^{\tau+1})$. The τth-order approximate model is computed recursively on the bases of (τ-1)th-order model starting with the first-order model which is equivalent to the classical Euler discretization scheme. These recursive formulas are especially appropriate for computing DT model via a computer algebra system such as REDUCE.

The results of [1, 3] can be obtained as a special case of the formulas (9).

REFERENCES

1. Котта, Ю. Изв. АН ЭССР. Физ., матем., 1986, **35**, 3, 326-328.
2. Monaco, S., Normand-Cyrot, D. Proc. 24th IEEE Cont. on Decision and Control, Fort Lauderdale, 1985, 90-95.
3. Котта, Ю. Изв. АН ЭССР. Физ., матем., 1989, **38**, 2, 222-224.
4. Monaco, S., Normand-Cyrot, D. Lect. Notes Contr. and Inf. Sci., 1986, **83**, 354-367.
5. Monaco, S., Normand-Cyrot, D. Lect. Notes Contr. and Inf. Sci., 1990, **144**, 788-797.
6. Barbot, J. P. Lect. Notes Comp. Sci., 1989, **357**, 74-88.
7. Fliess, M., Lamnabhi, M., Lamnabhi-Lagarrique, F. IEEE Trans. on Circuits and Syst., 1983, **CS-30**, 8, 554-570.
8. Hori, N., Mori, T., Nikiforuk, P. N. Proc. 1989 American Control Conference, 1989, **2**, 1659-1664.
9. Boothby, W. A. An Introduction to Differentiable Manifolds and Riemannian Geometry. Academic, New York, 1975.
10. McClamroch, N. H. State Models of Dynamic Systems. A Case Study Approach. Springer-Verlag, New York, 1980.
11. Moyon, P., Champetier, C., Reboulet, C. Lect. Notes Contr. and Inf. Sci., 1984, **63**, 327-338.

PIDEVATE MITTELINEAARSETE SÜSTEEMIDE DISKREETSED MUDELID

Ülle KOTTA

On vaadeldud pidevate mittelineaarsete juhtimissüsteemide diskreetsete mudelite konstrueerimise ülesannet. Nimetatud ülesanne seisneb niisuguste diferentsiaalvôrrandite leidmises, mille lahendid langevad diskretiseerimismomentidel kokku vastavate diferentsiaalvôrrandite (need kirjeldavad pideva juhtimissüsteemi dünaamikat) lahenditega eeldusel, et pideva süsteemi sisend on tükati konstantne funktsioon. Varasemad tulemused on üldistatud mitme sisendiga analüütiliste mittelineaarsete süsteemide klassile. On leitud nii täpne kui ka τ-ndat järku (diskretiseerimissammu suhtes) ligikaudne diskreetne mudel.

ДИСКРЕТНЫЕ МОДЕЛИ НЕЛИНЕЙНЫХ НЕПРЕРЫВНЫХ СИСТЕМ

Юлле КОТТА

Статья посвящена продолжению исследований по построению дискретных моделей непрерывных систем управления, т.е. нахождению таких разностных уравнений, решения которых совпадают в заданные дискретные моменты времени с решениями соответствующих дифференциальных уравнений (описывающих поведение непрерывной системы управления) в предположении, что вход дифференциальных уравнений ступенчатая функция. Ранние результаты обобщаются для класса аналитических нелинейных систем со многими входами. Построены точная, а также τ-го порядка приближенная (относительно шага дискретизации) дискретная модель.