Proc. Estonian Acad. Sci. Phys. Math., 1994, **43**, 2, 127 – 130
https://doi.org/10.3176/phys.math.1994.2.09

A GENERALIZATION OF THE ROGOSINSKI MEANS AND ITS APPLICATIONS

Andi KIVINUKK

Tallinna Tehnikaülikool (Tallinn Technical University), Ehitajate tee 5, EE-0026 Tallinn, Eesti (Estonia)

Presented by J. Engelbrecht

Received November 26, 1993; revised version received January 20, 1994; accepted April 6, 1994

ÜHEST ROGOSINSKI KESKMISTE ÜLDISTUSEST JA SELLE RAKENDUSTEST. Andi KIVINUKK

ОБОБЩЕННЫЕ СРЕДНИЕ РОГОЗИНСКОГО И ИХ ПРИМЕНЕНИЕ. Анди КИВИНУКК

Key words: trigonometrical approximation processes, Rogosinski means.

Main results. In this note a new variant of the subordination principle based on a generalization of the Rogosinski means is found. This approach, on the one hand, allows to cover some results obtained earlier by other methods and, on the other hand, it gives a promising method to attack some questions for many approximation processes which could not be handled until now.

Let $C_{2\pi}\,(AC_{2\pi})$ denote the set of all 2π- periodic functions which are continuous (absolutely continuous, respectively) everywhere. $\|\cdot\|_p$ is the norm in $L^p_{2\pi}\,(1 \le p \le \infty)$. W^1_∞ denotes the class of all functions $f \in AC_{2\pi}$ for which $\|f'\|_\infty \le 1$.

To be more precise, consider a continuous function $\varphi \in C_{[0,1]}$ with $\varphi(0) = 1$ and $\varphi(1) = 0$. Then φ is said to generate a singular integral U_n,

$$U_n = f * u_n$$

via the kernel

$$u_n(x) := 1 + 2\sum_{k=1}^{n} \varphi\left(\frac{k}{n}\right)\cos kx.$$

The operator U_n defines an approximation process on $C_{2\pi}$ if (cf. [1])

$$\|U_n\| := \|u_n\|_1 \le M.$$

We shall consider the quantity

$$e(W^1_\infty, U_n) := \sup_{f \in W^1_\infty} \|f - U_n f\|_C.$$

The **first** new observation in this connection is that for all $j \in \mathbf{N}$ the function $\varphi_j(t) := \cos(j - 1/2)\pi t$ generates a bounded approximation process, say $R_{n,j}$. The operator $R_{n,1}$ is known as the Rogosinski approximation method (cf. [1]).

T h e o r e m 1. *For all* $j \in \mathbf{N}$,

$$\sup_{n \in \mathbf{N}} \|R_{n,j}\| = \frac{2}{\pi} \sum_{k=0}^{2(j-1)} \int_0^\pi \frac{\sin t}{t + k\pi}\, dt = \frac{2}{\pi} \log j + O(1).$$

P r o o f. Our result is an immediate corollary from the formula (cf. [2], §24, Theorem 2)

$$\sup_{n \in \mathbf{N}} \|U_n\| = \frac{2}{\pi} \int_0^\infty \left| \int_0^1 \varphi(t) \cos tx\, dt \right| dx.$$

For the details compare [2], Example 3.

The **second** new observation in this connection is that the system $\{\varphi_j\}$ $(j \in \mathbf{N})$ is orthogonal on $[0,1]$ with the boundary conditions $\varphi_j(0) = 1, \varphi_j(1) = 0$ for all $j \in \mathbf{N}$. Our idea now consists in representing arbitrary $\varphi \in C_{[0,1]}$ with $\varphi(0) = 1, \varphi(1) = 0$ by the orthogonal system $\{\varphi_j\}$:

$$\varphi = \sum_{j=1}^\infty a_j \varphi_j, \qquad a_j := 2 \int_0^1 \varphi \cdot \varphi_j \qquad (j \in \mathbf{N}). \tag{*}$$

The representation (*) allows to generalize results – until now only known as the Rogosinski method – to various singular integrals. Here the main result reads

T h e o r e m 2. *Let* U_n *be generated by* φ, *where* φ *satisfies* (*) *with uniformly convergent series. If*

$$\Psi(t) := \int_t^\infty \left(\int_0^1 \varphi(u) \cos ux\, du \right) dx = O(t^{-1-\varepsilon}) \qquad (\varepsilon > 0, t \to \infty),$$

then

$$e(W^1_\infty, U_n) = \frac{2}{n\pi} \int_0^\infty |\Psi(t)|\, dt + O(n^{-1-\varepsilon}).$$

P r o o f. At first we must point out that our theorem is valid for the generalized Rogosinski means $R_{n,j}$ for all $j \in \mathbf{N}$. For the full technical details we may consult [3]. Now it is surprising that the same argument is true for arbitrary means U_n generated by φ, if only the representation (*) is valid.

It is clear that in general the evaluation of the integral $\int_0^\infty |\Psi(t)|\,dt$ is the crux of this statement; this nicely shows how numerical problems arise in these types of problems.

Examples. If one chooses $\varphi(t) := \cos(j-1/2)\pi t,\ (j \in \mathbf{N})$ in Theorem 2 then one obtains the following generalization of a result ($j = 1$) of Dzyadyk, Stepanets and Gavril'uk (cf. [3]) to arbitrary $j \in \mathbf{N}$:

$$e(W^1_\infty, R_{n,j}) = \frac{1}{n\pi}\int_0^\infty \left|si(t+m_j) + si(t-m_j)\right| dt + O(n^{-2}),$$

$$m_j := (j-1/2)\pi,$$

where the integral sine is defined by

$$si(t) := -\int_t^\infty \frac{\sin x}{x}\,dx.$$

Analogously this holds for the Jackson – de La Vallée Poussin method J_n generated by (cf. [1])

$$\varphi(t) := \begin{cases} 1-6t^2+6t^3, & 0 \le t \le 1/2, \\ 2(1-t)^3 & 1/2 \le t \le 1, \end{cases}$$

that

$$e(W^1_\infty, J_n) = \frac{12\log 2}{n\pi} + \frac{c_n}{n^3}, \qquad |c_n| \le 1/(3\pi^3).$$

For the typical means $Z_{n,r}$ generated by (cf. [1]) $\varphi_{(r)}(t) := (1-t^r)_+,\ (r \in \mathbf{N})$ there follows (r even)

$$e(W^1_\infty, Z_{n,r}) = \frac{2r!}{n\pi}\int_0^\infty \left| \int_t^\infty \sum_{\nu=1}^{r} \frac{1}{(r-\nu)!}\,\frac{\sin(x+\nu\pi/2)}{x^{\nu+1}}\,dx \right| dt + O(n^{-2})..$$

Final comments and acknowledgments. This paper is a short version of the new subordination principle based on a generalization of the Rogosinski means. The enlarged and more detailed version will appear in another journal in the future.

The paper was prepared during the author's EC-Fellowship at TH

Darmstadt, April 13–July 14, 1993 (Reference No. ERB3510PL925090). The author wishes to thank Prof. W. Trebels for several useful discussions and Mrs. G. Gehring and Dr. J. Lippus for the preparation of the manuscript.

REFERENCES

1. Butzer, P. L., Nessel, R. J. Fourier Analysis and Approximation 1. Birkhäuser Verlag, Basel, Stuttgart, 1971.
2. Жук В. В., Натансон Г. И. Тригонометрические ряды Фурье и элементы теорий аппроксимации. Изд.-во Ленингр. ун-та, Ленинград,1983.
3. Korneichuk, N. Exact Constants in Approximation Theory. Cambridge Univ. Press, Cambridge, 1991.