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BIFURCATION DIAGRAM OF THE DRIVEN ASYMMETRIC VAN DER POL EQUATION

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БИФУРКАЦИОННАЯ ДИАГРАММА ВЫНУЖДЕННОГО АСИММЕТРИЧНОГО УРАВНЕНИЯ ВАН ДЕР ПОЛЯ. Олав КОНГАС, Юри ЭНГЕЛЬБРЕХТ.

Key words: bifurcation, heart dynamics, chaos.

The celebrated van der Pol equation (VDPE) has played an important role in many physical applications, such as in electronic and mechanical as well as in biological systems. The original idea by van der Pol and van der Mark [1] to use this equation in heart dynamics has indeed been fruitful [2,3], explaining several biological phenomena in mathematical terms. Due to the existence of several oscillating subsystems in the heart, the main mathematical problem is to understand the dynamics of a driven VDPE. Most of the known results describe the behaviour of the standard symmetric VDPE with a driving function.

The early results by Cartwright and Littlewood [4] and Levinson [5] describe the dynamics of an acceleration-driven VDPE. It was shown that certain steady-state subharmonic responses with two different multiples of the forcing frequency can occur simultaneously. In contemporary understanding this transient response means transient chaos. Later, Hayashi [6] has shown the existence of a drifting phenomenon which topologically means a quasiperiodic motion. Based on these and a lot of other results, Guckenheimer and Holmes [7] gave a detailed description of specific phenomena for an acceleration-driven VDPE showing the phase-portraits and their blow-ups. To sum up, the following has been described in this connection [7–10]:

– hard-mode instability leading to the complete entrainment;
– soft-mode instability with two distinct modes;
– mode-locking;
– irregular response.

A velocity-driven symmetric VDPE has been studied by Shaw [11], resulting in the description of phase-mixing and its consequences. If the driving force is an asymmetric sine or bias, then the blue sky catastrophy can occur with the global stability loss [12].

A bias as a forcing means actually asymmetry. Beside the mentioned study [12], not much is known for an asymmetrically driven VDPE. Subharmonics (not irregular motion) for such a case were discussed by Cartwright [13]. A systematic study of asymmetric systems, i.e. systems with bias, has been started by Abraham and Simó [14]. First they have found that the acceleration bias only leads to periodic orbits ("cigars") as possible French duck effect, while the periodical driving leads to chaotic attractors. In the corresponding "symmetric" case only transient chaos was observed. In the "asymmetric" case the period doubling was described and in the chaotic regime a window of period 7 was found. In the case of double bias (acceleration and velocity), several types of periodic orbits have been found and bifurcations had been described.

As the full bifurcation diagram for the asymmetric VDPE is absent (up to the knowledge of the authors), the problem needs further analysis.

The acceleration-driven asymmetric VDPE

$$\ddot{w} + f(w)\,\dot{w} + w = A\sin Bt \tag{1}$$

is considered. Here $f(w)$ is an asymmetric (with respect to the origin) quadratic function with the roots $w_1 < 0$, $w_2 > 0$. It is expedient to emphasize

$$f(w) = \frac{-4h}{(w_2 - w_1)^2}\,(w - w_1)\,(w - w_2) \tag{2}$$

that determines $f(w)$ by the roots and ordinate $h < 0$ of its minimal value (Fig. 1). This form is suitable from the viewpoint of a mathematical model proposed by the authors for modelling the heart dynamics [15]. Equation (1) with $f(w)$ determined by (2) is solved numerically using standard Runge-Kutta procedure. The bifurcation diagram is found from the phase plane shown in Fig. 2. At every $t = (2n + 2/3)\pi$, $n = 1,2,...$, the distance from the origin

$$r = \left(w^2(t) + \dot{w}^2(t)\right)^{1/2} \tag{3}$$

is calculated and plotted versus the control parameter A in order to get the bifurcation diagram The transients are left out and the final result is shown in Fig. 3. The other parameters of Eqs. (1) and (2) are the following: $B = 1$, $w_1 = -0.2$, $w_2 = 1.9$, $h = -1.6$.

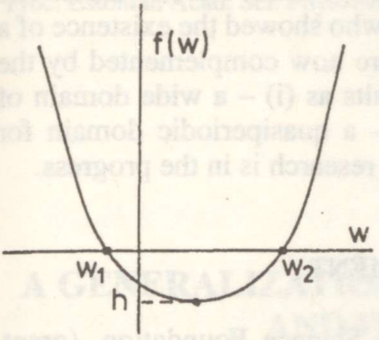
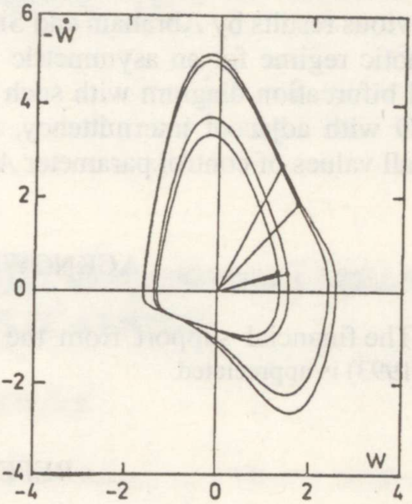


Fig. 1. Character of quadratic function *f*(*w*).

Fig. 2. Phase plane. →



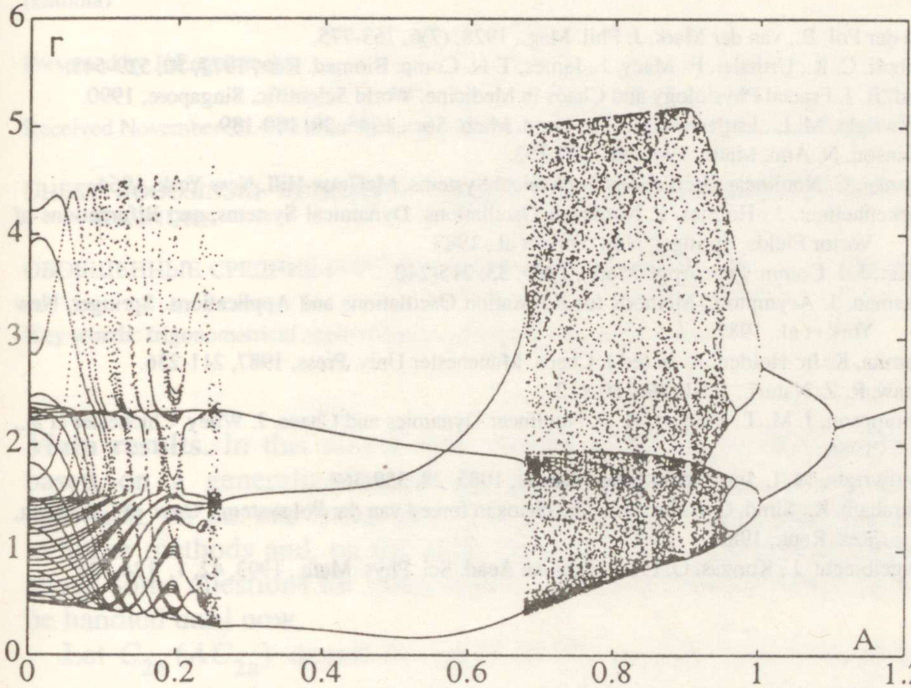


Fig. 3. Bifurcation diagram.

The following regions can be distinguished in the bifurcation diagram:

– simple entrainment with 1 periodical orbit (PO) for A > 1.1;
– period-doubling cascade according to the Feigenbaum scenario with 2PO, 4PO, 8PO, ... up to the accumulation point at $A \sim 0.96$;
– chaotic regime (strange attractor) with periodic windows 5PO ($A \sim 0.905$), 7PO ($A \sim 0.854$), 9PO ($A \sim 0.826$),..., the forming of which is due to an interior crisis; intermittency occurs around $A \sim 0.678$ that leads to a 3PO for a large interval $0.26 < A < 0.678$;
– a narrow range of chaotic regime again for $0.22 < A < 0.26$ preceded by a period-doubling;
– quasiperiodic regime for small values of $A < 0.22$.

This result enlarges essentially our knowledge about driven VDPE. The previous results by Abraham and Simó [14], who showed the existence of a chaotic regime for an asymmetric VDPE, are now complemented by the full bifurcation diagram with such new results as (i) – a wide domain of 3PO with adjacent intermittency, and (ii) – a quasiperiodic domain for small values of control parameter *A*. Further research is in the progress.

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