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Felix KAPLANSKY *

ON THE FLOW INDUCED BY TOROIDAL VORTICITY

(Presented by J. Engelbrecht)

Abstract. Recently toroidal vorticity distributions for the vortex ring were obtained analytically. But the stream function distribution for the pronounced toroidal structure of vorticity was not defined. The present paper is devoted to overcoming this inconsistency. The stream function distribution described below is found.

1. Introduction

Vortex motion or flow with vorticity plays an essential role in almost all kinds of fluid motion of interest. In order to explore the fundamental mechanisms of vortex dynamics, the motion of vortex tubes has been studied extensively, both experimentally and theoretically. Recently some analytical solutions were obtained in the form of vorticity distributions for this kind of problem [1, 2]. But these solutions will be complete only after the stream function has been determined. The paper deals with overcoming this inconsistency in the case when vorticity has a pronounced toroidal form. The approach is based on the idea of equating the values of a stream function on the boundary between the inner and outer regions under consideration, where the inner region is a vortex core and the outer one is arranged at a great distance from the centre of the vortex. The results have been found for the vortex ring in a viscous fluid [1].

2. Statement of the problem

In the frame of cylindrical coordinates (x, r, ϑ) for axially symmetric vorticity, the fundamental solution of the Poisson equation

$$-r\zeta=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial r^2}-\frac{1\partial\varphi}{r\partial r} \tag{1}$$

may be written in the form [3]

$$\varphi(r, x)=\frac{r}{4\pi}\int_0^\infty\int_0^\infty\int_0^{2\pi}\frac{\zeta(z, \varrho)\varrho\cos\theta\, dz\, d\varrho\, d\theta}{\sqrt{(x-z)^2+\varrho^2+r^2-2\varrho r\cos\theta}}, \tag{2}$$

where ζ and φ are vorticity and stream function.

* Eesti Teaduste Akadeemia Termofüüsika ja Elektrofüüsika Instituut (Institute of Thermophysics and Electrophysics, Estonian Academy of Sciences). Paldiski mnt. 1, EE-0001 Tallinn. Estonia.

For the isolated circular vortex filament the integral (2) is simplified and can be presented as follows [3]:

$$\varphi(r,x)=\frac{\Gamma}{4\pi}(rR_0)^{1/2}\int_0^{2\pi}\frac{\cos\theta\,d\theta}{\sqrt{x^2+R_0^2+r^2-2R_0r\cos\theta}}=$$

$$=\frac{\Gamma}{2\pi}(rR_0)^{1/2}\left(\left(\frac{2}{k}-k\right)K(k)-\frac{2}{k}E(k)\right), \qquad (3)$$

$$k^2=4\pi R_0 r/(x^2+(r+R_0)^2).$$

Here Γ is the given circulation of the vortex filament, $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and the second kind, respectively.

This expression is defined in the classical theory of inviscid fluid and is widely used in the vortex dynamics and in turbulence simulations. It is the basic relation for the determination of the flow induced by vorticity. The possibility of simplifying integral (2) is based on the presentation of vorticity as a delta-function. In the case where vorticity differs from the delta-function the stream function has not been found yet. We shall focus on the form of vorticity distribution

$$\zeta=C(t)\exp\left(-\frac{x^2+r^2+\tau^2}{2}\right)I_1(r\tau), \qquad (4)$$

where $I_1(r\tau)$ is a modified Bessel function.

The problem for the vortex ring in a viscous fluid was formulated in [1].

The solution of this problem is (4). Here $x=x^*/\sqrt{2\nu t}$, $r=r^*/\sqrt{2\nu t}$, $\tau=R_0/\sqrt{2\nu t}$ are dimensionless variables, $C=At^{-3/2}t_0^{1/2}$. The expression (4) tends to delta-function for $\tau\to\infty$ and in this case the distribution of the stream function can be obtained by (3). In the other limit case, where $\tau\to 0$, (4) yields

$$\zeta=Bt^{-1}\exp\left(-\frac{x^2+r^2}{2}\right),\quad B=\text{const}. \qquad (5)$$

The stream function in that case can be determined too

$$\varphi(r,x)=Bt^{-1}\left[\frac{1}{R}\int_0^R\exp\left(-\frac{t^2}{2}\right)dt-\exp\left(-\frac{R^2}{2}\right)\right]r^2/R^2,$$

$$R=(r^2+x^2)^{1/2}. \qquad (6)$$

Neither the first nor the second case describes pronounced toroidal vorticity. Our aim is to obtain the stream function distribution when vorticity has the form determined by (4).

3. Stream function for the vortex ring

We consider the case where τ is of the same order of magnitude as one and will expand:

$$\exp\left(-\frac{r^2}{2}\right)=\sum_{m=0}^{\infty} S_m r^{2m}, \tag{7}$$

$$I_1(r\tau)=\sum_{k=0}^{\infty} F_k (r\tau)^{2k+1}, \tag{8}$$

where S_m, F_k are constant coefficients.

The equation for the stream function in spherical coordinates is

$$\frac{\partial^2\varphi}{\partial R^2}+\frac{h_*^2\partial^2\varphi}{R^2\partial^2 h}=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} F_k S_m \tau^{2k+1} R^{2k+2m+2} h_*^{2k+2}, \tag{9}$$

$$h=\cos(\psi),\quad h_*=\sin(\psi),\quad x=Rh,\quad r=Rh_*.$$

We shall find the inhomogeneous solution of eq. (1) in the form

$$\varphi_1=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} R^{2k+2m+4} P_{k,m}(h_*). \tag{10}$$

This representation makes it possible to divide the variables and to get the equation for determining $P_{k,m}$

$$(2k+2m+4)(2k+2m+3)P_{k,m}+P''_{k,m}-\frac{h}{h_*}P_{k,m}=F_k S_m \tau^{2k+1} h_*^{2k+2}. \tag{11}$$

The solution of eq. (11) consists of recurrent expressions

$$P_{k,m}(h_*)=A^{(k)}_{k,m} h_*^{2(k+1)}+A^{(k-1)}_{k,m} h_*^{2k}+\ldots+A^{(i)}_{k,m} h_*^{2(i+1)}+\ldots+A^{(0)}_{k,m} h_*^2,$$

where

$$A^{(k)}_{k,m}=\frac{\tau^{2k+1}F_k S_m}{(2k+2m+4)(2k+2m+3)-(2k+2)(2k+1)},$$

$$A^{(i)}_{k,m}=\frac{2(2(i+1)+2)(i+1)A^{(i+1)}_{k,m}}{(2k+2m+4)(2k+2m+3)-(2i+2)(2i+1)}, \tag{12}$$

$$i=k-1,\ \ldots,\ 0.$$

The following solution of the homogeneous equation is known and can be presented in the form [4]

$$\varphi_0=\sum_{k=0}^{\infty}(M_n R^n+K_n R^{-n+1})\ \begin{matrix}\mathfrak{J}\\ \mathfrak{B}\end{matrix}(h). \tag{13}$$

Here $\mathfrak{J}_n(h)$ and $\mathfrak{B}_n(h)$ are Gegenbauer functions of the first and the second kinds, M_n and K_n are constants.

Due to the symmetry of the vortex ring

$$\varphi=0,\quad r=0 \tag{14}$$

the solution (13) was simplified

$$\varphi_0=\sum_{n=0}^{\infty} M_n R^n \mathfrak{J}_n(h). \tag{15}$$

The use of solution (12) is not convenient due to the variables being unlimited. Therefore the denominator in integral (2) was presented in the form of expansion [5] at a great distance R from the centre of the vortex ring

$$G^{-1}=\sqrt{(x-z)^2+\varrho^2+r^2-2\varrho r\cos\theta}=$$

$$=R^{-1}\left[1-\frac{2}{R}\left(\frac{r}{R}\varrho\cos(\theta)+\frac{x}{R}z\right)+\frac{1}{R^2}(z^2+\varrho^2)\right]^{-1/2}=$$

$$=R^{-1}(1+\alpha_1R^{-1}+\alpha_2R^{-2}+\alpha_3R^{-3}+O(R^{-4})).$$

Here

$$\alpha_1=\frac{r}{R}\varrho\cos(\theta)+\frac{x}{R}z, \tag{16}$$

$$\alpha_2=\frac{3}{2}\left[\left(\frac{x}{R}\right)^2z^2+\left(\frac{r}{R}\right)^2\varrho^2\cos^2(\theta)+\frac{2xr}{R^2}\varrho z\cos(\theta)\right]-\frac{1}{2}(\varrho^2+z^2),$$

$$\alpha_3=\frac{5}{2}\left(\frac{r}{R}\varrho\cos(\theta)+\frac{x}{R}z\right)\left[\left(\frac{x}{R}\right)^2z^2+\left(\frac{r}{R}\right)^2\varrho^2\cos^2(\theta)+\right.$$

$$\left.+\frac{2rx}{R^2}z\varrho\cos(\theta)-\frac{3}{5}(z^2+\varrho^2)\right].$$

Substitution of this expansion into eq. (2) and integration by θ gives the distribution of the stream function outside the circle of radius R

$$\varphi_*(x,r)=\frac{1}{4R}\left(\frac{r}{R}\right)^2\left\{\langle\zeta\varrho^2\rangle+\frac{3}{2}\left(\frac{x}{R}\right)\langle\zeta\varrho^2z\rangle+\right.$$

$$\left.+\frac{3}{2R^2}\left[\left(5\left(\frac{x}{R}\right)^2-1\right)\langle\zeta\varrho^2z^2\rangle+\left(\frac{5}{4}\left(\frac{r}{R}\right)^2-1\right)\langle\zeta\varrho^4\rangle\right]\right\}, \tag{17}$$

where integrals

$$\langle\zeta\varrho^mz^n\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta(\varrho,z)\varrho^mz^n\,dz\,d\varrho \tag{18}$$

take the values

$$\langle\zeta\varrho^2\rangle=\langle\zeta\varrho^2z^2\rangle=(2\pi)^{1/2}\tau,$$

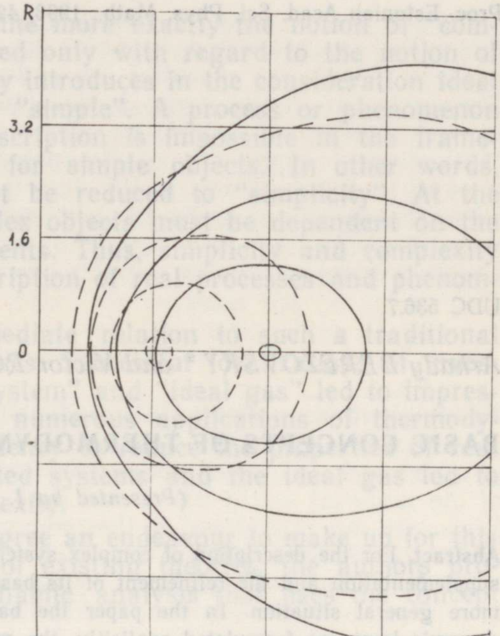
$$\langle\zeta\varrho^2z\rangle=2\tau,$$

$$\langle\zeta\varrho^4\rangle=(2\pi)^{1/2}4\tau\exp\left(-\frac{\tau^2}{2}\right)\left(\frac{2+\tau}{2}\right)^2M\left(1,1,\frac{\tau^2}{2}\right).$$

Here $M\left(1,1,\frac{\tau^2}{2}\right)$ is a tabulated confluent hypergeometric function.

The complete solution consists of homogeneous and inhomogeneous solutions and may be found as follows: We can get the system of linear equations for the determination of M_n by making $\varphi_0+\varphi_1$ equal to φ_* in the boundary between the vortex core and the region located at a great distance from the point of maximum value of vorticity. After solving it, we found φ_0 and then the sum $\varphi_0+\varphi_1$. The obtained solutions for $\tau=0.1$ (dashed curve) and $\tau=1.0$ (continuous curve corresponds to the pronounced toroidal vorticity form) are shown in the Figure. The approximation of (7), (8) from [6] was used and the number of points in the boundary was equal to fourteen.

The Figure shows that the isolines of the stream function corresponding to $\tau=1$ are also concentrated at an increased distance from the symmetry axis. The considered case lies between the case with a circular vortex filament and the one where the vortex ring is degenerated and occupies total space. The obtained results conform to this reasoning and make it possible to find the velocity field in this situation. The presented approach can also be used for different vorticity distributions.



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Felix KAPLANSKI

TOORIKUJULISE KEERISELISUSE INDUTSEERITUD VEDELIKU LIIKUMISEST

Varem analüütiliselt saadud selgelt toorina väljendatud keeriselisuse jaotusele on leitud voolufunktsiooni määramise metoodika. Voolufunktsiooni jaotus on esitatud isojoontena.

Феликс КАПЛАНСКИЙ

ОБ ИНДУЦИРОВАННОМ ТОРОИДАЛЬНОЙ ЗАВИХРЕННОСТЬЮ ДВИЖЕНИИ ЖИДКОСТИ

Предложена методика определения функции тока для аналитически найденного ранее распределения завихренности в виде ярко выраженного тора. Ее распределение представлено в виде изолиний.