Proc. Estonian Acad. Sci. Phys. Math., 1993, 42, 2, 144—156
https://doi.org/10.3176/phys.math.1993.2.02

UDC 512.542

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CONNECTION BETWEEN SYLOW SUBGROUPS OF SYMMETRIC GROUP AND THEIR SEMIGROUPS OF ENDOMORPHISM

(Presented by R.-K. Loide)

Abstract. Let m be a natural number and S_m be a symmetric group of the degree m. If p is a prime number which is a divisor of $|S_m|$, then all the Sylow p-subgroups of S_m are isomorphic to the direct product of groups

$$G(n,p) = (\dots((C_p \operatorname{Wr} C_p) \operatorname{Wr} C_p) \operatorname{Wr} \dots) \operatorname{Wr} C_p \quad (n \text{ factors}),$$

where C_p is the cyclic group of the order p.

The following results are proved.

Theorem. *The group $G(n,p)$ is determined for each natural n and prime p by its semigroup of endomorphisms in the class of all groups.*

Corollary 1. *Every Sylow subgroup of a finite symmetric group is determined by its semigroup of endomorphisms in the class of all groups.*

Corollary 2. *Let G be a finite p-group. Then G is imbeddable into a finite p-group $\overline{G}$ such that $\overline{G}$ is determined by its semigroup of endomorphisms in the class of all groups.*

1. Introduction

Let G be a fixed group. If for a suitable group H from the isomorphism of semigroups of all endomorphisms of groups G and H follows the isomorphism of groups G and H, then we say that the group G is determined by its semigroup of endomorphisms in the class of all groups. For example, every finite Abelian group is determined by its semigroup of endomorphisms in the class of all groups ([1], Theorem 4.2). There exist also examples of such groups which cannot be determined by their semigroups of endomorphisms ([2], Theorem 7). In this connection let us set a problem: for a given group G find a group $\overline{G}$ such that $G \subset \overline{G}$, $\overline{G}$ belongs to a well-known class of groups, and $\overline{G}$ is determined by its semigroup of endomorphisms in the class of all groups. Some results of this kind are well known. For example, if A is a suitable Abelian group, then there exists a divisible Abelian group D such that the direct sum $A \oplus D$ is determined by its semigroup of endomorphisms in the class of all groups ([3], Corollary). In this paper we show that every finite p-group G is imbeddable into a finite p-group $\overline{G}$ such that $\overline{G}$ is determined by its semigroup of endomorphisms in the class of all groups.

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$$G(n,p)=(\ldots((C_p \operatorname{Wr} C_p) \operatorname{Wr} C_p) \operatorname{Wr} \ldots) \operatorname{Wr} C_p \quad (n \text{ factors}),$$

where C_p is the cyclic group of the order p ([4]). Our aim is to prove the following theorem (Theorem 6.1).

Theorem. *The group $G(n,p)$ is determined for each natural n and prime p by its semigroup of endomorphisms in the class of all groups.*

Corollary 1. *Every Sylow subgroup of a finite symmetric group is determined by its semigroup of endomorphisms in the class of all groups.*

Corollary 2. *Let G be a finite p-group. Then G is imbeddable into a finite p-group $\overline{G}$ such that $\overline{G}$ is determined by its semigroup of endomorphisms in the class of all groups.*

Corollary 2 follows from the fact that every finite p-group is imbeddable into some Sylow subgroup of some finite symmetric group.

We shall use the following notations: $\operatorname{End} G$ denotes a semigroup of all endomorphisms of a group G; $I(G)$ — a set of all idempotents of $\operatorname{End} G$; $\langle a, b, \ldots\rangle$ — a subgroup generated by elements $a, b, \ldots$; $\langle A, B, \ldots\rangle$ — a subgroup generated by subsets $A, B, \ldots$; $\hat{g}$ — an inner automorphism generated by an element g; G' — a commutator-group of G; $[A,B]=\langle a^{-1}b^{-1}ab \mid a\in A, b\in B\rangle$; $K_G(x)=\{y\in \operatorname{End} G \mid yx=xy=y\}$.

2. Preliminaries

Let A and B be finite groups. Then the standard wreath product of A and B, denoted as $A \operatorname{Wr} B$, is the semidirect product $A^B\lambda B$ (here and henceforth, λ=semidirect product) of A^B by B, where A^B is the set of all functions $f: B\to A$ and

$$(fg)(b)=f(b)\cdot g(b), \quad c^{-1}fc=f^c,$$
$$f^c(b)=f(bc^{-1}) \tag{2.1}$$

for all $b, c\in B$ and $f, g\in A^B$. General properties of wreath products are presented in [5].

Let $A_0=\{f\in A^B \mid f(b)=1 \text{ for all } b\neq 1\}$. Then A_0 is a subgroup of $A \operatorname{Wr} B$ and from (2.1) it follows that A^B is a direct product of subgroups $b^{-1}A_0b=A_0^b$, $b\in B$. As A and A_0 are isomorphic, we identify below $A_0=A$. Therefore,

$$A \operatorname{Wr} B=A^B\lambda B, \quad A^B=\Pi_{b\in B}\, b^{-1}Ab=\Pi_{b\in B}\, A^b.$$

The following two lemmas are simple corollaries from the definition of the wreath product.

Lemma 2.1. *If $A=A_1\lambda A_2$, then $A \operatorname{Wr} B=(A_1\lambda A_2) \operatorname{Wr} B= A_1^B\lambda(A_2 \operatorname{Wr} B)$. If C is a subgroup of A, then $\langle C, B\rangle=C \operatorname{Wr} B$.*

Lemma 2.2. *Each endomorphism u of A induces an endomorphism $\tilde{u}$ of $A \operatorname{Wr} B$ by laws*

$$b\tilde{u}=b, \quad b\in B,$$
$$(b^{-1}ab)\tilde{u}=b^{-1}(au)b, \quad b\in B, \quad a\in A.$$

Lemma 2.3 ([1], Lemma 1.1). *If G is a group and $x \in I(G)$, then $G = \operatorname{Ker} x \lambda \operatorname{Im} x$ and $\operatorname{Im} x = \{g \in G \mid gx = g\}$.*

From Lemmas 2.1 and 2.3 follows

Lemma 2.4. *If $x \in I(A)$ and $\tilde{x}$ is induced by x, then*

$$A \operatorname{Wr} B = \operatorname{Ker} \tilde{x} \lambda \operatorname{Im} \tilde{x} = (\operatorname{Ker} x)^B \lambda (\operatorname{Im} x \operatorname{Wr} B),$$
$$\operatorname{Ker} \tilde{x} = (\operatorname{Ker} x)^B, \quad \operatorname{Im} \tilde{x} = \operatorname{Im} x \operatorname{Wr} B.$$

Lemma 2.5 ([6], Lemmas 4.2 and 4.3). *Suppose that x is a projection of $G = A \operatorname{Wr} B = A^B \lambda B$ onto B and $y \in \operatorname{End} G$ such that $yx = xy = x$. Then to y corresponds a family $\{Y_b\}_{b \in B}$ of endomorphisms of A such that*

$$(aY_b)(a_1 Y_c) = (a_1 Y_c)(aY_b) \tag{2.2}$$

for each $a, a_1 \in A$ and $b, c \in B$, $b \neq c$. If B is finite then this correspondence is one-to-one. The endomorphism Y_b of A is given by an equation $Y_b = y_A \tau_b \pi_b$ where $y_A = y|A$, τ_b is a projection of $A^B = \Pi_{b \in B}\, b^{-1}Ab$ onto $b^{-1}Ab$ and $\pi_b : b^{-1}Ab \to A$ is a natural isomorphism: $(b^{-1}ab)\pi_b = a$, $a \in A$.

Denote further $y = \{Y_b\}_{b \in B}$.

Lemma 2.6 ([1], Lemma 1.6). *If G is a group and $x \in I(G)$ then $K_G(x) \cong \operatorname{End}(\operatorname{Im} x)$.*

Lemma 2.7 ([1], Lemma 1.5). *If $x, y \in \operatorname{End} G$ and $xy = yx$ then $(\operatorname{Im} x)y \subset \operatorname{Im} x$ and $(\operatorname{Ker} x)y \subset \operatorname{Ker} x$.*

3. Some properties of the group $G(n)$

Let us fix a prime number p. Let $A_1, A_2, \ldots$ be cyclic groups of the order p. Define a group $G(n)$ by induction:

$$G(2) = A_1 \operatorname{Wr} A_2 = A_1^{A_2} \lambda A_2,$$
$$G(n) = G(n-1) \operatorname{Wr} A_n = (\ldots((A_1 \operatorname{Wr} A_2) \operatorname{Wr} A_3) \operatorname{Wr} \ldots) \operatorname{Wr} A_n.$$

Below we drop brackets, i. e. $G(n) = A_1 \operatorname{Wr} A_2 \operatorname{Wr} \ldots \operatorname{Wr} A_n$. Then $G(n) = \langle A_1, \ldots, A_n \rangle$ and

$$|G(n)| = p^{1+p+\ldots+p^{n-1}}. \tag{3.1}$$

Lemma 3.1 ([6], Lemma 2.6). *The group $G(2)$ splits up*

$$G(2) = A_1 \operatorname{Wr} A_2 = ([A_2, A_1^{A_2}] \times A_1) \lambda A_2 = ([A_2, A_1^{A_2}] \lambda A_2) \lambda A_1$$

and $A_1^{A_2} = [A_2, A_1^{A_2}] \times A_1$.

Lemma 3.2 ([7], Theorem 2.1). *Suppose $x, y, z \in I(G)$, $x, y \in K_G(z)$ and $\operatorname{Im} z = \operatorname{Im} y \operatorname{Wr} \operatorname{Im} x$, where G is some group, $\operatorname{Im} y$ and $\operatorname{Im} x$ are cyclic groups of the order p. If G^* is another group such that $\operatorname{End} G \cong \operatorname{End} G^*$ and x^*, y^*, z^* are idempotents which correspond to x, y, z by this isomorphism, then $\operatorname{Im} z^* = \operatorname{Im} y^* \operatorname{Wr} \operatorname{Im} x^*$ and the groups $\operatorname{Im} y^*$ and $\operatorname{Im} x^*$ are also cyclic groups of the order p.*

In view of Lemma 3.2 it is assumed that $n \geqslant 3$.

Lemma 3.3. *There exist the idempotents $x_1, x_2, y_1, \ldots, y_n, z_1, \ldots, z_{n-1}$ of $\operatorname{End} G(n)$ such that:*

(a) *$\operatorname{Im} y_j = A_j \subset \operatorname{Ker} y_i$ for each $i \neq j$;*

(b) *$A_2 \subset \operatorname{Ker} x_1$, $\operatorname{Im} x_1 = \langle A_i \mid i \neq 2 \rangle$;*

(c) $A_1 \subset \operatorname{Ker} x_2$, $\operatorname{Im} x_2 = \langle A_i \mid i \geqslant 2 \rangle$;

(d) $\operatorname{Ker} x_2 = (\dots((A_1^{A_2})^{A_3})\dots)^{A_n} = A_1^{A_2A_3\dots A_n}$;

(e) $y_n, y_j \in K_{G(n)}(z_j)$ *and* $\operatorname{Im} z_j = \operatorname{Im} y_j \operatorname{Wr} \operatorname{Im} y_n$ *for each* $j = 1, \dots, n-1$.

Proof. The proof is done by the induction on n. Suppose that $n=3$. By Lemmas 3.1 and 2.1

$$\begin{aligned} G(3) &= G(2)\operatorname{Wr} A_3 = G(2)^{A_3}\lambda A_3 = \\ &= ([A_2, A_1^{A_2}] \times A_1)^{A_3}\lambda(A_2 \operatorname{Wr} A_3) = \\ &= ([A_2, A_1^{A_2}]^{A_3} \times A_1^{A_3})\lambda(A_2 \operatorname{Wr} A_3) = \\ &= ([A_2, A_1^{A_2}]\lambda A_2)^{A_3}\lambda(A_1 \operatorname{Wr} A_3) = \\ &= ([A_2, A_1^{A_2}]^{A_3}\lambda A_2^{A_3})\lambda(A_1 \operatorname{Wr} A_3). \end{aligned} \quad (3.2)$$

Choose x_1 and x_2 as projections of $G(3)$ onto subgroups $A_1 \operatorname{Wr} A_3$ and $A_2 \operatorname{Wr} A_3$, respectively. Due to such a choice, (b) and (c) hold. It is easy to show that (d) also holds. Indeed,

$$\operatorname{Ker} x_2 = ([A_2, A_1^{A_2}] \times A_1)^{A_3} = (A_1^{A_2})^{A_3} = A_1^{A_2A_3}.$$

By Lemma 3.1

$$A_2 \operatorname{Wr} A_3 = ([A_3, A_2^{A_3}]\lambda A_3)\lambda A_2, \quad (3.3)$$

$$A_1 \operatorname{Wr} A_3 = ([A_3, A_1^{A_3}]\lambda A_3)\lambda A_1. \quad (3.4)$$

Choose y_1, y_2 and y_3 as projections of $G(3)$ onto subgroups A_1, A_2 and A_3, respectively. Then (a) holds. Finally, choose $z_1 = x_1$ and $z_2 = x_2$. Clearly, by such a choice (e) holds.

Assume now that $n > 3$ and for each group $G(k)$, where $k < n$, the statements of the lemma are true. Then there exist the idempotents $\bar{x}_1, \bar{x}_2, \bar{y}_1, \dots, \bar{y}_{n-1}$ of $\operatorname{End} G(n-1)$ such that

$$\operatorname{Im} \bar{y}_j = A_j \subset \operatorname{Ker} \bar{y}_i \quad \text{for each} \quad i \neq j, \quad (3.5)$$

$$A_2 \subset \operatorname{Ker} \bar{x}_1, \quad \operatorname{Im} \bar{x}_1 = \langle A_i \mid i \neq 2 \rangle, \quad (3.6)$$

$$A_1 \subset \operatorname{Ker} \bar{x}_2, \quad \operatorname{Im} \bar{x}_2 = \langle A_j \mid j \geqslant 2 \rangle, \quad (3.7)$$

$$\operatorname{Ker} \bar{x}_2 = A_1^{A_2 \dots A_{n-1}} \quad (3.8)$$

$(i, j = 1, \dots, n-1)$. These idempotents induce, by Lemma 2.2, endomorphisms $x_1, x_2, \tilde{y}_1, \dots, \tilde{y}_{n-1}$ of $G(n) = G(n-1) \operatorname{Wr} A_n$. By Lemma 2.4

$$\operatorname{Im} x_j = \operatorname{Im} \bar{x}_j \operatorname{Wr} A_n = \langle \operatorname{Im} \bar{x}_j, A_n \rangle, \quad (3.9)$$

$$\operatorname{Ker} x_j = (\operatorname{Ker} \bar{x}_j)^{A_n} = \Pi_{b \in A_n} b^{-1}(\operatorname{Ker} \bar{x}_j) b, \quad (3.10)$$

$$\operatorname{Im} \tilde{y}_i = \operatorname{Im} \bar{y}_i \operatorname{Wr} A_n, \quad \operatorname{Ker} \tilde{y}_i = (\operatorname{Ker} \bar{y}_i)^{A_n} \quad (3.11)$$

$(i = 1, \dots, n-1;\ j = 1, 2)$. From (3.8) and (3.10) it follows that the statement (d) holds. Statements (b) and (c) follow from the formulas (3.6), (3.7), (3.9) and (3.10).

In view of Lemma 2.3, $G(n) = \operatorname{Ker} \tilde{y}_i \lambda \operatorname{Im} \tilde{y}_i$ and, by Lemma 3.1,

$$\operatorname{Im} \tilde{y}_i = \operatorname{Im} \bar{y}_i \operatorname{Wr} A_n = A_i \operatorname{Wr} A_n = ([A_n, A_i^{A_n}]\lambda A_n)\lambda A_i. \quad (3.12)$$

Let y_i be a projection of $G(n)$ onto subgroup A_i $(i = 1, \dots, n-1)$. Choose y_n as a projection of $G(n) = G(n-1)^{A_n}\lambda A_n$ onto A_n. Clearly, for such $y_1, \dots, y_n$ (a) holds. Finally, choose $z_i = \tilde{y}_i$ for each $i = 1, \dots, n-1$. Then the statement (e) follows from the equations (3.5), (3.11) and (3.12). The lemma is proved.

Fix now for the next reasonings the idempotents $x_1, x_2, y_1, \ldots, y_n, z_1, \ldots, z_{n-1}$ as in Lemma 3.3.

Lemma 3.4. $\operatorname{Ker} x_1 \cap \operatorname{Ker} x_2 = [A_2, A_1^{A_2}]^{A_3 \ldots A_n} \subset G(n)'$.

Proof. The proof is again based on the induction on n. If $n=3$, then, by the construction of x_1 and x_2, we have

$$\operatorname{Ker} x_1 = [A_2, A_1^{A_2}]^{A_3} \lambda A_2^{A_3}, \quad \operatorname{Ker} x_2 = [A_2, A_1^{A_2}]^{A_3} \lambda A_1^{A_3},$$
$$\operatorname{Ker} x_1 \cap \operatorname{Ker} x_2 = [A_2, A_1^{A_2}]^{A_3} \subset G(3)'$$

and the statement of the lemma is true.

Assume now that $n>3$ and for each group $G(k)$, where $k<n$, the statement of the lemma is true. As in the proof of Lemma 3.3 the idempotents x_1 and x_2 are induced by idempotents $\bar{x}_1$ and $\bar{x}_2$ of $\operatorname{End} G(n-1)$. By assumption of the induction

$$\operatorname{Ker} \bar{x}_1 \cap \operatorname{Ker} \bar{x}_2 = [A_2, A_1^{A_2}]^{A_3 \ldots A_{n-1}}. \tag{3.13}$$

From (3.10) and (3.13) it now follows that

$$\begin{aligned}\operatorname{Ker} x_1 \cap \operatorname{Ker} x_2 &= (\operatorname{Ker} \bar{x}_1)^{A_n} \cap (\operatorname{Ker} \bar{x}_2)^{A_n} = \\ &= (\operatorname{Ker} \bar{x}_1 \cap \operatorname{Ker} \bar{x}_2)^{A_n} = \\ &= [A_2, A_1^{A_2}]^{A_3 \ldots A_n} \subset G(n)'.\end{aligned}$$

The lemma is proved.

Since $\operatorname{Im} y_j \cong C_p$, then $G(n)' \subset \operatorname{Ker} y_j$ and from Lemma 3.4 follows

Lemma 3.5. $\operatorname{Ker} x_1 \cap \operatorname{Ker} x_2 \subset \operatorname{Ker} y_j$ *for each* $j=1, \ldots, n$.

Lemma 3.6. $\operatorname{Ker} y_n = ((\operatorname{Ker} x_1 \cap \operatorname{Ker} x_2) \times A_1^{A_3 \ldots A_n}) \lambda$
$\lambda (A_2 \operatorname{Wr} \ldots \operatorname{Wr} A_{n-1})^{A_n}$.

Proof. Let us prove the lemma by induction. If $n=3$, then the statement holds due to (3.2), Lemma 3.1 (applied to $A_2 \operatorname{Wr} A_3$) and Lemma 3.4. Assume that $n>3$ and for $G(k)$, where $k<n$, the statement of the lemma is true. Suppose that the idempotents $\bar{y}_1, \ldots, \bar{y}_{n-1}$ correspond to $G(n-1)$. Then $G(n-1) = \operatorname{Ker} \bar{y}_{n-1} \lambda \operatorname{Im} \bar{y}_{n-1} = \operatorname{Ker} \bar{y}_{n-1} \lambda A_{n-1}$ and by assumption of the induction

$$\begin{aligned}\operatorname{Ker} \bar{y}_{n-1} = ((\operatorname{Ker} \bar{x}_1 \cap \operatorname{Ker} \bar{x}_2) \times A_1^{A_3 \ldots A_{n-1}}) \lambda \\ \lambda (A_2 \operatorname{Wr} \ldots \operatorname{Wr} A_{n-2})^{A_{n-1}}.\end{aligned} \tag{3.14}$$

As $\operatorname{Ker} y_n = G(n-1)^{A_n}$, then from (3.13) and (3.14) it follows that

$$\begin{aligned}\operatorname{Ker} y_n &= G(n-1)^{A_n} = (\operatorname{Ker} \bar{y}_{n-1} \lambda A_{n-1})^{A_n} = \\ &= ((([A_2, A_1^{A_2}]^{A_3 \ldots A_{n-1}} \times A_1^{A_3 \ldots A_{n-1}}) \lambda \\ &\lambda (A_2 \operatorname{Wr} \ldots \operatorname{Wr} A_{n-2})^{A_{n-1}}) \lambda A_{n-1})^{A_n} = \\ &= ([A_2, A_1^{A_2}]^{A_3 \ldots A_n} \times A_1^{A_3 \ldots A_n}) \lambda \\ &\lambda (A_2 \operatorname{Wr} \ldots \operatorname{Wr} A_{n-1})^{A_n} = ((\operatorname{Ker} x_1 \cap \operatorname{Ker} x_2) \times \\ &\times A_1^{A_3 \ldots A_n}) \lambda (A_2 \operatorname{Wr} \ldots \operatorname{Wr} A_{n-1})^{A_n}.\end{aligned}$$

The lemma is proved.

Lemma 3.7. *If* $\tilde{y}_1, \ldots, \tilde{y}_n \in I(G(n))$ *and*

$$\begin{aligned} y_j \tilde{y}_j = \tilde{y}_j, \quad \tilde{y}_j y_j = y_j, \\ \tilde{y}_j x_1 = y_j x_1, \quad \tilde{y}_j x_2 = y_j x_2 \end{aligned} \tag{3.15}$$

for each $j=1, \ldots, n$, *then* $G(n) = \langle \operatorname{Im} \tilde{y}_1, \ldots, \operatorname{Im} \tilde{y}_n \rangle$.

Proof. Suppose that the assumptions of the lemma are true. Denote $M_0 = \langle \operatorname{Im} \tilde{y}_1, \ldots, \operatorname{Im} \tilde{y}_n \rangle$. Due to (3.15), we have $(g\tilde{y}_j)^{-1}(gy_j) \in \operatorname{Ker} x_1 \cap \operatorname{Ker} x_2$ for each $g \in G(n)$. Consequently, $\operatorname{Im} y_j \subset \operatorname{Im} \tilde{y}_j \cdot M$ and $G(n) = \langle \operatorname{Im} y_1, \ldots, \operatorname{Im} y_n \rangle = M_0 \cdot M$, where $M = \operatorname{Ker} x_1 \cap \operatorname{Ker} x_2$.

If $G(n) \neq M_0$, then by Sylow theorems there exists an invariant subgroup N of $G(n)$ such that $M_0 \subset N \neq G(n)$ and the factor-group $G(n)/N$ is Abelian. Then $G(n)' \subset N$ and by Lemma 3.4 $M \subset G(n)'$. Therefore, $M \subset N$ and $MN = N \neq G(n)$. On the other hand, since $M_0 \subset N$, then $G(n) = M_0 \cdot M \subset NM = MN = N$. The obtained contradiction shows that $G(n) = M_0$. The lemma is proved.

4. Further properties of x_1, x_2 and $y_1, \ldots, y_n$

In this section denote $G = G(n)$ and assume that $x_1, x_2, y_1, \ldots, y_n$ and $z_1, \ldots, z_{n-1}$ have the previous meaning. Suppose that a_j is a generator of $A_j = \operatorname{Im} y_j \cong C_p$. Then $A_j = \langle a_j \rangle$.

P r o p e r t y 4.1. *$K_G(y_j) \cong \operatorname{End} C_p$ for each $j = 1, \ldots, n$.*

Property 4.1 follows from Lemmas 2.6 and 3.3.

P r o p e r t y 4.2. *$x_1 x_2 = x_2 x_1$ and $K_G(x_1) \cong K_G(x_2) \cong \operatorname{End} G(n-1)$.*

P r o o f. By Lemma 3.3 $\operatorname{Im} x_1 = \langle A_i \mid i \neq 2 \rangle$ and $\operatorname{Im} x_2 = \langle A_j \mid j \geqslant 2 \rangle$. In view of Lemma 2.3, $x_1 x_2$ and $x_2 x_1$ act identically on the subgroups $A_3, \ldots, A_n$. Since by Lemma 3.3 $A_2 \subset \operatorname{Ker} x_1$ and $A_1 \subset \operatorname{Ker} x_2$, then $A_2 x_1 x_2 = A_2 x_2 x_1 = A_1 x_1 x_2 = A_1 x_2 x_1 = \langle 1 \rangle$. Consequently, $x_1 x_2 = x_2 x_1$.
From Lemma 2.1 it follows that $\operatorname{Im} x_1 = \langle A_i \mid i \neq 2 \rangle \cong G(n-1)$ and $\operatorname{Im} x_2 = \langle A_j \mid j \geqslant 2 \rangle \cong G(n-1)$. By Lemma 2.6, $K_G(x_1) \cong K_G(x_2) \cong \cong \operatorname{End} G(n-1)$. The property is proved.
The following four properties follow from Lemma 3.3.

P r o p e r t y 4.3. *$x_2 y_1 = y_1 x_2 = 0$.*

P r o p e r t y 4.4. *$x_1 y_1 = y_1 x_1 = y_1$.*

P r o p e r t y 4.5. *$x_2 y_j = y_j x_2 = y_j$ for each $j = 2, \ldots, n$.*

P r o p e r t y 4.6. *$y_n y_j = y_j y_n = 0$ for each $j = 1, \ldots, n-1$.*

P r o p e r t y 4.7. *The idempotent y_n has no orthogonal complement.*

Property 4.7 follows from the definition of the wreath product.

P r o p e r t y 4.8. *There exist $z_1, \ldots, z_{n-1} \in I(G)$ such that for each $j = 1, \ldots, n-1$ the following statements are true:* (a) $y_j, y_n \in K_G(z_j)$; (b) $\operatorname{Im} z_j = \operatorname{Im} y_j \operatorname{Wr} \operatorname{Im} y_n$.

Property 4.8 follows from Lemma 3.3.

P r o p e r t y 4.9. *If $u, v \in \operatorname{End} G$ and $y_n u = u$, $y_{n-1} v = v$, $u x_2 = v x_2 = 0$, then there exists $w \in \operatorname{End} G$ such that:* (a) $y_n w = u$; (b) $y_{n-1} w = v$; (c) *if $y \in \operatorname{End} G$ and $y y_n = y y_{n-1} = 0$, then $y w = 0$;* (d) $z_{n-1} w = w$.

P r o o f. By Property 4.6 $\operatorname{Im} y_n \subset \operatorname{Ker} y_{n-1}$ and $\operatorname{Im} y_{n-1} \subset \operatorname{Ker} y_n$. Basing on Lemmas 2.3 and 2.7 we have $G = (M \lambda \operatorname{Im} y_{n-1}) \lambda \operatorname{Im} y_n = (M \lambda \lambda \operatorname{Im} y_n) \lambda \operatorname{Im} y_{n-1}$, where $M = \operatorname{Ker} y_{n-1} \cap \operatorname{Ker} y_n$. Therefore, $G/M = = \langle a_n M \rangle \times \langle a_{n-1} M \rangle \cong C_p \times C_p$. If $u, v \in \operatorname{End} G$ and $y_n u = u$, $y_{n-1} v = v$, $u x_2 = v x_2 = 0$, then $a_n u$, $a_{n-1} v \in \operatorname{Ker} x_2$. As $\operatorname{Ker} x_2$ is by Lemma 3.3 an elementary Abelian p-group, we can define an endomorphism w of G by setting $w = \pi u_0$, where $\pi : G \to G/M$ is a natural homomorphism and $(a_n M) u_0 = a_n u$, $(a_{n-1} M) u_0 = a_{n-1} v$. From the definition of w it follows that $y_n w = y_n u$ and $y_{n-1} w = y_{n-1} v$. Since $y_n u = u$ and $y_{n-1} v = v$, then (a) and (b) are true. If $y \in \operatorname{End} G$ and $y y_n = y y_{n-1} = 0$, then $\operatorname{Im} y \subset M \subset \subset \operatorname{Ker} w$, $y w = 0$ and so (c) is also true.

For the proof of (d) observe that by Property 4.8, $y_n, y_{n-1} \in K_G(z_{n-1})$. Hence, $\operatorname{Ker} z_{n-1} \subset M \subset \operatorname{Ker} w$, $(\operatorname{Ker} z_{n-1})(z_{n-1}w) = (\operatorname{Ker} z_{n-1}) w = \langle 1 \rangle$ and so z_{n-1}w and w act equally on the subgroup $\operatorname{Ker} z_{n-1}$. Since by Property 4.8 $\operatorname{Im} z_{n-1} = \langle a_{n-1}, a_n \rangle$ and

$$a_n (z_{n-1} w) = (a_n y_n)(z_{n-1} w) = a_n (y_n w) = a_n w,$$

$$a_{n-1} (z_{n-1} w) = (a_{n-1} y_{n-1})(z_{n-1} w) = a_{n-1} (y_{n-1} w) = a_{n-1} w,$$

then $z_{n-1}w$ and w coincide on the subgroup $\operatorname{Im} z_{n-1}$. In view of the equation $G = \operatorname{Ker} z_{n-1} \leftthreetimes \operatorname{Im} z_{n-1}$, $z_{n-1}w = w$ holds. The property is proved.

Define for each $y_1, \ldots, y_n$ a set

$$[y_j] = \{z \in I(G) \mid z y_j = y_j,\ y_j z = z,\ z x_1 = y_j x_1,\ z x_2 = y_j x_2\}.$$

Property 4.10. *If $z \in \operatorname{End} G$, $\tilde{y}_j \in [y_j]$ and $\tilde{y}_j z = \tilde{y}_j$ for each $j = 1, \ldots, n$, then $z = 1$.*

Property 4.10 follows directly from Lemma 3.7.

Property 4.11. *If $u \in \operatorname{End} G$ and $y_j u = 0$ for each $j = 1, \ldots, n$, then $u = 0$.*

This property follows from the fact that $G = \langle \operatorname{Im} y_1, \ldots, \operatorname{Im} y_n \rangle$.

Property 4.12. *If $z \in K_G(x_2)$ and $y_j z = 0$ for each $j = 2, \ldots, n$, then $z = 0$.*

Property 4.12 is evident. Indeed, by Lemma 3.3 $\operatorname{Im} x_2 = \langle \operatorname{Im} x_i \mid i \geqslant 2 \rangle$.

Property 4.13. *There exists $z \in \operatorname{End} G$ such that: (a) $y_j z = y_j z y_{j-1} \neq 0$ for each $j = 2, \ldots, n$; (b) if $u \in K_G(x_2)$ and $y_n u = u \neq 0$, then $uz \neq 0$.*

Proof. By the definition of the wreath product and Lemma 3.3 it is clear that $\operatorname{Im} x_2 = \langle A_2, \ldots, A_n \rangle = A_2 \operatorname{Wr} \ldots \operatorname{Wr} A_n$ and $\langle A_1, \ldots, A_{n-1} \rangle = A_1 \operatorname{Wr} \ldots \operatorname{Wr} A_{n-1}$. Consequently, a map z defined by

$$(\operatorname{Ker} x_2) z = \langle 1 \rangle, \qquad a_j z = a_{j-1}; \qquad j = 2, \ldots, n,$$

induces an endomorphism of $G(n)$ such that z is injective on the subgroup $\operatorname{Im} x_2$. By this definition, (a) is true.

Suppose that $u \in K_G(x_2)$ and $y_n u = u \neq 0$. Then $\operatorname{Im} u = \langle a_n u \rangle \subset \operatorname{Im} x_2$, $a_n u \neq 1$, $a_n (uz) = (a_n u) z \neq 1$, $uz \neq 0$ and so (b) is also true. The property is proved.

Property 4.14. *$|[y_j]|$ is a power of p for each $j = 1, \ldots, n$.*

Proof. Assume that $z \in [y_j]$. From the definition of $[y_j]$ we have $\operatorname{Ker} y_j = \operatorname{Ker} z$ and $\operatorname{Im} y_j \cong \operatorname{Im} z$. Since $G = \operatorname{Ker} y_j \leftthreetimes \operatorname{Im} y_j$, z is determined by its action on the element a_j. As $z y_j = y_j$ then $a_j^{-1} \cdot a_j z \in \operatorname{Ker} y_j$, i. e., $a_j z = a_j c$ for some $c \in \operatorname{Ker} y_j$. From the equations $z x_1 = y_j x_1$ and $z x_2 = y_j x_2$ it follows that $c \in \operatorname{Ker} x_1 \cap \operatorname{Ker} x_2$. In addition, $a_j c$ is an element of the order p. Conversely, if $c \in \operatorname{Ker} x_1 \cap \operatorname{Ker} x_2$ such that $a_j c$ is an element of the order p, then by Lemma 3.5 $c \in \operatorname{Ker} y_j$ and a map z, defined by $a_j z = a_j c$, $(\operatorname{Ker} y_j) z = \langle 1 \rangle$, is an endomorphism of G and $z \in [y_j]$. Consequently, $|[y_j]|$ is equal to the number of elements $a_j c$ of the order p where $c \in \operatorname{Ker} x_1 \cap \operatorname{Ker} x_2$. This is a basic fact for the proof of Property 4.14.

The proof is by induction on n. Suppose that $n = 3$. In view of Lemma 3.4

$$\operatorname{Ker} x_1 \cap \operatorname{Ker} x_2 = [A_2, A_1^{A_2}]^{A_3} = \Pi_{b \in A_3}\, b^{-1} [A_2, A_1^{A_2}]\, b$$

and therefore $\operatorname{Ker} x_1 \cap \operatorname{Ker} x_2$ is an elementary Abelian p-group. Since a_1 commutes by Lemma 3.1 with each element of $[A_2, A_1^{A_2}]$, then a_1

commutes with each element of $\operatorname{Ker} x_1 \cap \operatorname{Ker} x_2$ and a_1c is an element of the order p for each $c \in \operatorname{Ker} x_1 \cap \operatorname{Ker} x_2$. Consequently, $|[y_1]|$ is a power of p.

Every element a_2c_1 of $G(2)$, where $c_1 \in [A_2, A_1^{A_2}]$, is an element of the order p. This a_2c_1 commutes with each element $c_2 \in b^{-1}[A_2, A_1^{A_2}]b$, $b \in A_3 \setminus \langle 1 \rangle$. Hence, $a_2c = a_2c_1c_2$ is an element of the order p for each $c \in \operatorname{Ker} x_1 \cap \operatorname{Ker} x_2$ and $|[y_2]|$ is a power of p.

Every element a_3c of the order p of $G = G(3) = G(2) \operatorname{Wr} A_3$, where $c \in \operatorname{Ker} x_1 \cap \operatorname{Ker} x_2$, is conjugate with a_3 ([5], Theorem 10.1), i.e. there exists $d \in \operatorname{Ker} y_3$ such that $a_3c = d^{-1}a_3d$. By Lemma 3.6

$$\operatorname{Ker} y_3 = ((\operatorname{Ker} x_1 \cap \operatorname{Ker} x_2) \times A_1^{A_3}) \lambda A_2^{A_3}. \tag{4.1}$$

From (4.1) it follows that $c \in \operatorname{Ker} x_1 \cap \operatorname{Ker} x_2$ if and only if $d \in \operatorname{Ker} x_1 \cap \operatorname{Ker} x_2$. Therefore, the number of elements a_3c of the order p is equal to $[(\operatorname{Ker} x_1 \cap \operatorname{Ker} x_2) : C_{\operatorname{Ker} x_1 \cap \operatorname{Ker} x_2}(a_3)]$. This number is a power of p and so is $|[y_3]|$. Consequently, for $n=3$ the property is true.

Assume now that $n>3$ and for $G(k)$, where $k<n$, the statement of the property is true. Suppose that $\bar{x}_1$, $\bar{x}_2$ and $\bar{y}_1, \ldots, \bar{y}_{n-1}$ are similar idempotents for $G(n-1)$ as x_1, x_2 and $y_1, \ldots, y_n$ are for $G(n)$. By assumption of the induction the number of elements a_jc_1 of the order p, where $c_1 \in \operatorname{Ker} \bar{x}_1 \cap \operatorname{Ker} \bar{x}_2 = [A_2, A_1^{A_2}]^{A_3 \cdots A_{n-1}}$, is a power of p $(j=1, \ldots, n-1)$. Since

$$\operatorname{Ker} x_1 \cap \operatorname{Ker} x_2 = [A_2, A_1^{A_2}]^{A_3 \cdots A_n} = (\operatorname{Ker} \bar{x}_1 \cap \operatorname{Ker} \bar{x}_2)^{A_n} =$$
$$= (\operatorname{Ker} \bar{x}_1 \cap \operatorname{Ker} \bar{x}_2) \times (\Pi_{b \in A_n \setminus \langle 1 \rangle} b^{-1} (\operatorname{Ker} \bar{x}_1 \cap \operatorname{Ker} \bar{x}_2) b) =$$
$$= (\operatorname{Ker} \bar{x}_1 \cap \operatorname{Ker} \bar{x}_2) \times S$$

and S is an elementary Abelian p-group, all the elements a_jc of the order p, where $c \in \operatorname{Ker} x_1 \cap \operatorname{Ker} x_2$, can be expressed in the form $a_jc_1c_2$ where $c_1 \in \operatorname{Ker} \bar{x}_1 \cap \operatorname{Ker} \bar{x}_2$, a_jc_1 is of the order p and c_2 is a suitable element of S. Hence, the number of elements a_jc is a power of p and so is $|[y_j]|$. This holds for $j=1, \ldots, n-1$. Similar reasoning as in the case $n=3$ shows that $|[y_n]|$ is a power of p. The property is proved.

Property 4.15. $|\{u \in \operatorname{End} G \mid y_nu = u,\ ux_2 = 0\}| = (p)^{p^{n-1}}$.

Proof. Since $G = \operatorname{Ker} y_n \lambda \operatorname{Im} y_n$, the equations $y_nu = u$ and $ux_2 = 0$ are equivalent to conditions $(\operatorname{Ker} y_n)u = \langle 1 \rangle$ and $(\operatorname{Im} y_n)u \subset \operatorname{Ker} x_2$. Therefore, the number of such endomorphisms u is equal to the number of homomorphisms $\operatorname{Im} y_n \to \operatorname{Ker} x_2$. As $\operatorname{Im} y_j = A_j \cong C_p$ for each $j=1, \ldots, n$ and by Lemma 3.3 $\operatorname{Ker} x_2 = A_1^{A_2 \cdots A_n}$, the number of the mentioned homomorphisms is $(p)^{p^{n-1}}$. The property is proved.

5. A property of an automorphism of the order p of $G(n)$

Suppose that $x_1, x_2, y_1, \ldots, y_n$ have the previous meaning and $z_1, \ldots, z_{n-1}$ are chosen as in Property 4.8. Then $A_j = \operatorname{Im} y_j = \langle a_j \rangle \cong C_p$. Denote $G = G(n)$ and

$$b = a_n, \quad B = A_n = \langle b \rangle, \quad A = \langle a_1, \ldots, a_{n-1} \rangle = G(n-1).$$

Hence, $G = G(n-1) \operatorname{Wr} A_n = A \operatorname{Wr} B = A^B \lambda B$ and

$$A^B = \Pi_{k=0}^{p-1} b^{-k} A b^k. \tag{5.1}$$

By Lemma 3.3

$$\operatorname{Im} z_k = A_k \operatorname{Wr} A_n = A_k \operatorname{Wr} B \tag{5.2}$$

for each $k=1, \ldots, n-1$.

L e m m a 5.1. *Let α be an automorphism of the order p of G and $\alpha y_n = y_n\alpha = y_n$, $y_j\alpha z_j = y_j\alpha$ for each $j=1, \ldots, n-1$. Then for each $i, j = 1, \ldots, n-1$; $i \neq j$, there exists an endomorphism u of G such that the following statements are true:* (a) $y_j u = y_j\alpha$; (b) $y_n u y_i = y_n u \neq 0$; (c) *if $v \in \operatorname{End} G$ and $vy_n = vy_j = 0$, then $vu = 0$.*

P r o o f. Suppose that the assumptions of the lemma are true. Choose $i, j \in \{1, \ldots, n-1\}$, $i \neq j$. First we show that $a_i \cdot a_j\alpha = a_j\alpha \cdot a_i$.

From $y_j\alpha z_j = y_j\alpha$ and $y_i\alpha z_i = y_i\alpha$ it follows that $a_j\alpha = a_j(y_j\alpha) \in \operatorname{Im} z_j$ and $a_i\alpha = a_i(y_i\alpha) \in \operatorname{Im} z_i$. On the other hand, by Lemma 3.3 $a_j, a_i \in \operatorname{Ker} y_n$ and by Lemma 2.7 $(\operatorname{Ker} y_n)\alpha \subset \operatorname{Ker} y_n$. Therefore, $a_j\alpha \in \operatorname{Im} z_j \cap \operatorname{Ker} y_n$ and $a_i\alpha \in \operatorname{Im} z_i \cap \operatorname{Ker} y_n$. From the construction of y_k and z_k in the proof of Lemma 3.3, it is clear that $\operatorname{Im} z_k \cap \operatorname{Ker} y_n = A_k^{A_n} = A_k^B$. Consequently, $a_j\alpha \in A_j^B \subset A^B$ and $a_i\alpha \in A_i^B \subset A^B$. From the direct product decomposition (5.1) follow

$$a_i\alpha = a_i^{t_0} \cdot b^{-1} a_i^{t_1} b \cdot \ldots \cdot b^{-(p-1)} a_i^{t_{p-1}} b^{p-1},$$

$$a_j\alpha = a_j^{s_0} \cdot b^{-1} a_j^{s_1} b \cdot \ldots \cdot b^{-(p-1)} a_j^{s_{p-1}} b^{p-1}$$

for some integers $t_0, \ldots, t_{p-1}, s_0, \ldots, s_{p-1}$.

In view of Lemma 2.5 $\alpha = \{Y_k\}_{k=0, 1, \ldots, n-1}$; $Y_k \in \operatorname{End} A$. By definition of such endomorphisms

$$a_i Y_k = a_i^{t_k}, \quad a_j Y_k = a_j^{s_k} \quad \text{for each } k = 0, 1, \ldots, p-1$$

and

$$a_i Y_k \cdot a_j Y_l = a_j Y_l \cdot a_i Y_k \quad \text{for each } k \neq l.$$

Hence,

$$a_i^{t_k} \cdot a_j^{s_l} = a_j^{s_l} \cdot a_i^{t_k} \tag{5.3}$$

for each $k, l = 0, 1, \ldots, p-1$; $k \neq l$. Since $a_i a_j \neq a_j a_i$, from (5.3) we obtain that

$$t_k s_l = 0 \quad \text{for each } k \neq l. \tag{5.4}$$

Assume that $s_0 \neq 0$. Then by (5.4) $t_1 = \ldots = t_{p-1} = 0$, $a_i\alpha = a_i^{t_0}$, $t_0 \neq 0$ and again by (5.4) $s_1 = \ldots = s_{p-1} = 0$, $a_j\alpha = a_j^{s_0}$, $s_0 \neq 0$. In view of Fermat Theorem $s_0^p \equiv s_0 \pmod p$ and $t_0^p \equiv t_0 \pmod p$. Hence,

$$a_i\alpha^p = a_i^{t_0^p} = a_i^{t_0}; \quad a_j\alpha^p = a_j^{s_0^p} = a_j^{s_0}.$$

But $\alpha^p = 1$. Therefore, $s_0 = t_0 = 1$ and $a_i\alpha = a_i$, $a_j\alpha = a_j$. As i is a suitable element of $\{1, \ldots, n-1\} \setminus \{j\}$ and $b\alpha = a_n\alpha = a_n(y_n\alpha) = a_n y_n = a_n = b$, then $a_k\alpha = a_k$ for each $k = 1, \ldots, n$, i. e. $\alpha = 1$. This contradiction shows that $s_0 = 0$. Due to the direct product decomposition (5.1) we have

$$a_i \cdot a_j\alpha = a_j\alpha \cdot a_i. \tag{5.5}$$

Let us construct now an endomorphism u. Apply Lemma 3.1 to the wreath product (5.2) (take $k=j$):

$$\operatorname{Im} z_j = (N \lambda A_j) \lambda A_n = (N \lambda A_n) \lambda A_j,$$

where $N = [A_n, A_j^{A_n}] = \operatorname{Im} z_j \cap \operatorname{Ker} y_j' \cap \operatorname{Ker} y_n$. Then the factor-group $\operatorname{Im} z_j / N$ is Abelian and

$$\operatorname{Im} z_j/N = (A_j N/N) \times (A_n N/N) = \langle a_j N \rangle \times \langle a_n N \rangle \cong C_p \times C_p.$$

Choose $u = z_j \pi z$ where $\pi : \operatorname{Im} z_j \to \operatorname{Im} z_j/N$ is a natural homomorphism and $z : \langle a_j N \rangle \times \langle a_n N \rangle \to G(n)$, $(a_n N) z = a_i$, $(a_j N) z = a_j\alpha$. Due to (5.5) the endomorphism u is correctly defined.

Since $a_j u = a_j(z_j\pi z) = a_j(\pi z) = (a_j N) z = a_j\alpha$, then $y_j u = y_j\alpha$ and (a) is true. Similarly, $a_n u = a_i$ and $a_n(uy_i) = a_i y_i = a_i = a_n u$, i. e. $y_n u y_i = y_n u \neq 0$ and (b) is also true.

Suppose that $v \in \operatorname{End} G(n)$ and $vy_n = vy_j = 0$. Then

$$\operatorname{Im} v \subset \operatorname{Ker} y_n \cap \operatorname{Ker} y_j. \tag{5.6}$$

By Property 4.8 (a), $\operatorname{Ker} z_j \subset \operatorname{Ker} y_n \cap \operatorname{Ker} y_j$. As $G(n) = \operatorname{Ker} z_j \leftthreetimes \operatorname{Im} z_j$, then $\operatorname{Ker} y_n \cap \operatorname{Ker} y_j = \operatorname{Ker} z_j \leftthreetimes N \subset \operatorname{Ker} u$ and by (5.6) $\operatorname{Im} v \subset \operatorname{Ker} u$. Hence, $vu = 0$ and (c) is true. This completes the proof of Lemma 5.1.

6. Main Theorem

Theorem 6.1. *The group $G(n)$ is determined by its semigroup of endomorphisms in the class of all groups for each $n \geqslant 2$.*

Proof. Let us use induction on n. The group $G(2)$ is determined by its semigroup of endomorphisms in the class of all groups ([7], Theorem 3.1). Assume now that $n > 2$ and the group $G(k)$ is determined for each $k < n$ by its semigroup of endomorphisms.

Let G be a suitable group such that

$$\operatorname{End} G \cong \operatorname{End} G(n). \tag{6.1}$$

We will show that the groups G and $G(n)$ are isomorphic.

As the semigroup $\operatorname{End} G$ is finite then so is the group G ([8], Theorem 2). In the semigroup $\operatorname{End} G(n)$ there exist idempotents $x_1, x_2, y_1, \ldots, y_n, z_1, \ldots, z_{n-1}$ for which Properties 4.1—4.15 and Lemma 5.1 are true. In view of (6.1) and Lemma 3.2 there exist such idempotents also in the semigroup $\operatorname{End} G$. Suppose further that $x_1, x_2, y_1, \ldots, y_n, z_1, \ldots, z_{n-1}$ are idempotents of $\operatorname{End} G$ such that Properties 4.1—4.15 and Lemma 5.1 are true.

From Lemma 2.6, Property 4.2 and the assumption of the induction we conclude that $\operatorname{Im} x_1 \cong G(n-1)$ and $\operatorname{Im} x_2 \cong G(n-1)$. Therefore, $\operatorname{Im} x_1$ and $\operatorname{Im} x_2$ are p-groups. Since every finite Abelian group is determined by its semigroup of endomorphisms in the class of all groups ([1], Theorem 4.2), then, by Property 4.1 and Lemma 2.6, $\operatorname{Im} y_j \cong C_p$ for each $j = 1, \ldots, n$. Suppose next that $\operatorname{Im} y_j = \langle a_j \rangle$.

Further proof is developed in the following lemmas.

Lemma 6.2. *The group G is a p-group.*

Proof. By Lemma 2.7 and Property 4.3, $(\operatorname{Ker} x_2) x_1 \subset \operatorname{Ker} x_2$. Hence, $\operatorname{Ker} x_2 = (\operatorname{Ker} x_1 \cap \operatorname{Ker} x_2) \leftthreetimes (\operatorname{Im} x_1 \cap \operatorname{Ker} x_2)$ and $G = ((\operatorname{Ker} x_1 \cap \operatorname{Ker} x_2) \leftthreetimes (\operatorname{Im} x_1 \cap \operatorname{Ker} x_2)) \leftthreetimes \operatorname{Im} x_2$. Since $\operatorname{Im} x_1$ and $\operatorname{Im} x_2$ are p-groups, then all p'-elements of G are contained in the subgroup $\operatorname{Ker} x_1 \cap \operatorname{Ker} x_2$.

From Properties 4.4 and 4.5 it follows that

$$\operatorname{Ker} x_1 \cap \operatorname{Ker} x_2 \subset \operatorname{Ker} y_j \tag{6.2}$$

for each $j = 1, \ldots, n$. Assume that q is a prime different from p and h is a suitable q-element of G. Then $h \in \operatorname{Ker} x_1 \cap \operatorname{Ker} x_2$. Fix $j \in \{1, \ldots, n\}$. If $z \in [y_j]$, then by (6.2) $z\hat{h} \in [y_j]$. Therefore, $\hat{h}$ acts on the set $[y_i]$ so that the image is again contained in $[y_j]$. As by Property 4.14 $|[y_j]|$ is a power of p and h is a q-element, there exists $\tilde{y}_j \in [y_j]$ such that $\tilde{y}_j \hat{h} = \tilde{y}_j$. Since j is a suitable element of $\{1, \ldots, n\}$, then, by Property 4.10, $\hat{h} = 1$. Hence, all p'-elements of G are contained in its centre. Consequently, $G = G_p \times G_{p'}$ where G_p and $G_{p'}$ are Sylow p-subgroup and Hall p'-subgroup of G, respectively.

Assume that π is a projection of G onto its subgroup $G_{p'}$. Since $\operatorname{Im} y_j \subset G_p$, then $y_j\pi = 0$ for each $j=1, \ldots, n$. By Property 4.11 $\pi = 0$. Therefore, $G = G_p$ and G is a p-group. The lemma is proved.

Lemma 6.3. $\operatorname{Im} x_2 = \langle a_2, \ldots, a_n\rangle$.

Proof. Denote $A_0 = \langle a_2, \ldots, a_n\rangle$. By Property 4.5, $\operatorname{Im} y_j \subset \operatorname{Im} x_2$ for each $j = 2, \ldots, n$. Hence, $A_0 \subset \operatorname{Im} x_2$. If $\operatorname{Im} x_2 \neq A_0$, then there exists an invariant subgroup N_0 of $\operatorname{Im} x_2$ such that $A_0 \subset N_0$ and $\operatorname{Im} x_2/N_0 \cong C_p$. Define now an endomorphism z of G by setting $z = x_2uv$, where $u : \operatorname{Im} x_2 \to \operatorname{Im} x_2/N_0$ is a natural homomorphism and v is some isomorphism $\operatorname{Im} x_2/N_0 \cong \langle a_n\rangle$. Then $z \neq 0$ and $y_jz = 0$ for each $j = 2, \ldots, n$. By Property 4.12, $z = 0$. This contradiction shows that $\operatorname{Im} x_2 = A_0$. The lemma is proved.

Similarly, it follows from Property 4.11:

Lemma 6.4. $G = \langle a_1, \ldots, a_n\rangle$.

Lemma 6.5. *The group* $\operatorname{Ker} x_2$ *is an elementary Abelian p-group.*

Proof. Denote $P = \{g \in \operatorname{Ker} x_2 \mid g^p = 1\}$. We show first that P is an Abelian subgroup of $\operatorname{Ker} x_2$. Choose $a, b \in P$. By Property 4.8 $y_n, y_{n-1} \in K_G(z_{n-1})$ and $\operatorname{Im} z_{n-1} = \operatorname{Im} y_{n-1} \operatorname{Wr} \operatorname{Im} y_n$. In view of Property 4.6 and Lemmas 2.3, 2.7, 3.1

$$\operatorname{Im} z_{n-1} = (N \lambda \operatorname{Im} y_{n-1}) \lambda \operatorname{Im} y_n = (N \lambda \operatorname{Im} y_n) \lambda \operatorname{Im} y_{n-1},$$

where $N = \operatorname{Im} z_{n-1} \cap \operatorname{Ker} y_{n-1} \cap \operatorname{Ker} y_n$ and N is an elementary Abelian p-group. Hence, $\operatorname{Im} z_{n-1}/N = \langle a_{n-1}N\rangle \times \langle a_nN\rangle \cong C_p \times C_p$ and there exist endomorphisms $u = z_{n-1}y_nu_0$ and $v = z_{n-1}y_{n-1}v_0$ of G, where $a_nu_0 = a$ and $a_{n-1}v_0 = b$. By this definition $y_nu = u$, $y_{n-1}v = v$ and $ux_2 = vx_2 = 0$. In view of Property 4.9 there exists $w \in \operatorname{End} G$ such that the conditions (a)—(d) of Property 4.9 are true. By condition (d) $\operatorname{Im} w = (\operatorname{Im} z_{n-1})w$.

Suppose that c is a suitable element of subgroup N. Then there exists an endomorphism y of G such that $y = z_{n-1}y_nw_0 = y_nw_0$ and $a_nw_0 = c$. Then $yy_n = yy_{n-1} = 0$ and, by Property 4.9 (c), $yw = 0$. Therefore, $N \subset \operatorname{Ker} w$ and the group $\operatorname{Im} w = (\operatorname{Im} z_{n-1})w$ is Abelian as $\operatorname{Im} z_{n-1}/N$ is Abelian. Since $a, b \in \operatorname{Im} w$, then $ab = ba$. Consequently, P is an Abelian subgroup of $\operatorname{Ker} x_2$.

Next we show that $P = \operatorname{Ker} x_2$. Clearly, P is an invariant subgroup of G. Hence, $\langle P, \operatorname{Im} x_2\rangle = P \lambda \operatorname{Im} x_2$. By Property 4.3, $\operatorname{Im} y_1 \subset \operatorname{Ker} x_2$. Therefore, $a_1 \in P$. In view of Lemma 6.3, $a_1, \ldots, a_n \in P \lambda \operatorname{Im} x_2$. From Lemma 6.4 it now follows that $G = P \lambda \operatorname{Im} x_2$. As $P \subset \operatorname{Ker} x_2$, then $P = \operatorname{Ker} x_2$. Consequently, $\operatorname{Ker} x_2$ is an elementary Abelian p-group. The lemma is proved.

Denote $A = \langle \operatorname{Im} y_1, \ldots, \operatorname{Im} y_{n-1}\rangle = \langle a_1, \ldots, a_{n-1}\rangle$.

Lemma 6.6. $A \cong G(n-1)$.

Proof. By Property 4.13 there exists $z \in \operatorname{End} G$ for which the conditions (a) and (b) are true. Hence, $a_jzy_{j-1} = a_jz \neq 1$ for each $j = 2, \ldots, n$. In view of Lemma 2.3, $a_jz \in \operatorname{Im} y_{j-1} = \langle a_{j-1}\rangle$. Therefore, $(\operatorname{Im} x_2)z = \langle a_2, \ldots, a_n\rangle z \subset A$. Since $a_jz \neq 1$, then $(\operatorname{Im} x_2)z = A$.

It is sufficient to show that z is injective on the subgroup $\operatorname{Im} x_2$. By contradiction assume that there exists $a \in \operatorname{Im} x_2$ for which $az = 1$. Since $\operatorname{Im} y_n = \langle a_n\rangle \subset \operatorname{Im} x_2$ and by Property 4.5, $y_n \in K_G(x_2)$, it follows from Lemmas 2.3 and 2.7 that $\operatorname{Im} x_2 = (\operatorname{Im} x_2 \cap \operatorname{Ker} y_n) \lambda \operatorname{Im} y_n$. Hence, there exists an endomorphism $u = x_2y_nu_0 = y_nu_0$ of G such that $a_nu_0 = a$. Then $\operatorname{Im} u = \langle a\rangle$ and $uz = 0$. On the other hand, by definition $u \in K_G(x_2)$, $y_nu = u$, $u \neq 0$. In view of Property 4.13 (c), $uz \neq 0$. This contradiction shows that z is injective on $\operatorname{Im} x_2$. Consequently, $A = (\operatorname{Im} x_2)z \cong \operatorname{Im} x_2 \cong G(n-1)$. The lemma is proved.

In view of Lemma 2.3, $G = \mathrm{Ker}\, y_n \lambda \mathrm{Im}\, y_n = \mathrm{Ker}\, y_n \lambda \langle a_n \rangle$. By Property 4.6, $\mathrm{Im}\, y_j = \langle a_j \rangle \subset \mathrm{Ker}\, y_n$ for each $j = 1, \ldots, n-1$. Hence, $A \subset \subset \mathrm{Ker}\, y_n$. Our aim is to show that $\mathrm{Ker}\, y_n = A^B$ and $G = A^B \lambda B$, where $B = \mathrm{Im}\, y_n = \langle a_n \rangle$. In this connection we use Lemma 5.1. Let $\alpha = \hat{a}_n$ be an inner automorphism of G generated by a_n. Since $\langle a_n \rangle \cong C_p$, then by Property 4.7 the order of α is p. Clearly, $\alpha y_n = y_n \alpha = y_n$.

Lemma 6.7. *$a_i \cdot a_j \alpha = a_j \alpha \cdot a_i$ for each $i, j = 1, \ldots, n-1$.*

Proof. As in the proof of Lemma 6.5, it follows from Property 4.8 that

$$\mathrm{Im}\, z_j = \mathrm{Im}\, y_j \,\mathrm{Wr}\, \mathrm{Im}\, y_n = (N_j \lambda \mathrm{Im}\, y_j) \lambda \mathrm{Im}\, y_n = (N_j \lambda \mathrm{Im}\, y_n) \lambda \mathrm{Im}\, y_j, \qquad (6.3)$$

where $N_j = \mathrm{Im}\, z_j \cap \mathrm{Ker}\, y_j \cap \mathrm{Ker}\, y_n$, $j = 1, \ldots, n-1$. Hence, $\mathrm{Im}\,(y_j \alpha) = \mathrm{Im}\,(y_j \hat{a}_n) \subset \mathrm{Im}\, z_j$ and by Lemma 2.3 $y_j \alpha z_j = y_j \alpha$. Therefore, α satisfies the assumptions of Lemma 5.1.

Fix $i, j = 1, \ldots, n-1$; $i \neq j$. There exists, by Lemma 5.1, an endomorphism u of G such that the conditions (a), (b) and (c) are true. By (a), $a_j u = a_j \alpha$. In view of Lemma 2.3 and (b), $a_n u \in \langle a_i \rangle = \mathrm{Im}\, y_i$, $a_n u \neq 1$. Since $\mathrm{Im}\, z_j = \langle a_j, a_n \rangle$, then $(\mathrm{Im}\, z_j) u = \langle a_j \alpha, a_i \rangle$.

Let c be a suitable element of N_j and $v = z_j y_n v_0 = y_n v_0$, $a_n v_0 = c$. Then $v \in \mathrm{End}\, G$, $v y_n = v y_j = 0$ and by Lemma 5.1 (c), $vu = 0$, i. e. $cu = 1$. Hence, $N_j \subset \mathrm{Ker}\, u$ and by (6.3) the group $(\mathrm{Im}\, z_j) u = \langle a_j \alpha, a_i \rangle$ is Abelian. Therefore, $a_i \cdot a_j \alpha = a_j \alpha \cdot a_i$ for each $i, j = 1, \ldots, n-1$; $i \neq j$. If $i = j$, then the equation $a_i \cdot a_i \alpha = a_i \alpha \cdot a_i$ follows from the fact that $\mathrm{Im}\, z_i = \langle a_i \rangle \,\mathrm{Wr}\, \langle a_n \rangle$ and from the definition of wreath product. The lemma is proved.

As α^t is for each $t = 1, \ldots, p-1$ also an automorphism of the order p of the group G for which the assumptions of Lemma 5.1 are true, then from Lemma 5.1 it follows that

$$(a_i \alpha^s) \cdot (a_j \alpha^t) = (a_j \alpha^t) \cdot (a_i \alpha^s)$$

for each $i, j = 1, \ldots, n-1$ and $s, t = 0, 1, \ldots, p-1$; $s \neq t$. Hence, $ab = ba$ for each $a \in A\alpha^s = a_n^{-s} A a_n^s$ and $b \in A\alpha^t = a_n^{-t} A a_n^t$, $s \neq t$. Denote

$$C = A \cdot (a_n^{-1} A a_n) \cdot \ldots \cdot (a_n^{-(p-1)} A a_n^{p-1}). \qquad (6.4)$$

Then C is a subgroup of G and $C \subset \mathrm{Ker}\, y_n$. By Lemma 6.4, C is an invariant subgroup of G. Consequently, $C = \mathrm{Ker}\, y_n$ and

$$G = \mathrm{Ker}\, y_n \lambda \mathrm{Im}\, y_n = C \lambda \langle a_n \rangle. \qquad (6.5)$$

We shall find the number of elements of G. Note that $|\{u \in \mathrm{End}\, G \mid y_n u = u,\ u x_2 = 0\}|$ is equal to the number of all homomorphisms from $\mathrm{Im}\, y_n \cong C_p$ into $\mathrm{Ker}\, x_2$. By Lemma 6.5, $\mathrm{Ker}\, x_2$ is an elementary Abelian p-group. Hence, the number of mentioned homomorphisms is equal to $|\mathrm{Ker}\, x_2|$. By Property 4.15, $|\mathrm{Ker}\, x_2| = (p)^{p^{n-1}}$. On the other hand, by (3.1) $|\mathrm{Im}\, x_2| = |G(n-1)| = p^{1+p+p^2+\ldots+p^{n-2}}$. As $G = \mathrm{Ker}\, x_2 \lambda \mathrm{Im}\, x_2$, then $|G| = |\mathrm{Ker}\, x_2| \cdot |\mathrm{Im}\, x_2| = p^{1+p+\ldots+p^{n-1}}$. Therefore, $|C| = |G| : p = p^{p+\ldots+p^{n-1}}$. Since $|A| = |a_n^{-j} A a_n^j| = |G(n-1)| = p^{1+p+\ldots+p^{n-2}}$, then $|C| = |A|^p$. Consequently, the decomposition (6.4) is a direct product decomposition, i. e.

$$C = \Pi_{j=0}^{p-1} a_n^{-j} A a_n^j = A^{\langle a_n \rangle}. \qquad (6.6)$$

From (6.5) and (6.6) it follows that $G = A \,\mathrm{Wr}\, \langle a_n \rangle$. Since $A \cong G(n-1)$ and $\langle a_n \rangle \cong C_p$, then $G \cong G(n)$. The theorem is proved.

REFERENCES

1. *Пуусемп П.* Уч. зап. Тартуск. ун-та, 1975, 366, 76—104.
2. *Пуусемп П.* Тр. Таллинск. политех. ин-та, 1981, 511, 129—140.
3. *Пуусемп П.* Изв. АН ЭССР. Физ. Матем., 1989, 38, 2, 139—145.
4. *Weir, A. J.* Proc. Amer. Math. Soc., 1955, 6, 534—541.
5. *Neumann, P. M.* Math. Z., 1964, 84, 4, 343—373.
6. *Пуусемп П.* Тр. Таллинск. полит. ин-та, 1988, 671, 154—163.
7. *Puusemp, P.* Proc. Estonian Acad. Sci. Phys. Math., 1992, **41**, 4, 241—247.
8. *Alperin, J. L.* Pacif. J. Math., 1962, **12**, 1, 1—5.

Received
November 20, 1992

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SEOS SÜMMEETRILISE RÜHMA SYLOW' ALAMRÜHMADE JA NENDE ENDOMORFISMIPOOLRÜHMADE VAHEL

Olgu S_m *m*-astme sümmeetriline rühm ja *p* suvaline algarv, mis on rühma S_m järgu teguriks. Siis rühma S_m iga Sylow' *p*-alamrühm on isomorfne *p*. järku tsüklilise rühma C_p korduvalt võetud (*n* korda) standardsete põimikute

$$G(n,p)=(\ldots((C_p \operatorname{Wr} C_p)\operatorname{Wr} C_p)\operatorname{Wr}\ldots)\operatorname{Wr} C_p$$

otsekorrutisega.

Artiklis on tõestatud järgmised väited.

Teoreem. *Rühm $G(n,p)$ on määratud oma endomorfismipoolrühmaga kõigi rühmade klassis iga naturaalarvu n ja algarvu p korral.*

Järeldus 1. *Lõpliku sümmeetrilise rühma iga Sylow' alamrühm on määratud oma endomorfismipoolrühmaga kõigi rühmade klassis.*

Järeldus 2. *Iga lõplik p-rühm G on sisestatav sellisesse lõplikku p-rühma $\bar{G}$, mis on määratud oma endomorfismipoolrühmaga kõigi rühmade klassis.*

Пеэтер ПУУСЕМП

СВЯЗЬ МЕЖДУ СИЛОВСКИМИ ПОДГРУППАМИ СИММЕТРИЧЕСКОЙ ГРУППЫ И ИХ ПОЛУГРУППАМИ ЭНДОМОРФИЗМОВ

Пусть *p* — произвольное простое число и C_p — циклическая группа порядка *p*. Рассмотрим кратное стандартное сплетение

$$G(n,p)=(\ldots((C_p \operatorname{Wr} C_p)\operatorname{Wr} C_p)\operatorname{Wr}\ldots)\operatorname{Wr} C_p$$

(*n* раз) группы C_p. Известно, что каждая силовская *p*-подгруппа конечной симметрической группы степени *m* изоморфна прямому произведению групп $G(n,p)$ для подходящих *n*. В статье доказываются следующие результаты.

Теорема. *Группа $G(n,p)$ определяется своей полугруппой эндоморфизмов в классе всех групп.*

Следствие 1. *Каждая силовская подгруппа конечной симметрической группы определяется ее полугруппой эндоморфизмов в классе всех групп.*

Следствие 2. *Каждая конечная p-группа вложима в такую конечную p-группу, которая определяется своей полугруппой эндоморфизмов в классе всех групп.*