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A PROBLEM OF FLOW IN CYLINDRICAL GEOMETRY

(Presented by H. Keres)

Introduction

The description of flow of fluids is one of the richest problems that can be treated using computational art. The non-linearity of the equations and the complexity of phenomena nearly always leads to a point where numerical methods have to be used for quantitative results.

In this paper, the problem of a time-independent flow of an incompressible fluid past the outside thin wall of a hollow cylinder is considered. The flow is assumed to have the direction of the axis of the cylinder, its length being infinite. It seems strange, but the author has so far not succeeded to find preceding work on this «classical» problem. On the other hand, it may have been «forgotten» as there seems not to be an immediate technical application.

First the classical problem of laminar flow past a thin plate is briefly considered as an illustration. The quite similar analysis in cylindrical geometry is then shown to lead — as in the above case — to an ordinary differential equation. So the solution of elliptic partial differential equations can be avoided. However, some additional considerations concerning the boundary conditions are required when solving the equation numerically.

Fluid Dynamics

The flow of a fluid through space is described at least by the following two fundamental equations of hydrodynamics:

the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{v} = 0 \quad (1a)$$

which expresses the conservation of mass and the Navier-Stokes equation

$$\frac{\partial \vec{v}}{\partial t} = -(\vec{v} \cdot \nabla) \vec{v} - \frac{1}{\rho} \nabla p + \vec{v} \Delta \vec{v} \quad (1b)$$

which expresses the conservation of momentum. Here the mass density ρ and the velocity vector \vec{v} of a fluid element at each point in space is considered.

As usual $\Delta = \nabla^2$ is the Laplacian. If temperature varies as well, an additional equation embodying the conservation of energy is required.

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The pressure p is in general given in terms of some «equation of state», and the kinematic viscosity ν will be assumed constant in what follows. Further we will consider time-independent flows, and assume that the fluid is incompressible. The time derivations can then be set to zero and the density can be a constant. Eqs. (1) then become

$$\nabla \cdot \vec{v} = 0, \quad (2a)$$

$$(\vec{v} \cdot \nabla) \vec{v} = - \frac{1}{\rho} \nabla p + \nu \vec{\alpha}. \quad (2b)$$

A Classical Problem

One of the most known problems in fluid dynamics is that of a laminar flow past a thin plate — infinite in z -direction — as shown in Fig. 1. It is assumed that the temperature is constant throughout the fluid and that the incident speed u_0 is constant as well. The further assumptions are that the flow change in x -direction is negligible as compared to the change in y -direction, and that the pressure in x -direction is constant. Denoting u and v , the x and y components of the velocity vector \vec{v} , respectively, Eqs. (2) can readily be written in the two-dimensional form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (3a)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = v \frac{\partial^2 u}{\partial y^2}. \quad (3b)$$

By means of a similarity transformation, that is introducing in this case the variable

$$\eta = y \sqrt{\frac{u_0}{vx}}, \quad (4)$$

the partial differential equations (3) can — with the above assumptions — be condensed into the ordinary differential equation

$$f' f^{(3)} = f''^2 - \frac{1}{2} ff'^2, \quad (5)$$

where derivations refer to the variable η . The x component of the velocity vector is then given by

$$u = u_0 f(\eta).$$

A somewhat simpler equation was given by Blasius [1] in his doctoral thesis at Göttingen already in 1908:

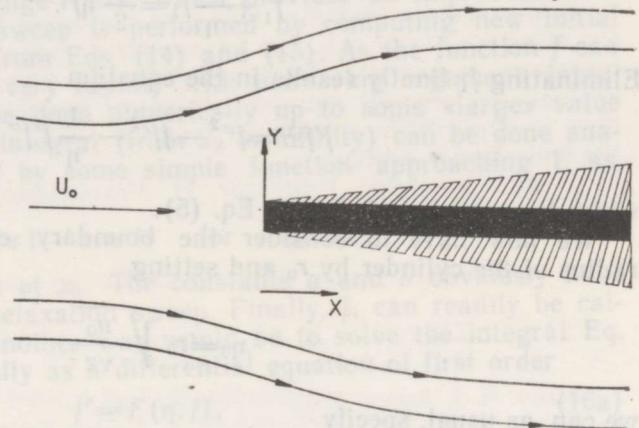
$$\varphi^{(3)} + \frac{1}{2} \varphi \varphi'' = 0 \quad (6)$$

f being the derivative of φ : $\varphi' = f$.

Similar equations are obtained for the y -component which we omit for brevity [2] and, also, because we will consider a similar analysis in more detail in what follows.

After earlier work, Eq. (6) was probably at last solved numerically by Howarth in 1938 [3]. His results differ only by about 0.01% from those obtained with modern facilities.

Fig. 1.



where

Flow Dynamics in Cylindrical Geometry

Assuming cylindrical symmetry we introduce the z and the radial component r of the velocity vector, u and v , respectively, with the incident flow having z -direction and speed U_0 as before. The analysis is quite similar to that of the thin plate. With the same assumptions we get instead of Eqs. (3)

$$\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} + v/r = 0, \quad (7a)$$

$$u \frac{\partial u}{\partial z} + v \frac{\partial u}{\partial r} = v \frac{\partial^2 u}{\partial r^2} + \frac{v}{r} \frac{\partial u}{\partial r}. \quad (7b)$$

Obviously Eq. (7b) can be made formally equal to Eq. (3b) simply by introducing the transformed radial component of the velocity

$$v' = v - v/r.$$

Again we use the similarity transform

$$\eta = r \sqrt{\frac{u_0}{vz}}$$

as given by Eq. (4) (with x replaced by z , and y by r). Writing the velocity components in the form

$$u = u_0 f(\eta) \quad \text{and} \quad v' = \sqrt{\frac{u_0 v}{z}} f_1(\eta), \quad (8)$$

and inserting into Eqs. (7) then leads to the following two ordinary differential equations

$$f_1 = \frac{1}{2} \eta f + f''/f', \quad (9a)$$

$$f'_1 + \frac{1}{\eta} f_1 = \frac{1}{2} \eta f'. \quad (9b)$$

Eliminating f_1 finally results in the equation

$$f' f^{(3)} = f''^2 - f f'^2 - \frac{1}{\eta} f' f'' \quad (10)$$

which is somewhat similar to Eq. (5).

We now have to consider the boundary conditions. Denoting the radius of the cylinder by r_0 and setting

$$\eta_0 = r_0 \sqrt{\frac{u_0}{vz}}$$

we can, as usual, specify

$$u(r_0) = v(r_0) = 0 \quad \text{and} \quad u = u_0 \quad \text{for} \quad r \rightarrow \infty,$$

from which follows

$$f(\eta_0) = 0, \quad f = 1 \quad \text{for} \quad \eta \rightarrow \infty, \quad f_1(\eta_0) = -1/\pi_0. \quad (11)$$

We obviously have to solve Eq. (10) as a boundary value problem with one of the boundaries being at infinity. Any of the well-known iterative procedures, as, for example, the shooting method, can therefore not be applied straightforwardly. By means of the following additional considerations, however, the problem can be readily solved.

An Integral Equation

We consider Eq. (9b) as a linear differential equation of first order. Its solution can readily be given in the form

$$f_1 = \frac{1}{2} \eta f - \frac{1}{\eta} \left[1 + \int_{\eta_0}^{\eta} x f(x) dx \right] \quad (12)$$

performing a simple integration in parts. Inserting this into Eq. (9a), and integrating both sides gives, after some algebra,

$$\eta f'(\eta) = \eta_0 f'(\eta_0) - \int_{\eta_0}^{\eta} x f(x) [f(\eta) - f(x)] dx, \quad (13)$$

where the boundary condition $f(\eta_0) = 0$ has been used. Finally, if η approaches infinity, from Eq. (13) we have

$$f'(\eta_0) = \frac{1}{\eta_0} \int_{\eta_0}^{\infty} x f(x) [1 - f(x)] dx \quad (14)$$

as $\eta f''$ — and of course f' — vanish as $\eta \rightarrow \infty$.

Eq. (14) makes it possible to treat the problem as an initial value instead of a boundary value problem. Of course still an iterative procedure has to be applied. One possible iteration scheme goes as follows. We begin by choosing a trial function $f(\eta)$. By means of Eq. (14) a first value of $f'(\eta_0)$ is obtained. Taking into account Eqs. (11), from Eq. (9a) we immediately have

$$f''(\eta_0) = -\frac{1}{\eta_0} f'(\eta_0). \quad (15)$$

With these initial values Eq. (10) can be solved using one of the common methods, say Runge-Kutta, which provides an improved value of $f(\eta)$. One relaxation sweep is performed by computing new initial values $f'(\eta_0)$ and $f''(\eta_0)$ from Eqs. (14) and (15). As the function f can be shown to approach 1 very rapidly with increasing argument, integration in Eq. (14) can be done numerically up to some «large» value x_0 , say 5. The remaining integral (from x_0 to infinity) can be done analytically, approximating f by some simple function approaching 1, as, for instance,

$$f(\eta) \approx 1 - a \cdot e^{-b\eta} \quad \text{for } \eta \geq x_0,$$

connecting both solutions at x_0 . The constants a and b obviously have to be computed at each relaxation sweep. Finally, f_1 can readily be calculated from Eq. (12). Another way would be to solve the integral Eq. (13) considering it formally as a differential equation of first order

$$f' = F(\eta, f), \quad (16a)$$

where

$$F = \frac{\eta_0}{\eta} f'(\eta_0) - \frac{1}{\eta} \int_{\eta_0}^{\eta} x f(x) [f(\eta) - f(x)] dx. \quad (16b)$$

The iteration scheme is quite similar to that described above. Again equation (14) can be used to calculate improving values of $f'(\eta_0)$.

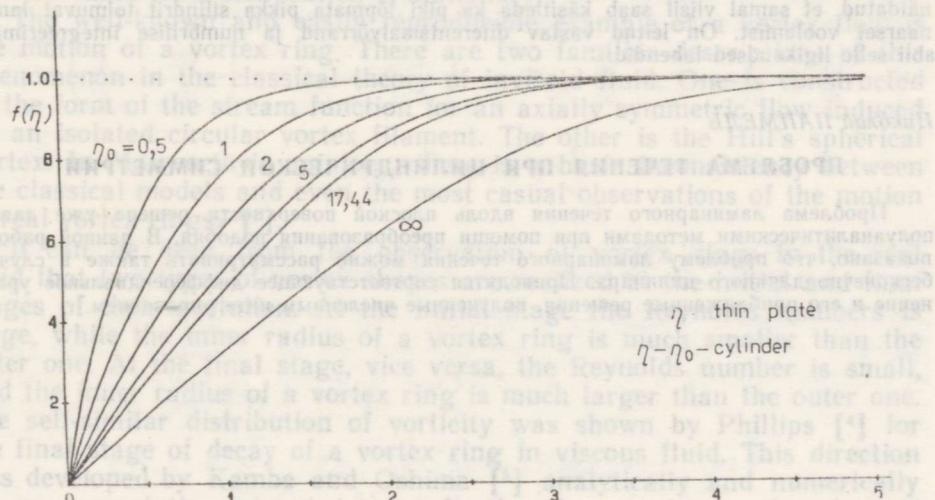


Fig. 2.

Computational results are shown in Fig. 2. They can easily be interpreted. For small values of η_0 the flow in the direction of the axis (i.e. the z -component $u = u_f(\eta)$) rapidly approaches the undisturbed value u_0 . As η_0 increases there is also an increase in the range where the flow is disturbed. The function f of the thin plate is also drawn and denoted by ∞ . In fact, it signifies a limit with respect to the cylindrical functions as the radius of the cylinder approaches infinity.

Conclusion

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It was shown that the flow past the outer thin wall of a cylinder in the direction of his axis can be computed using the well-known similarity transform technique. There are different ways to solve the resulting ordinary differential equation. It seems however that the optimal procedure is to solve the equivalent integral equation as it naturally embodies the boundary conditions.

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VOOLAMISE PROBLEEM SILINDRILISES GEOMEETRIAS

Piki õhukest tasast plati toimuva laminaarse voolamise probleem on lahendatud juba ammu poolanalüütiliste meetoditega sarnasustaisenduse abil. Käesolevas töös on näidatud, et samal viisil saab käsitleda ka piki lõpmata pikka silindrit toimuvat laminaarset voolamist. On leitud vastav diferentsiaalvörrand ja numbrilise integreerimise abil selle ligikaudsed lahendid.

Николай ПАПМЕЛЬ

ПРОБЛЕМА ТЕЧЕНИЯ ПРИ ЦИЛИНДРИЧЕСКОЙ СИММЕТРИИ

Проблема ламинарного течения вдоль плоской поверхности решена уже давно полуаналитическими методами при помощи преобразования подобия. В данной работе показано, что проблему ламинарного течения можно рассматривать также в случае бесконечно длинного цилиндра. Приводятся соответствующее дифференциальное уравнение и его приближенные решения, полученные численным интегрированием.