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FOURIER MULTIPLIERS OF GENERALIZED LIPSCHITZ CLASSES ON THE REAL LINE

(Presented by J. Engelbrecht)

We find necessary and sufficient conditions for multipliers on $\text{Lip}(\omega, L)$ under the assumption that the majorant ω is slowly decreasing. In the periodic case the result was known earlier [1].

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1. Introduction

Let $L(\mathbf{R}) = L$ be the Banach space of functions f , integrable on the real axis \mathbf{R} , endowed with the usual norm

$$\|f\| = (2\pi)^{-1/2} \int_{-\infty}^{\infty} |f(x)| dx.$$

Let

$$\hat{f}(v) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(u) e^{-ivu} du$$

be the Fourier transform of f and let $A, B \subset L$. We say that a complex-valued function $\lambda = \lambda(v)$ is a multiplier from A to B if $\lambda(v)\hat{f}(v)$ is the Fourier transform of a function $f_\lambda \in B$ whenever $f \in A$. The set of all multipliers from A to B will be denoted by (A, B) .

Let ω be an abstract modulus of continuity — a function continuous on $[0, \infty)$ with

$$0 \leq \omega(s) \leq \omega(s+t) \leq \omega(s) + \omega(t) \quad (s, t > 0).$$

Standard examples of ω are supplied by the powers $\omega(\delta) = \delta^\alpha$ ($0 < \alpha \leq 1$). Let $\omega(f, \delta)$ denote the first modulus of continuity of $f \in L$

$$\omega(f, \delta) = \sup_{|h| \leq \delta} \|f(\cdot + h) - f(\cdot)\|.$$

Consider the class $\text{Lip}(\omega, L)$ of all $f \in L$ for which we have the estimate

$$\omega(f, \delta) = O(\omega(\delta)).$$

If $\omega(\delta) = \delta^\alpha$ ($0 < \alpha \leq 1$) we get the standard integral Lipschitz classes $\text{Lip}(\alpha, L)$.

The problem of multipliers of Lipschitz classes of functions has been studied in the case of periodic functions, continuous ($C_{2\pi}$) and integrable ($L_{2\pi}$). A. Zygmund proved in 1959 [2] that if $0 < \alpha < 1$, then a sequence of complex numbers is a multiplier $(\text{Lip}(\alpha, C_{2\pi}), \text{Lip}(\alpha, C_{2\pi}))$ if and

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only if the indefinite integral of the kernel of the multiplier belongs to the Zygmund class L_* (see also [3] 16.3.9). In [4] and also [5] it was shown that the same assertion may be extended to a wider class of moduli of continuity, namely for those satisfying the condition

$$\int_0^\delta \frac{\omega(t)}{t} dt + \delta \int_\delta^{2\pi} \frac{\omega(t)}{t^2} dt = O(\omega(\delta)) \quad (\delta \rightarrow 0+), \quad (1)$$

but if this condition does not hold, then the theorem is no longer true. In [1] we obtained a necessary and sufficient condition for a sequence of complex numbers to be a $(\text{Lip } (\omega, C_{2\pi}), \text{Lip } (\omega, C_{2\pi}))$ multiplier under the assumption that ω is a slowly decreasing modulus of continuity, i.e. $\omega(\delta)\delta^{-1/2}$ monotonely tends to infinity with δ tending to zero.

The purpose of the present note is to extend this result to classes of nonperiodic functions integrable on the real axis. The general ideas employed are the same that were used in the periodic case, they are based on Jackson- and Bernstein-type inequalities. The main difference lies in the choice of approximating operators and appropriate test functions. We shall also make use of the gliding hump technique an extensive study of which may be found in [6].

2. Background

Let $W^2 = W^2(\mathbf{R})$ denote the Wiener class of measurable functions $f(x)$ ($x \in \mathbf{R}$) such that $f(x)/(1+|x|)$ is square integrable on the real axis. Let E_τ denote the set of entire functions of exponential type $\leq \tau$ (see e.g. [7] p. 174).

Let $f \in L$. The singular integral of Jackson—de la Vallée Poussin is defined through ($\varrho > 0$)

$$N(f; x; \varrho) = \frac{12}{\pi \varrho^3} \int_{-\infty}^{\infty} f(x-u) \frac{\sin^4(\varrho u/2)}{u^4} du.$$

The generalized integral of Fejér is defined by

$$V(f; x; \varrho) = \frac{1}{\pi \varrho} \int_{-\infty}^{\infty} f(x-u) \frac{\cos \varrho u - \cos 2\varrho u}{u^2} du$$

with kernel

$$V(x) = \sqrt{\frac{2}{\pi}} \frac{\cos x - \cos 2x}{x^2} = 4F(2x) - F(x),$$

where

$$F(x) = (2\pi)^{-1/2} \left(\frac{\sin(x/2)}{(x/2)} \right)^2$$

is the Fejér kernel. We shall take $N(f; x; 0) = 0$ and $V(f; x; 0) = 0$. The generalized integral of Fejér is a nonperiodic analog of the de la Vallée Poussin means and possesses several of their important properties. We list them here for further reference (see e.g. [7], p. 253–258 for the proofs).

$$P1 \quad \dot{V}(v) = \begin{cases} 1, & |v| \leq 1; \\ 2 - |v|, & 1 \leq |v| \leq 2; \\ 0, & |v| > 2. \end{cases}$$

P2. If $f \in E_\tau \cap W^2$ and $\varrho > \tau$, then

$$V(f; x; \varrho) \equiv f(x).$$

P3. $\|V(f; \cdot; \varrho)\| \leq 3\|f\|.$

P4. If $f \in \text{Lip}(\omega, L)$, then

$$\|f - V(f; \cdot; \varrho)\| = O(\omega(1/\varrho)).$$

The last property needs some comment. We have

$$\|f - N(f; \cdot; \varrho)\| = O(\omega(1/\varrho)), \quad (2)$$

(see e.g. [6] p. 142, 145). Since $N(f; x; \varrho) \in E_{2\varrho} \cap W^2$ [8] p. 205 we may write in view of P2.

$$\|f - V(f; \cdot; \varrho)\| = \|f - N(f; \cdot; \varrho/2) - V(f - N(f; \cdot; \varrho/2); \cdot; \varrho)\|.$$

Hence by P3 and (2)

$$\|f - V(f; \cdot; \varrho)\| = O(\omega(2/\varrho)) = O(\omega(1/\varrho)).$$

Let $f, g \in L$. By $f * g$ we shall denote the convolution

$$(f * g)(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x-u)g(u) du.$$

Let $\lambda(v)$ be measurable on \mathbf{R} and integrable over every finite interval. By $V(\Lambda; x; \varrho)$ we shall denote the integral

$$V(\Lambda; x; \varrho) = (2\pi)^{-1/2} \int_{-2\varrho}^{2\varrho} \lambda(v) \hat{V}(v/\varrho) e^{ivx} dv.$$

If $\lambda(v)$ is a Fourier transform, then

$$\Lambda(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \lambda(v) e^{ivx} dv$$

and $V(\Lambda; x; \varrho)$ is the usual generalized integral of Fejér of Λ .

3. Results

Let $\omega(\delta)$ be an abstract modulus of continuity. Consider the sequence $D(\omega) = \{\delta_k\}_{k=0}^{\infty}$ defined by induction ($k \geq 0$)

$$\delta_0 = 1,$$

$$\delta_{k+1} = \min \left\{ \delta : \max \left(\frac{\omega(\delta)}{\omega(\delta_k)}, \frac{\delta \omega(\delta_k)}{\delta_k \omega(\delta)} \right) = \frac{1}{2} \right\}. \quad (3)$$

(see [9]). The sequence $D(\omega)$ has, among others, the following properties (c is a positive constant):

$$1. \quad (1/c)\omega(\delta) \leq \sum_{k=0}^{\infty} \omega(\delta_k) \min(1, \delta/\delta_k) \leq c\omega(\delta); \quad (4)$$

$$2. \quad \delta_k/\delta_{k+1} = O(1) \quad (k \rightarrow \infty) \text{ if and only if} \\ \omega(\delta) \text{ satisfies the condition (1).} \quad (5)$$

Theorem. Let $\lambda = \lambda(v)$ be measurable on \mathbf{R} and integrable over every finite interval. Let $\omega(\delta)$ be a slowly decreasing modulus of continuity, i.e. $\omega(\delta)\delta^{-1/2}$ monotonely increases ($\delta \rightarrow 0+$). Let $D(\omega) = \{\delta_k\}_{k=0}^{\infty}$ be defined by (3) and let $\varrho_k = 1/\delta_k$. Then $\lambda = \lambda(v)$ is of the type $(\text{Lip}(\omega, L), \text{Lip}(\omega, L))$ if and only if

$$\|V(\Lambda; \cdot; \varrho_{k+1}) - V(\Lambda; \cdot; \varrho_k)\| = O(1) \quad (k \rightarrow \infty). \quad (6)$$

P r o o f. Sufficiency. Let $f \in \text{Lip}(\omega, L)$ and suppose (6) holds. Then $V(f_\lambda; x; \varrho)$ is well defined. Consider the series ($\varrho_{-1}=0$)

$$\begin{aligned} & \sum_{k=-1}^{\infty} (V(f_\lambda; x; \varrho_{k+1}) - V(f_\lambda; x; \varrho_k)) \\ &= \sum_{k=-1}^{\infty} (V(f; x; 2\varrho_{k+1}) - V(f; x; \varrho_k/2)) \\ &\quad * (V(\Lambda; x; \varrho_{k+1}) - V(\Lambda; x; \varrho_k)). \end{aligned}$$

As $f \in \text{Lip}(\omega, L)$ we have by P4 that

$$\|V(f; \cdot; 2\varrho_{k+1}) - V(f; \cdot; \varrho_k/2)\| = O(\omega(2/\varrho_k)) = O(\omega(\delta_k)).$$

Hence, by (3) this series converges in integral metrics. It follows that $f_\lambda \in L$. Next let us show that $f_\lambda \in \text{Lip}(\omega, L)$. Through $\Delta_h f(x) = f(x+h) - f(x)$ we shall denote the first difference of f . If $0 < h < \delta$ we may write in view of P1 and P2

$$\begin{aligned} \|\Delta_h f_\lambda\| &= \left\| \sum_{k=-1}^{\infty} \{\Delta_h\} * \{V(\Lambda; \cdot; \varrho_{k+1}) - V(\Lambda; \cdot; \varrho_k)\} \right\| \\ &= \left\| \sum_{k=-1}^{\infty} \{\Delta_h(V(f; \cdot; 2\varrho_{k+1}) - V(f; \cdot; \varrho_k/2))\} \right. \\ &\quad \left. * \{V(\Lambda; \cdot; \varrho_{k+1}) - V(\Lambda; \cdot; \varrho_k)\} \right\| \\ &\leq \sum_{k=-1}^{\infty} \|\Delta_h(V(f; \cdot; 2\varrho_{k+1}) - V(f; \cdot; \varrho_k/2))\| \\ &\quad \cdot \|V(\Lambda; \cdot; \varrho_{k+1}) - V(\Lambda; \cdot; \varrho_k)\|. \end{aligned}$$

The second factors in the last sum are bounded by (6). For the first factors we have the estimates

$$\begin{aligned} &\|\Delta_h(V(f; \cdot; 2\varrho_{k+1}) - V(f; \cdot; \varrho_k/2))\| \\ &\leq 2\|V(f; \cdot; 2\varrho_{k+1}) - V(f; \cdot; \varrho_k/2)\| \\ &= O(\omega(2/\varrho_k)) = O(\omega(\delta_k)) \end{aligned}$$

and

$$\begin{aligned} &\|\Delta_h(V(f; \cdot; 2\varrho_{k+1}) - V(f; \cdot; \varrho_k/2))\| \\ &= O(\varrho_{k+1}h\|V(f; \cdot; 2\varrho_{k+1}) - V(f; \cdot; \varrho_k/2)\|) \\ &= O(h\omega(\delta_k)/\delta_{k+1}). \end{aligned}$$

Together with the assumption that $\omega(\delta)$ is slowly decreasing, these two inequalities imply

$$\|\Delta_h(V(f; \cdot; 2\varrho_{k+1}) - V(f; \cdot; \varrho_k/2))\| = O(\omega(\delta_k) \min\{1, \delta/\delta_k\}).$$

Thus in view of (4) we obtain

$$\|\Delta_h f_\lambda\| = O(\sum \omega(\delta_k) \min\{1, \delta/\delta_k\}) = O(\omega(\delta)),$$

and hence $f_\lambda \in \text{Lip}(\omega, L)$.

Necessity. Suppose (6) does not hold. Then there exists a sequence of indices $\{k(j)\}$ such that

$$\|V(\Lambda; \cdot; \varrho_{k(j)+1}) - V(\Lambda; \cdot; \varrho_{k(j)})\| \geq j. \quad (7)$$

We may also assume that this sequence is sufficiently rare, that is

$$\varrho_{k(j+1)} > 8\varrho_{k(j)+1}. \quad (8)$$

Let

$$\varphi_j(x) = \varrho_{k(j)+1} V(x/\varrho_{k(j)+1}) - \varrho_{k(j)} V(x/\varrho_{k(j)}).$$

Consider the function

$$f(x) = \sum_{k=1}^{\infty} \omega(\delta_{k(j)}) \varphi_{k(j)}(x).$$

From $\|\varphi_j\| = O(1)$ and (3) we conclude that this definition is correct. In view of Bernstein's inequality we have

$$\omega(\varphi_j, \delta) = O(\min\{1, Q_{k(j)+1}\delta\}) = O(\min\{1, \delta/\delta_{k(j)+1}\}).$$

Applying (4) we get $f \in \text{Lip}(\omega, L)$.

Now let us demonstrate that $f_\lambda \notin \text{Lip}(\omega, L)$. If f_λ is to be in $\text{Lip}(\omega, L)$ we should have

$$\|V(f_\lambda; \cdot; 2Q_{k+1}) - V(f_\lambda; \cdot; Q_k/2)\| = O(\omega(\delta_k)).$$

However, considering the construction of f , the choice of φ_j and (8) we see that by (7)

$$\begin{aligned} & \|V(f_\lambda; \cdot; 2Q_{k(j)+1}) - V(f_\lambda; \cdot; Q_{k(j)}/2)\| \\ &= \omega(\delta_{k(j)}) \|V(\Lambda; \cdot; Q_{k(j)+1}) - V(\Lambda; \cdot; Q_{k(j)})\| \\ &\geq j \omega(\delta_{k(j)}) \neq O(\omega(\delta_{k(j)})). \end{aligned}$$

This contradiction concludes the proof of the theorem.

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ÜLDISTATUD LIPSCHITZI KЛАSSIDE FOURIER' MULTIPLIKAATORITEST REAALTELJEL

On leitud tarvilikud ja piisavad tingimused selleks, et komplekssete väärustega funktsioon $\lambda = \lambda(v)$ oleks multiplikaatoriks klassil $\text{Lip}(\omega, L)$ eeldusel, et pidevuse moodul ω on aeglaselt kahanev. Perioodilisel juhul oli tulemus varem teada.

Юри ЛИППУС

О МНОЖИТЕЛЯХ ФУРЬЕ ОБОБЩЕННЫХ КЛАССОВ ЛИПШИЦА НА ВЕЩЕСТВЕННОЙ ОСИ

Находятся необходимые и достаточные условия для того, чтобы комплекснозначная функция $\lambda = \lambda(V)$ являлась мультипликатором на классе $\text{Lip}(\omega, L)$ при ограничении, что модуль непрерывности ω убывает медленно. В периодическом случае эти условия были известны заранее.