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ETALON EVOLUTION EQUATIONS

The possibility of deriving evolution equations from theoretically or experimentally determined dispersion relations is analysed. If the full dispersion relations are simplified for a certain frequency range then the corresponding evolution equations are also simplified and thereafter called etalon evolution equations. This paper is aimed at stressing the importance of such an approximate approach which can be used in cases when the traditional reductive methods do not work. The basic notions are given and explained at the physical level of presentation. Two examples are presented in more detail: Love waves in a solid layer and longitudinal waves in a liquid layer.

1. Introduction

Nonlinear evolution equations, together with the analytical and numerical methods derived for solving them, form a remarkable chapter of contemporary mathematical physics [1–3]. The number of solvable problems in the mathematical theory of nonlinear wave motion is large and involves many cases of weakly dispersive, weakly nonhomogeneous and stratified media. In most cases the starting point is a governing system of wave equations to which a certain reductive method (asymptotic, iterative or spectral) is then applied [4, 5]. As a result of a perturbative analysis, the leading physical effects are established and the process is decomposed into separate waves. Each separate wave is described by its own single equation called the evolution equation that involves all the physical effects of the same order. Such is a usual way to derive the celebrated Korteweg-de Vries, Burgers and other equations. However, there are cases when due to complexity of the wave process the starting point is not the system of wave equations but a known dispersion relation, and then the usual reductive methods simply do not work. This case should be analysed separately because the dispersion relations could be obtained not only by certain asymptotic procedures but also by experimental techniques. In the linear case, the correspondence between a dispersion relation and a wave equation is well understood [1], provided the Fourier transform is applicable and the respective integration possible. For complicated dispersion relations and nonlinearity involved, this straight-forward approach is not possible and needs a certain asymptotic procedure to be established.

In this paper, an attempt is made to describe the formalism needed for constructing asymptotic evolution equations on the basis of asymptotic dispersion relations. In order to distinguish the evolution equations derived in this way from those obtained by the reductive methods [2, 4, 5], these are called *etalon evolution equations*. An etalon evolution equation may satisfactorily describe the process, provided certain additional conditions are fulfilled which relate the asymptotic dispersion relations to the basic ones.

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In Section 2 the basic formalism of etalon evolution equations is described with needed assumptions and conditions pointed out. This forms the ground for Section 3 where two examples are given: Love waves in a solid layer and longitudinal waves in a liquid layer. In Section 4 some conclusions are given.

2. Basic Equations

The basic idea is given by Whitham [1] for a linear problem. Let us consider the one-dimensional evolution equation

$$\frac{\partial u}{\partial \tau} + \int_{-\infty}^{\infty} K(\xi - z) \frac{\partial u}{\partial z} dz = 0, \quad (1)$$

where τ and ξ are the independent variables, u is a certain wave variable (particle velocity, deformation, etc.) and the kernel function $K(z)$ describes the dispersion properties. This equation has elementary solutions of the form

$$u = u_0 \exp(ik_*\xi + i\omega_*\tau), \quad u_0 = \text{const}, \quad (2)$$

provided the dispersion relation

$$c = \omega_* k_*^{-1} = \int_{-\infty}^{\infty} K(\zeta) \exp(ik_*\zeta) d\zeta \quad (3)$$

is satisfied. As the right hand side of expression (3) is the Fourier transform of the given kernel $K(\zeta)$, the inverse transform

$$K(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(k_*) \exp(ik_*\xi) dk_* \quad (4)$$

allows us to determine the kernel for every given phase velocity $c(k)$. The trivial example of deriving the linearized Korteweg-de Vries (KdV) equation according to this scheme is well known. In this case the phase velocity has a polynomial form with even terms in it, and the kernel function is given by the delta-function and its consecutive derivatives [1]. It is appropriate that some remarks based on physical ideas should now be made. First, the independent variables in an evolution equation correspond to a moving frame, i.e., usually $\xi = c_0 t - x$, $\tau = \varepsilon x$ where ε is a small parameter and $c_0 = \text{const}$. is the "translation" velocity. Usually c_0 is either the phase velocity c determined from the corresponding linear associated problem or the sound velocity close to c [6]. Secondly, the wave number k_* and the frequency ω_* in (2) correspond to the variables ξ and τ . In real space-time coordinates x and t the wave number k and the frequency ω are different from those. It is easily concluded that [7]

$$k_* = -\omega c_0^{-1}, \quad \omega_* = (k - \omega c_0^{-1}) \varepsilon^{-1}. \quad (5)$$

Thirdly, evolution equations are usually derived for weakly dispersive systems [2, 3, 6]. In the case of strongly dispersive system, the group velocities must be used instead of the phase velocities when constructing the moving frame [2].

Based on these comments, and the expressions given above, a simple asymptotic approach for deriving evolution equations may be developed. Whitham [1] has used it when deriving the approximate evolution equation for Stokes' waves. Here we present the main idea in a more general form and demonstrate its applicability by two examples.

Suppose we know the dispersion relation

$$G(\omega, k) = 0 \quad (6)$$

for a certain wave. Let an asymptotic expression

$$G_e(\omega, k) = 0 \quad (7)$$

exist provided the additional condition

$$f(\omega, k) = 0 \quad (8)$$

is satisfied. Condition(s) (8) may be easily interpreted physically; for example, it might characterize high or low frequency processes. The main idea is to determine the kernel function in (4) not from the exact dispersion relation (6) but from its asymptotic expression (7). This may considerably simplify the evaluation of the inverse transform (4) and, consequently, may result in an explicit form of the kernel function which often cannot be obtained by using the exact dispersion relation. We denote the kernel function corresponding to the asymptotic expression (7) by $K_e(\xi)$. The evolution equation

$$\frac{\partial v}{\partial \tau} + \int_{-\infty}^{\infty} K_e(\xi - z) \frac{\partial v}{\partial z} dz = 0 \quad (9)$$

describes the process provided the condition (8) is satisfied, and will be called the *etalon evolution equation*. Notice once more that the difference between k_* , ω_* and k , ω must be clearly stated in order to obtain physically admissible results.

The evolution equations are usually derived from the system of equations of motion, and they actually form a sequence of n -th order evolution equations. The traditional analysis deals with the 1st-order evolution equations only [4, 5]. Matching the physical and mathematical requirements, it is easily concluded that the 1st-order evolution equation describes the situation where the dispersive, nonlinear, dissipative e. a. terms on the level of the evolution equation are additive [1]. The solutions to these nonlinear evolution equations are by no means additive, reflecting effectively the possible balance between different physical effects. In this context the main question is always how to describe the dispersion properties of the media [1]. All these ideas may be summed up in the following successive scheme for constructing nonlinear evolution equations:

- derive or measure the general dispersion relation (6);
- derive the asymptotic dispersive relation (7) under the condition (8);
- determine the kernel $K_e(\xi)$ from (4);
- construct the linear evolution equation (9);
- add the nonlinear term according to the nonlinear wave motion in the corresponding nondispersing medium;
- get the final nonlinear etalon evolution equation which describes the process under conditions (8).

3 Waves in Layers

3.1. Love Waves in a Solid Layer. The propagation of Love waves in a solid layer $0 \geq z \geq h$ (medium 1) resting on a solid halfspace $z > h$ (medium 2) has been analysed by Bataille and Lund [7]. The Love waves localized in medium 1 and decaying exponentially for $z > h$, obey the dispersion relation

$$\tan K_1 k = (\mu_2 K_2) / (\mu_1 K_1), \quad (10a)$$

$$K_1 = (k^2 - \omega^2/c_{s1}^2)^{1/2}, \quad K_2 = (\omega^2/c_{s2}^2 - k^2)^{1/2}. \quad (10b)$$

Subscripts 1 and 2 refer to media 1 and 2, respectively, μ_i is the Poisson ratio, c_{si} is the velocity of shear waves, k is the wave number and ω is frequency. For

$$hk \gg 1, \quad (11)$$

i.e., for the wave lengths that are large as compared to the thickness h of the layer 1, expression (10a) may be approximated by

$$\omega^2 = c_{s2}^2 (k^2 - \beta k^4), \quad (12)$$

$$\beta = h^2 (c_{s2}^2/c_{s1}^2 - 1)^2 \mu_1^2/\mu_2^2 > 0. \quad (13)$$

The plots of the phase velocity c over hk determined from (10) and (12) in a rather large interval are shown in Fig. 1. Here $c_{s1} = 4000 \text{ m} \cdot \text{s}^{-1}$, $c_{s2} = 4400 \text{ m} \cdot \text{s}^{-1}$ and $c_{s1}/c_{s2} \cdot \mu_2/\mu_1 = 1.09$. It is obvious that if the condition (11) is satisfied then the expression (12) is a very good approximation to (10). There is even no need to calculate the inverse Fourier transform (4). A straight-forward analysis immediately shows that the dispersion relation (12) corresponds to the equation of motion

$$\frac{\partial^2 U}{\partial t^2} = c_{s2}^2 \frac{\partial^2 U}{\partial x^2} + c_{s2}^2 \beta \frac{\partial^4 U}{\partial x^4}. \quad (14)$$

Here we have the possibility of adding the nonlinear terms immediately to the equation of motion. Following [8, 9] we have

$$\frac{\partial^2 U}{\partial t^2} = c_{s2}^2 \frac{\partial^2 U}{\partial x^2} + c_{s2}^2 \alpha \left(\frac{\partial U}{\partial x} \right)^2 \frac{\partial^2 U}{\partial x^2} + c_{s2}^2 \beta \frac{\partial^4 U}{\partial x^4}, \quad (15)$$

where α is the nonlinear parameter involving the elastic moduli of the second, third and fourth orders. The etalon evolution equation is easily derived by the standard methods [2, 5]

$$\frac{\partial u}{\partial \tau} - \frac{\alpha}{2\varepsilon^2 c_{s2}^2} u^2 \frac{\partial u}{\partial \xi} + \frac{1}{2} \frac{\beta}{\varepsilon^2} \frac{\partial^3 u}{\partial \xi^3} = 0, \quad (16)$$

where $\xi = c_{s2}t - x$, $\tau = \varepsilon^2 x$, $u = \partial U/\partial t$. This result — the modified KdV equation — was actually obtained by Bataille and Lund [8]; here we have only employed the notions of the general theory, demonstrating also the validity of approximation (12).

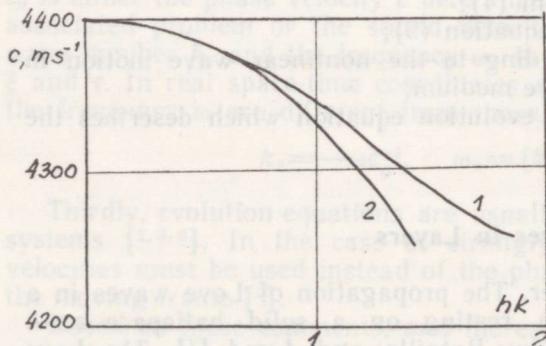


Fig. 1. Dispersion curves for Love waves: 1 — expression (10), 2 — expression (12).

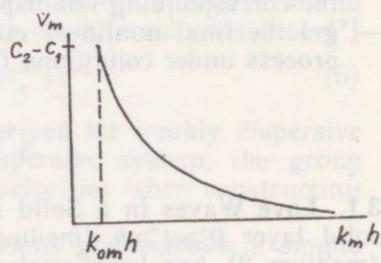


Fig. 2. Phase velocity region used for the etalon evolution equation in case of a liquid layer.

3.2. Longitudinal Waves in a Liquid Layer. Here we analyse the propagation of longitudinal waves in a liquid layer $0 \geq z \geq h$ (medium 1) resting on a liquid halfspace $z > h$ (medium 2). This is the well-known Pekeris problem [10]. We are interested in modes decaying exponentially in the halfspace $z > h$. For $0 \geq z \geq h$ the solution for the potential or sound pressure is given by the relation

$$\Phi_m = q_m \sin (\gamma_{1m} z) I_0(k_m x), \quad q_m = \text{const.}, \quad (17)$$

where $m = 1, 2, 3, \dots$ is the mode number, I_0 is the Bessel function and k_m is the wave number in the horizontal direction x . The dispersion relation takes now the form

$$\gamma_{1m} h + \arctan \frac{1}{a} \frac{\gamma_{1m}}{g_{2m}} = m\pi, \quad (18)$$

where

$$\gamma_{1m} = k_m (v_m^2/c_1^2 - 1)^{1/2}, \quad (19a)$$

$$g_{2m} = k_m (1 - v_m^2/c_2^2)^{1/2}, \quad (19b)$$

and $a = \rho_1/\rho_2$. Here and below, c_i , $i = 1, 2$ is the sound velocity, ρ_i the density and v_m the phase velocity for a certain mode m . Wave number k_m can be determined as

$$k_m = \left(\frac{v_m^2}{c_1^2} - 1 \right)^{1/2} \frac{1}{h} \left\{ m\pi - \arctan \frac{1}{a} \left[\frac{v_m^2/c_1^2 - 1}{1 - v_m^2/c_2^2} \right]^{1/2} \right\}, \quad (20)$$

taking the real values for $c_1 \leq v_m \leq c_2$. The critical frequencies ω_{0m} at $v_m \rightarrow c_2$ are governed by [10]

$$\omega_{0m} = c_2 k_{0m} = \frac{c_2 \pi (m - 1/2)}{h (c_2^2/c_1^2 - 1)^{1/2}}. \quad (21)$$

For frequencies below the critical, the modes are fast decaying and not taken into account.

We suppose that the approximation for the phase velocity

$$v_m(\bar{k}_m) = (c_2 - c_1) \frac{v_m^2}{v_m^2 + \bar{k}_m^2}, \quad \bar{k}_m = k_m h - k_{0m} h \quad (22)$$

is justified for

$$\omega > \omega_{0m} \quad (23)$$

(c.f. expressions (7) and (8)). Note that expression (22) is given with respect to the velocity difference $c_2 - c_1$, $c_2 > c_1$, and k_{0m} is determined by the critical frequency $\omega_{0m} = c_2 k_{0m}$. In this case the qualitative dispersion curve is shown in Fig. 2 from which the applicability of the Fourier transform is easily seen. The constants v_m can be determined for each mode m by the least-square method on the basis of the exact dispersion relation (20). For example, if $c_1 = 1500 \text{ m} \cdot \text{s}^{-1}$, $c_2 = 1650 \text{ m} \cdot \text{s}^{-1}$, $a = 0.9$, then $v_1 = 3.65$; $v_2 = 6.59$; $v_3 = 9.52$.

The next step is to determine the asymptotic kernel function

$$K_e(\xi) = \frac{c_2 - c_1}{2\pi} \int_{-\infty}^{\infty} \frac{v_m^2}{v_m^2 + k_m^2} \exp(i k_m \xi) dk_m = \\ = \frac{(c_2 - c_1)v_m}{2h} \exp\left(-\frac{v_m |\xi|}{h}\right) \quad (24)$$

which can be used for frequencies satisfying (23).

The corresponding linear etalon evolution equation is now

$$2c_1 \frac{\partial u}{\partial \tau} + \frac{(c_2 - c_1)v_m}{2h} \int_{-\infty}^{\infty} \exp\left(-\frac{v_m |\xi - x|}{h}\right) \frac{\partial u}{\partial z} dz = 0, \quad (25)$$

where $\tau = \varepsilon x$, $\xi = c_1 t - x$, i.e., c_1 is the velocity of the moving frame (c.f. our assumption (22), including changes with respect to this velocity). Here ε is the small parameter, and the coefficient $2c_1$ at the first term appears due to the standard procedures of perturbative methods [5]. The nonlinear variant can be easily constructed and the final equation with the convective nonlinearity takes the form

$$\frac{\partial u}{\partial \tau} - \frac{\alpha}{2\varepsilon c_1} u \frac{\partial u}{\partial \xi} + \frac{(c_2 - c_1)v_m}{4c_1 h} \int_{-\infty}^{\infty} \exp\left(-\frac{v_m |\xi - x|}{h}\right) \frac{\partial u}{\partial z} dz = 0, \quad (26)$$

where $\alpha = 1 + \gamma$ and γ is the adiabate [11]. Such an evolution equation was described earlier by Whitham [1] and Fornberg and Whitham [12].

4. Conclusions

The approach demonstrated above is applicable when the traditional methods [2, 4, 5] in nonlinear wave analysis cannot be used because of the physical complexity involved. In this case even the statement of the problem is different, because instead of the usual initial system of wave equations the starting point is a known dispersion relation. It must be stressed that this approach may also be of importance when the experimental dispersion curves are used for restoring the governing equation. Two examples analysed briefly show explicitly how a complicated dispersion relation can be handled in order to construct the corresponding evolution equation. In order to stress the approximation, the latter is called etalon evolution equation.

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ETALON-EVOLUTSIOONIVÖRRANDID

On vaadeldud meelevaldse profiiliga lainete levi kirjeldavate evolutsioonivörrandite tuletuskäiku lähtudes etteantud dispersiooniseadusest. Kuna üldjuhul osutub vajalikuks auproksimeerida keerulisi dispersiooniseadusi lihtsamate valdistega, mis kehtivad teatud tingimustel, on kasutatud mõistet «etalon-evolutsioonivörrandid», et eristada neid «täpsustest» evolutsioonivörranditest. Seesugune lähenemine võimaldab konstrueerida evolutsioonivörrandeid ka juhul, kui traditsioonilised, lainevörrandite süsteemi redutseerimisel põhinevad meetodid ei tööta. Artiklis on esitatud etalon-evolutsioonivörrandite konstrueerimise põhimõisted ning neid illustreeritud kahe näitega. Näited haaravad Love'i laineid elastses kihis ja pikilaineid vedelukukihis.

Юрий ЭНГЕЛЬБРЕХТ

ЭТАЛОННЫЕ ЭВОЛЮЦИОННЫЕ УРАВНЕНИЯ

Анализируется возможность построения эволюционных уравнений на основе известных дисперсионных соотношений. Особенность постановки заключается в том, что громоздкие дисперсионные соотношения, часто встречающиеся в практике, заменяются их аппроксимациями при соблюдении определенных условий. Это обстоятельство является причиной трактовки построенных «эволюционных» уравнений как эталонных в отличие от точных. Предлагаемый подход применим в случае, когда известные редукционные методы не работают. Показаны основные этапы такого подхода и приведены два примера: для волн Лява в упругом слое и продольных волн в жидким слое.