

UDC 539.12

Jaak LÖHMUS *, Eugen PAAL **, and Leo SORGSEPP ***

MOUFANG SYMMETRIES AND CONSERVATION LAWS

(Presented by H. Keres)

Concept of the Moufang symmetry is clarified through elementary field theoretical context. Corresponding infinitesimal symmetry operators obey the open Moufang-Mal'tsev algebra. By closing the latter, the complete collection of conservation laws generated by continuous Moufang symmetries has been found.

Introduction

A story is being told about Princeton physicists having revised, at the turn of the century, the mathematical disciplines with even slightest perspective for physics. A famous man is said to have remarked then about the group theory that this would hardly ever be of any concern to physics. By that time, the foundations of the theory of continuous transformation groups had been laid down by Sophus Lie, and the classification of simple Lie algebras had been given by Wilhelm Killing and Elie Cartan. It is the irony of Fate that afterwards the applied group theory for physics was to a great extent elaborated by Princeton professors.

The Noether theorem (1918) binds invariance (symmetry) principles and conservation laws in classical physics. Through the fundamental works of Hermann Weyl, Bartel van der Waerden, Eugene Wigner and John von Neumann the group theory developed into a powerful tool for quantum theory as well. By the 1940 there were already in existence the general representation theory of semisimple Lie groups and Wigner's famous representation theory of the Poincaré group. As a valuable by-product, the mathematical theory of semisimple Lie groups and also their representation theory was rewritten for physics in detail and in a less abstract form.

Nevertheless, nowadays we are witnessing that the classical group theory has been essentially exhausted and physicists are forced to look for generalizations and extensions of the group concept — such as supergroups, infinite-dimensional groups, quantum groups and some *non-associative* systems.

The theory of *Moufang loops* [1, 2] and *Mal'tsev algebras* [3, 4] has been well elaborated, also some indications [5, 6] appear that these algebraic systems are knocking at the door of physics as well. Application of Moufang loops and Mal'tsev algebras in physics may nevertheless

* Eesti Teaduste Akadeemia Füüsika Instituut (Institute of Physics, Estonian Academy of Sciences). EE2400 Tartu, Riia 142. Estonia.

** Tallinna Tehnikaülikool (Tallinn Technical University). EE0108 Tallinn, Akadeemia tee 1. Estonia.

*** Eesti Teaduste Akadeemia Astrofüüsika ja Atmosfäärifüüsika Instituut (Institute of Astrophysics and Atmospheric Physics, Estonian Academy of Sciences). EE2444 Tartu-maa, Tõravere. Estonia.

be complicated, for we do not know how to translate the language of Nature written in terms of these systems into our common language of groups and Lie algebras. To overcome the *group theoretic geocentrism* of which one is so fond nowadays, the *representation theory* of Moufang loops and Mal'tsev algebras is needed. Meanwhile, the representation theory of Moufang loops is actually missing, and the *Eilenberg representation theory* [7–10] of the Mal'tsev algebras is not quite comfortable for physical purposes. Motivated by this, a nonassociative extension (generalization) of the group geocentric method based on the Moufang loops and Mal'tsev algebras has been initiated [11–19].

In this paper, the concept (method) of the Moufang symmetry is clarified through elementary field theoretical context, i.e. we show how the Moufang loops and Mal'tsev algebras (Sec. 1) can be elementarily but naturally exploited in classical field theory. Continuous Moufang transformations are introduced (Sec. 2) and then used to create their own symmetries, Noether currents, charges and conservation laws without hurting the conventional field theoretical formalism. The corresponding infinitesimal symmetry operators obey an *open algebra* called the Moufang-Mal'tsev algebra (Sec. 3). By closing the latter (Sec. 4), the complete set of Noether's identities (conservation laws) is found (Sec. 5). The notions of the weak and hidden Moufang symmetries are introduced (Sec. 6 and 7) and shown to revive the weak and generalized representations of the Mal'tsev algebras in the sense of K. Yamaguti [8]. We hope that the main features of this method easily survive more sophisticated contexts.

1. Moufang loops and Mal'tsev algebras

A *Moufang loop* [1, 2] is a quasigroup G with the two-sided identity element e in which the *Moufang identity*

$$(ag)(ha) = a(gh)a \quad (1.1)$$

holds. The Moufang loop G is said to be *analytic* [3] if G is a real, analytic manifold so that both the Moufang loop operation $G \times G \rightarrow G$: $(g, h) \rightarrow gh$ and the inversion map $G \rightarrow G$: $g \rightarrow g^{-1}$ are analytic ones. We shall denote the dimension of G by r . The *local coordinates* of $g \in G$ are denoted (in a fixed chart of e) by g^1, \dots, g^r , and the local coordinates of the identity element e of G are supposed to be zero: $e^i = 0$ ($i = 1, \dots, r$). As in the case of Lie groups, we can consider the Taylor expansions

$$(gh)^i = g^i + h^i + a_{jk}^i g^j h^k + \dots, \quad i = 1, \dots, r,$$

and introduce the anti-symmetric quantities

$$c_{jk}^i := a_{jk}^i - a_{kj}^i = -c_{kj}^i, \quad i, j, k = 1, \dots, r,$$

called the *structure constants* of G .

The *tangent algebra* of G can be defined [3, 20] similarly to the tangent (Lie) algebra of the Lie [21] group and we denote it by Γ . Geometrically, this algebra is the tangent space of G at e . The product of X, Y in Γ will be denoted by $[X, Y]$:

$$[X Y]^i := c_{jk}^i X^j Y^k = -[Y X]^i, \quad i = 1, \dots, r.$$

The tangent algebra of G need not be a Lie algebra. In other words, there may be a triple $X, Y, Z \in \Gamma$, such that the Jacobi identity fails in Γ :

$$J(X, Y, Z) := [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] \neq 0.$$

Instead, for all $X, Y, Z \in \Gamma$, we have [3] a more general identity

$$[[X, Y], [Z, X]] + [[[X, Y], Z], X] + [[[Y, Z], X], X] + [[[Z, X], X], Y] = 0, \quad (1.2)$$

called the *Mal'tsev identity*. The tangent algebra Γ of G is hence said to be the *Mal'tsev algebra*. This identity concisely reads [4]

$$J(X, Y, [X, Z]) = [J(X, Y, Z), X], \quad (1.3)$$

from which it can be easily seen that every Lie algebra is a Mal'tsev algebra as well. Every finite-dimensional real Mal'tsev algebra is proved [22–24] to be the tangent algebra of some analytic Moufang loop.

2. Moufang symmetries and currents (I)

Let us now introduce some necessary field theoretic notations. The *coordinates* of a *space-time* point x are labeled as x^μ ($\mu = 0, 1, \dots, D-1$) with a *time* coordinate $x^0 = t$. The *Lagrangian* (density) $L(\psi)$ is supposed to depend on a set of independent *fields* $\psi^A(x)$ ($A = 1, \dots, N$) and their derivatives $\partial_\mu \psi^A(x) = \psi_\mu^A(x)$. The canonical *D-momenta* are denoted as $\pi_A^\mu := \partial L / \partial \psi_\mu^A$. The *field equations* read

$$L_A := \partial_\mu \partial L / \partial \psi_\mu^A - \partial L / \partial \psi^A = 0, \quad A = 1, \dots, N. \quad (2.1)$$

In the following, we shall denote the *row*-vector of canonical *D-momenta* as π^μ , and ψ will label the *column*-vector of fields.

Let us consider a pair (S, T) of the linear transformations

$$\psi(x) \rightarrow S_g \psi(x) = \psi(x) + g^i S_i \psi(x) + O(g^2), \quad (2.2a)$$

$$\psi(x) \rightarrow T_g \psi(x) = \psi(x) + g^j T_j \psi(x) + O(g^2), \quad (2.2b)$$

with S_i and T_j as the *infinitesimal operators* of (S, T) . Here, contrary to the group geocentric expectations, g is not assumed to be an element of a Lie group, instead, it is supposed to be an element of an analytic Moufang loop G . Once more discarding the group geocentric prejudices, the following identities are assumed to hold for all g, h of G :

$$S_g T_h S_h = S_{gh} T_g, \quad S_g T_g T_h = T_{hg} S_g. \quad (2.3)$$

Nevertheless, $S_e = T_e = E$ (*Identity transformation*) are assumed to survive. The pair (S, T) with such properties is called [12] the *birepresentation* of G , and the transformations (2.2a, b) are said to be *G-transformations* (*Moufang transformations*).

Algebraic properties of such transformations were recently studied in [12, 17–19]. Let us praise here only the following most elementary ones:

$$S_g T_g = T_g S_g, \quad S_g^{-1} = S_{g^{-1}}, \quad T_g^{-1} = T_{g^{-1}}, \quad \forall g \in G.$$

The action of $g \in G$ on the Lagrangian $L(\psi)$ is defined by

$$L(\psi) \rightarrow L(S_g^{-1} \psi) = L(\psi) + g^i S'_i L(\psi) + O(g^2), \quad (2.4a)$$

$$L(\psi) \rightarrow L(T_g^{-1} \psi) = L(\psi) + g^j T'_j L(\psi) + O(g^2), \quad (2.4b)$$

where S'_i and T'_j are the corresponding infinitesimal operators. Such pair of transformations is said to be *induced* by (S, T) and denoted as (S', T') . It is quite easy to check [15] that the latter turns out to be a birepresentation of G as well.

In general, the Moufang transformations need not be the symmetries of $L(\psi)$. The Lagrangian $L(\psi)$ is said to be *G-invariant* (*Moufang invariant*) if

$$L(S_g\psi) = L(T_g\psi) = L(\psi), \quad \forall g \in G, \quad (2.5)$$

which infinitesimally read

$$S'_j L(\psi) = T'_j L(\psi) = 0, \quad j=1, \dots, r. \quad (2.6)$$

By rearranging the terms according to the canonical prescription, the latter can be in turn rewritten as *Noether identities*,

$$-S'_j L(\psi) = S_{jB}^A \psi^B L_4 + \partial_\mu s_j^\mu = 0, \quad (2.7a)$$

$$-T'_j L(\psi) = T_{jB}^A \psi^B L_4 + \partial_\mu t_j^\mu = 0, \quad (2.7b)$$

with s_j^μ and t_j^μ as the Noether *currents* generated by the Moufang transformations:

$$s_j^\mu(x) := \pi^\mu(x) S_j \psi(x), \quad t_j^\mu(x) := \pi^\mu(x) T_j \psi(x). \quad (2.8)$$

At this point, it is useful to remind that every infinitesimal transformation $\delta\psi$ of ψ generates its Noether current $\pi^\mu \delta\psi$. Thus, continuous Moufang transformations produce their own currents without the slightest hurt to the field theoretic formalism. In our case, the Noether *charges* read

$$\sigma_j(t) := -i \int s_j^0(x) dx^1 \dots dx^{D-1}, \quad (2.9a)$$

$$\tau_j(t) := -i \int t_j^0(x) dx^1 \dots dx^{D-1}, \quad (2.9b)$$

$$j=1, \dots, r,$$

and the corresponding *conservation laws*

$$\frac{d}{dt} \sigma_j(t) = \frac{d}{dt} \tau_j(t) = 0, \quad j=1, \dots, r. \quad (2.10)$$

can be obtained from (2.7a, b) under the ordinary assumptions that field equations (2.1) hold and *G-currents* s_j^μ , t_j^μ vanish on the spatial integration boundary.

3. Moufang-Mal'tsev algebra

Suppose for a moment that the birepresentation (S, T) of G obeys the common group geocentric identities

$$S_g S_h = S_{gh}, \quad T_g T_h = T_{hg}, \quad S_g T_h = T_h S_g, \quad \forall g, h \in G.$$

Such birepresentation of G is called *associative*. It can be easily shown that these identities are *equivalent*. The Moufang loop G may nevertheless remain non-associative, but one must remember [12] that *non-associative* Moufang loops may not have *faithful* associative birepresentations at all.

It is well known [21] that infinitesimal associative transformations obey the *Lie algebra* commutation relations (*CR*)

$$[S_j, S_k] = c_{jk}^n S_n, \quad (3.2a)$$

$$[T_j, T_k] = -c_{jk}^n T_n, \quad (3.2b)$$

$$[S_j, T_k] = 0 \quad , \quad j, k = 1, \dots, r, \quad (3.2c)$$

with the structure constants c_{jk}^n of G . Likewise, the induced infinitesimal symmetry transformations of the Lagrangian $L(\psi)$ follow this algebra. Now, if the Moufang transformations are not assumed to be associative, a question about the corresponding modification of CR (3.2a—c) arises. Answer [13] is given by

Theorem 3.1. *Let (S, T) be a differentiable birepresentation of an analytic Moufang loop G and c_{jk}^n be the structure constants of G . Then, the infinitesimal operators of (S, T) obey the CR*

$$[S_j, S_k] = c_{jk}^n S_n - 2[S_j, T_k], \quad (3.3a)$$

$$[T_j, T_k] = -c_{jk}^n T_n - 2[T_j, S_k], \quad (3.3b)$$

$$[S_j, T_k] = [T_j, S_k], \quad j, k = 1, \dots, \dim G \quad (3.3c)$$

called the Moufang-Mal'tsev algebra.

We can see that the Moufang-Mal'tsev algebra (3.3a—c) is in some sense a *minimal* though an *open* extension (generalization) of the Lie algebra (3.2a—c). The induced infinitesimal Moufang transformations of the Lagrangian $L(\psi)$ obey this algebra as well. Since c_{jk}^n are structure constants of the tangent Mal'tsev algebra of the analytic Moufang loop G , continuous Moufang symmetries are natural to call the *Moufang-Mal'tsev symmetries*.

4. Yamaguti operators and closure conditions

Infinitesimal G -invariance conditions (2.6) can be considered as a constraining set of partial differential equations for the Lagrangian $L(\psi)$ to have truly the G -symmetry. CR (3.2a—c) are the *closure* conditions for the system (2.6) when G -transformations are *associative*. To close the system in *general (non-associative)* case, we must extend it with equations

$$[S_j, T_k] L(\psi) = 0, \quad j, k = 1, \dots, r. \quad (4.1)$$

Now, a way of closing [11, 13] the Moufang-Mal'tsev algebra (which in fact means its *embedding* into a *Lie algebra*) is outlined.

Start by rewriting the Moufang-Mal'tsev algebra as follows:

$$[S_j, S_k] = 2F_{jk} + 1/3c_{jk}^n S_n + 2/3c_{jk}^n T_n, \quad (4.2a)$$

$$[S_j, T_k] = -F_{jk} + 1/3c_{jk}^n S_n - 1/3c_{jk}^n T_n, \quad (4.2b)$$

$$[T_j, T_k] = 2F_{jk} - 2/3c_{jk}^n S_n - 1/3c_{jk}^n T_n \quad (4.2c)$$

Here, (4.2b or 2a or 2c) can be assumed to be the defining identity for the *Yamaguti operator* F_{jk} . It turns out that these operators are not linearly independent, since

$$c_{jk}^n F_{nl} + c_{kl}^n F_{nj} + c_{lj}^n F_{nk} = 0, \quad (4.3)$$

$$F_{jk} + F_{kj} = 0, \quad j, k, l = 1, \dots, r. \quad (4.4)$$

Constraints (4.4) trivially descend from the *anti-symmetry* of the commutator bracketing, but the proof of (4.3) is not so trivial [11]. The following two theorems are of crucial meaning.

Theorem 4.1. Let (S, T) be a differentiable birepresentation of an analytic Moufang loop G and c_{jk}^n be the structure constants of G . Then, the infinitesimal operators of (S, T) obey the CR

$$[F_{jk}, S_l] = d_{jkl}^n S_n , \quad (4.5a)$$

$$\begin{aligned} [F_{jk}, T_l] &= d_{jkl}^n T_n , \\ j, k, l &= 1, \dots, \dim G, \end{aligned} \quad (4.5b)$$

called the reductivity conditions of (S, T) , where the Yamaguti constants d_{jkl}^n are defined by

$$6d_{jkl}^n := c_{js}^n c_{kl}^s - c_{ks}^n c_{jl}^s + c_{sl}^n c_{jk}^s , \quad (4.6)$$

and F_{jk} are defined by (4.2b or 2a or 2c).

Theorem 4.2. Let (S, T) be a differentiable birepresentation of an analytic Moufang loop G and d_{jkl}^n be the Yamaguti constants of G . Then, the Yamaguti operators of (S, T) obey the Lie algebra

$$\begin{aligned} [F_{jk}, F_{ln}] &= d_{jkl}^s F_{sn} + d_{jkn}^s F_{ls} , \\ j, k, l, n &= 1, \dots, \dim G. \end{aligned} \quad (4.7)$$

Computations which in fact prove these theorems were carried out in [11, 13]. Dimension of the Lie algebra (4.2–7) does not exceed $2r+r(r-1)/2$, meanwhile the dimension of its subalgebra (4.7) does not exceed $r(r-1)/2$. Consistency (Jacobi identities) of this Lie algebra is guaranteed [11] by the identities of the Lie [25] and general Lie triple systems [8, 26] associated with the tangent Mal'tsev algebra of G .

5. Moufang symmetries and currents (II)

We now turn to our discussion of G -symmetry by noting that equations (2.6) can be in fact closed by

$$F'_{jk} L(\psi) = 0 , \quad j, k = 1, \dots, r, \quad (5.1)$$

where the Yamaguti operators F'_{jk} are defined as

$$F'_{jk} := -[S'_j, T'_k] + 1/3 c_{jk}^n S'_n - 1/3 c_{jk}^n T'_n = -F'_{kj} . \quad (5.2)$$

Also, to find the closing set of conservation laws, (5.1) must be rewritten as the Noether identities as well:

$$F'_{jk} L(\psi) = F'_{jkB} \Psi^B L_A + \partial_\mu f'_{kj} = 0 \quad (5.3)$$

where the additional currents

$$f'_{jk}(x) := \pi^\mu(x) F'_{jk} \psi(x) = -f'_{kj}(x) \quad (5.4)$$

obey the linear constraints

$$c_{jk}^n f'_{nl}(x) + c_{kl}^n f'_{nj}(x) + c_{lj}^n f'_{nk}(x) = 0 ,$$

which are due to (4.3). By introducing now the charges

$$\Phi_{jk}(t) := -i \int f_{jk}^0(x) dx^1 \dots dx^{D-1} = -\Phi_{kj}(t) , \quad (5.5)$$

with the obvious constraints

$$c_{jk}^n \Phi_{nl}(t) + c_{kl}^n \Phi_{nj}(t) + c_{lj}^n \Phi_{nk}(t) = 0 ,$$

we can obtain from (5.3) the desired *closing* set of conservation laws:

$$\frac{d}{dt} \Phi_{jk}(t) = 0 , \quad j, k = 1, \dots, r . \quad (5.6)$$

Note that the charges Φ_{jk} and conservation laws (5.6) descend from *non-associativity* phenomenon. Our method used up the fact that infinitesimal Moufang transformations generate the *Lie* algebra (4.2–7). In this sense, we can state that the collection of conservation laws (2.10) and (5.6) is closed (complete) as well.

6. Weak Moufang symmetries

The structure of the algebra of infinitesimal Moufang transformations enables to introduce some natural generalizations [15, 16] of the Moufang symmetry.

The Lagrangian $L(\psi)$ is said to be *weakly G-invariant* if

$$L(S_g \psi) = L(T_g \psi) \quad \forall g \in G , \quad (6.1)$$

which infinitesimally reads

$$S'_j L(\psi) = T'_j L(\psi) , \quad j = 1, \dots, r . \quad (6.2)$$

Rearranging the terms according to the canonical prescription, we can get from (6.2) the *weak Noether identities*

$$S'_{jB} \psi^B L_A + \partial_\mu s_j^\mu = T'_{jB} \psi^B L_A + \partial_\mu t_j^\mu = 0 , \quad (6.3)$$

from which the *weak* conservation laws

$$\frac{d}{dt} \sigma_j(t) = \frac{d}{dt} \tau_j(t) , \quad j = 1, \dots, r . \quad (6.4)$$

can in turn be obtained. Denoting

$$Q'_j := S'_j - T'_j , \quad Q_j := S_j - T_j , \quad (6.5)$$

$$q_j^\mu := s_j^\mu - t_j^\mu , \quad \theta_j := \sigma_j - \tau_j , \quad (6.6)$$

it follows from (6.2 and 3) that

$$-Q'_j L(\psi) = Q'_{jB} \psi^B L_A + \partial_\mu q_j^\mu = 0 , \quad j = 1, \dots, r , \quad (6.7)$$

and the weak conservation laws (6.4) read

$$\frac{d}{dt} \theta_j(t) = 0 , \quad j = 1, \dots, r . \quad (6.8)$$

To close the system of the weak *G*-invariance conditions (6.2), we must extend it with equations

$$[Q'_j, Q'_k] L(\psi) = 0 , \quad j, k = 1, \dots, r . \quad (6.9)$$

Now, we can proceed by computing

$$\begin{aligned}
 [Q_j, Q_k] &= [S_j - T_j, S_k - T_k] \\
 &= [S_j, S_k] - 2[S_j, T_k] + [T_j, T_k] \\
 &= 2F_{jk} + 1/3c_{jk}^n (S_n + 2T_n) + \\
 &\quad + 2F_{jk} - 1/3c_{jk}^n (2S_n + T_n) \\
 (6.8) \quad &\quad + 2F_{jk} - 2/3c_{jk}^n (S_n - T_n) + \\
 &\quad = 6F_{jk} - c_{jk}^n (S_n - T_n) \\
 &= 6F_{jk} - c_{jk}^n Q_n
 \end{aligned}$$

The required algebra reads

$$[Q_j, Q_k] = 6F_{jk} - c_{jk}^n Q_n, \quad (6.10)$$

$$[F_{jk}, Q_l] = d_{jkl}^n Q_n, \quad j, k, l = 1, \dots, r, \quad (6.11)$$

which in the terms of K. Yamaguti [8] means that the infinitesimal *weak* Moufang symmetry operators realize a *weak representation* of the tangent Mal'tsev algebra of G . CR (6.11) easily follow from (4.5a, b), meanwhile (6.10) arises as a new re-defining identity for F_{jk} . Note that the algebra (6.10, 11) can be also closed by (4.7), and the dimension of the resulting Lie algebra does not exceed $r+r(r-1)/2$. The closed collection of infinitesimal *weak G*-invariance conditions (weak Noether identities) consists of the equations (6.7) and (5.3).

7. Hidden Moufang symmetries

The Lagrangian $L(\psi)$ is said to be *hiddenly G*-invariant if $F'_{jk} L(\psi) = 0$ ($j, k = 1, \dots, r$). The operators F'_{jk} are defined by (5.2) and obey the Lie algebra

$$[F'_{jk}, F'_{ln}] = d_{jkl}^s F'_{sn} + d_{jkn}^s F'_{ls}, \quad j, k, l, n = 1, \dots, r.$$

The corresponding Noether identities (5.3) give rise to *hidden* conservation laws (5.6). Following K. Yamaguti [8], it can be said that the operators F'_{jk} realize a *generalized representation* of the tangent Mal'tsev algebra of G .

Let us finally remark that all these Moufang symmetry considerations are well acceptable from the point of view of alternative algebras and octonions [27-30]. Also, it is quite trivial to foresee the Noether *charge (density) algebras* generated by continuous Moufang transformations.

REFERENCES

1. Moufang, R. Math. Ann., 1935, **110**, 416-430.
2. Bruck, R. H. A Survey of Binary Systems. Berlin; Heidelberg; New York, Springer, 1971.
3. Мальцев А. И. Матем. сб., 1955, **36**, 3, 569-576.
4. Sagle, A. A. Trans. Amer. Math. Soc., 1961, **101**, 3, 426-458.
5. Jo S.-G. Phys. Lett., 1985, **B163**, 5/6, 353-359.
6. Niemi, A.; Semenoff, G. W. Phys. Rev. Lett., 1985, **55**, 9, 927-930.
7. Eilenberg, S. Ann. Soc. Polon. Math., 1948, **21**, 1, 125-134.

8. Yamaguti, K. Kumamoto J. Sci. Ser. A., 1963, **6**, 1, 9–45.
9. Кузьмин Е. Н. Алгебра и логика, 1968, **7**, 4, 48–69.
10. Carlsson, R. J. Reine Angew. Math., 1976, **281**, 199–210.
11. Паал Э. Препринт Ф-42. АН ЭССР. Тарту, 1987.
12. Паал Э. Тр. Ин-та физики АН ЭССР, 1987, **62**, 142–158.
13. Paal, E. Prepr. F-46. Tartu, 1988.
14. Паал Э. Междунар. конф. по алгебре, посвящ. памяти А. И. Мальцева (Новосибирск 21–26 авг. 1989). Тез. докл. по теории моделей и алгебраических систем. Новосибирск, 1989, 97.
15. Löhmus, I., Paal, E., Sorgsepp, L. Prepr. F-50. Tartu, 1989.
16. Paal, E. Trans. Inst. of Phys. Estonian Acad. Sci., 1990, **66**, 98–106.
17. Паал Э. Тр. Ин-та физики АН ЭССР, 1989, **65**, 104–124.
18. Paal, E. Group Theoretical Methods in Physics. Proc. 18th Int. Colloq., Moscow, June 4–9, 1990. Berlin; Heidelberg; New York, Springer, 1991, 573–574.
19. Paal, E. Proc. Estonian Acad. Sci. Phys. Math., 1991, **40**, 2, 105–111.
20. Sagle, A. A. Can. Math. J., 1965, **17**, 4, 550–558.
21. Pontryagin, L. S. Topological Groups. Gordon and Breach, New York, 1966.
22. Кузьмин Е. А. Алгебра и логика, 1971, **10**, 1, 3–22.
23. Акимов М. А., Шелехов А. М. Сиб. мат. ж., 1971, **12**, 6, 1181–1191.
24. Кердман Ф. С. Докл. АН СССР, 1979, **249**, 3, 533–536.
25. Loos, O. Pacific J. Math., 1966, **18**, 3, 533–562.
26. Yamaguti, K. Kumamoto J. Sci. Ser. A, 1962, **5**, 4, 203–207.
27. Schafer, R. D. Trans. Amer. Math. Soc., 1952, **72**, 1, 1–17.
28. Schafer, R. D. An Introduction to Nonassociative Algebras. New York; London, Academic Press, 1966.
29. Lukierski, J., Minnaert, P. Phys Lett., 1983, **B129**, 6, 392–396.
30. Sudbery, A. J. Phys.: Math. Gen., 1984, **17**, 5, 939–955.

Received
Oct. 18, 1991

Jaak LÖHMUS, Eugen PAAL, Leo SORGSEPP

MOUFANGI SUMMEETRIAD JA JÄAVUSSEADUSED

Lihtsas väljateoreetilises kontekstis on selgitatud Moufangi sümmeetria ideed. Sümmeetriaisendustele vastavad infinitesimaloperaatorid moodustavad avatud Moufangi-Maltsevi algebra. Viimase sulgemisega on leitud pidevate Moufangi teisendustele poolt genereeritud täielik jäavusseaduste süsteem.

Яак ЛЫХМУС, Эуген ПААЛ, Лео СОРГСЕПП

МУФАНГОВЫЕ СИММЕТРИИ И ЗАКОНЫ СОХРАНЕНИЯ

В простом теоретико-полевом контексте объяснена идея Муфанговой симметрии. Соответствующие операторы инфинитезимальных симметрий образуют открытую Муфанг-Мальцевскую алгебру. Замыкая последнюю, найдена полная система законов сохранения, порожденная непрерывными Муфанговыми симметриями.

(1.1)

(2.1)

$$(\beta\gamma - \beta\delta + \alpha\gamma - \alpha\delta) \exp(\beta\gamma - \beta\delta + \alpha\gamma - \alpha\delta) = 1$$

$$(\beta\gamma - \beta\delta + \alpha\gamma - \alpha\delta) \exp(\beta\gamma - \beta\delta + \alpha\gamma - \alpha\delta) = 1$$

$$(\beta\gamma - \beta\delta + \alpha\gamma - \alpha\delta) \exp(\beta\gamma - \beta\delta + \alpha\gamma - \alpha\delta) = 1$$