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MOUFLANG LOOPS AND GENERALIZED LIE EQUATIONS

(Presented by H. Keres)

The generalized Lie equations (GLE) for linear birepresentations of the analytic Moufang loops are proposed and discussed. It is shown how the integrability conditions of GLE can be related to the non-Eilenbergian birepresentations of the Mal'tsev algebras.

1. Moufang loops and Mal'tsev algebras

A *Moufang loop* [1, 2] is a quasigroup G with the two-sided *identity element* e in which the *Moufang identity*

$$(ag)(ha)=a(gh)a \quad (1.1)$$

holds. The Moufang loop G is said to be *analytic* if G is a real, analytic manifold so that both the Moufang loop operation $G \times G \rightarrow G: (g, h) \rightarrow gh$ and the inversion map $G \rightarrow G: g \rightarrow g^{-1}$ are analytic ones. We denote the dimension of G by r . The *local coordinates* of $g \in G$ are denoted (in a fixed chart of e) by g^1, \dots, g^r , and the local coordinates of the identity element e of G are supposed to be zero: $e^i=0$ ($i=1, \dots, r$). As in the case of Lie groups, we can consider the Taylor expansions

$$\begin{aligned} (gh)^n &= g^n + v_j^n(g)h^j + O(h^2) \\ &= g^n + h^n + a_{kj}^n g^k h^j + \dots, \quad n=1, \dots, r, \end{aligned}$$

and introduce the anti-symmetric quantities

$$c_{jk}^i := a_{jk}^i - a_{kj}^i = -c_{kj}^i, \quad i, j, k = 1, \dots, r,$$

called the *structure constants* of G .

The *tangent algebra* of G is defined similarly to the tangent (Lie) algebra of the Lie group [3, 4] and we shall denote it by Γ . Geometrically, this algebra is the tangent space $T_e(G)$ of G at e . The product of $x, y \in \Gamma$ will be denoted by $[x, y]$. In component form,

$$[x, y]^i := c_{jk}^i x^j y^k = -[y, x]^i, \quad i=1, \dots, r. \quad (1.2)$$

The tangent algebra of G need not be a Lie algebra. In other words, there may be a triple $x, y, z \in \Gamma$, such that the Jacobi identity fails in Γ :

$$J(x, y, z) := [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \neq 0.$$

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Instead, for all $x, y, z \in \Gamma$, we have [3] a more general identity

$$[[x, y], [z, x]] + [[[x, y], z], x] + [[[y, z], x], x] + [[[[z, x], x], y]] = 0 \quad (1.3)$$

called the *Mal'tsev identity*. The tangent algebra Γ of G is hence said to be the *Mal'tsev algebra*. This identity reads concisely [5]

$$J(x, y, [x, z]) = [J(x, y, z), x], \quad (1.4)$$

from which it can be easily seen that every Lie algebra is a Mal'tsev algebra as well. Every finite-dimensional real Mal'tsev algebra is proved [6-8] to be the tangent algebra of some analytic Moufang loop.

2. Birepresentations of Moufang loops

Let \mathfrak{X} be a set and let $\mathfrak{T}(\mathfrak{X})$ be *transformation group* of \mathfrak{X} , i.e. group of bijective maps of \mathfrak{X} onto \mathfrak{X} . Elements of $\mathfrak{T}(\mathfrak{X})$ are called the *transformations* of \mathfrak{X} . Multiplication in $\mathfrak{T}(\mathfrak{X})$ is defined as the *composition* of transformations, and *identity element* of $\mathfrak{T}(\mathfrak{X})$ coincides with identity transformation E of \mathfrak{X} .

A pair (S, T) of the maps $g \rightarrow S_g$, $g \rightarrow T_g$ of a Moufang loop G into the group $\mathfrak{T}(\mathfrak{X})$ is said [9] to be an *action* of G on \mathfrak{X} if

$$S_e = T_e = E, \quad (2.1)$$

and

$$S_g T_g S_h \stackrel{(a)}{=} S_{gh} T_g, \quad S_g T_g T_h \stackrel{(b)}{=} T_{hg} S_g \quad (2.2)$$

hold for all $g, h \in G$. The pair (S, T) is called also a *birepresentation* of G . The transformations $S_g, T_g \in \mathfrak{T}(\mathfrak{X})$ ($g \in G$) are called *G-transformations* (*Moufang transformations*) of \mathfrak{X} .

The properties of such transformations were recently discussed in [9-16]. Let us raise here only the following most elementary ones:

$$\begin{aligned} S_g T_g &= T_g S_g, & S_g^{-1} &= S_{g^{-1}}, & T_g^{-1} &= T_{g^{-1}}, \\ S_g S_h T_h T_g &= T_h T_g S_g S_h, & \forall g, h \in G. \end{aligned} \quad (2.3)$$

A birepresentation (S, T) of G is said to be *linear* if \mathfrak{X} is a linear space and S_g, T_g ($g \in G$) are linear transformations of \mathfrak{X} . It is an easy exercise to show that if (S, T) is an action of G on a set \mathfrak{X} , and $\mathfrak{F}(\mathfrak{X})$ denotes the linear space of functions on \mathfrak{X} , then the pair (S', T') of the maps

$$g \rightarrow S'_g : (S'_g f)(x) = f(S_g^{-1}x), \quad g \rightarrow T'_g : (T'_g f)(x) = f(T_g^{-1}x), \\ g \in G, \quad x \in \mathfrak{X}, \quad f \in \mathfrak{F}(\mathfrak{X}),$$

defines a *linear action* of G on $\mathfrak{F}(\mathfrak{X})$.

3. Generalized Lie equations

Let G be an analytic Moufang loop and let \mathfrak{X} be a differentiable manifold. The dimensions of G and \mathfrak{X} will be denoted by r and n , respectively.

An action (S, T) of G on \mathfrak{X} is said to be *differentiable* (*smooth, analytic*) if the local coordinates of the points $S_g A$ and $T_g A$ are differentiable (*smooth, analytic*) functions of the local coordinates of the points $g \in G$ and $A \in \mathfrak{X}$. In this case, (S, T) is called *differentiable* (*smooth, analytic*) *birepresentation* as well.

Now, let GL_n denote the *general linear group* of n -dimensional (real, or complex) vector space V_n . Let G be an analytic Moufang loop with the identity element e , and let (S, T) be a differentiable linear birepresentation of G in GL_n . By fixing a base in V_n , one can represent an element g of G by two nonsingular matrices $S_g, T_g \in GL_n$ which will further be assumed to be differentiable with respect to g , as many times as needed. It is obvious that we should define the *generators* of (S, T) as follows:

$$S_j = \partial_j S_g|_{g=e}, \quad T_j = \partial_j T_g|_{g=e}, \quad j=1, 2, \dots, r.$$

We have here denoted $\partial_j := \partial/\partial g^j$.

With the following theorem, the *generalized Lie equations* for differentiable linear Moufang transformations are stated.

Theorem 3.1. *Let (S, T) be a differentiable linear birepresentation of an analytic Moufang loop G . Then the birepresentation matrices S_g, T_g ($g \in G$) satisfy the system of simultaneous differential equations (generalized Lie equations)*

$$v_j^n(g) \partial_n S_g = S_g T_g S_j T_g^{-1}, \quad (3.1a)$$

$$v_j^n(g) \partial_n T_g = S_g^{-1} T_j S_g T_g, \quad (3.1b)$$

$$j=1, 2, \dots, r.$$

Proof. We obtain (3.1a) at once by differentiating (2.2a) with respect to the local coordinates h^j ($j=1, \dots, r$) of $h \in G$ at $h=e$. To show (3.1b), we must first note that $S_h T_h T_g = T_{gh} S_h$. Differentiating the latter with respect to h^j at $h=e$, we get

$$S_j T_g + T_j T_g = v_j^n(g) \partial_n T_g + T_g S_j,$$

from which it follows that

$$v_j^n(g) \partial_n T_g = S_j T_g - T_g S_j + T_j T_g. \quad (3.2)$$

We must thus show that

$$S_j T_g - T_g S_j + T_j T_g = S_g^{-1} T_j S_g T_g.$$

This identity reads

$$S_g^{-1} T_j S_g + T_g S_j T_g^{-1} = S_j + T_j, \quad j=1, \dots, r, \quad (3.3)$$

and can easily be obtained by differentiating (2.3) with respect to h^j at $h=e$. \square

Corollary 3.1. *By means of (3.3), we can rewrite (3.1a, b) as follows:*

$$v_j^n(g) \partial_n S_g = S_g S_j + [S_g, T_j], \quad (3.4a)$$

$$v_j^n(g) \partial_n T_g = T_j T_g + [S_j, T_g], \quad (3.4b)$$

$$j=1, 2, \dots, r.$$

Remark. In fact, equation (3.4b) coincides with (3.2); the brackets $[\cdot, \cdot]$ are used for the commutators of matrices as well. Unlike (3.1a, b), equations (3.4a, b) are *linear* with respect to S_g and T_g . If $S_g T_h = T_h S_g$ for all $g, h \in G$, then (3.4a, b) and also (3.1a, b) return the familiar *Lie equations*.

4. Integrability conditions

Let us now rewrite the generalized Lie equations (3.1a, b) as

$$v_j^n(g) \partial_n S_g = S_g S'_j(g), \quad (4.1a)$$

$$v_j^n(g) \partial_n T_g = T'_j(g) T_g, \quad (4.1b)$$

with

$$S'_j(g) := {}^{(a)} T_g S_j T_g^{-1}, \quad T'_j(g) := {}^{(b)} S_g^{-1} T_j S_g. \quad (4.2)$$

Keeping close to the terminology of [11, 12], the matrices $S'_j(g)$, $T'_j(g)$ are called the *derivative generators* of (S, T) . We have the obvious initial conditions

$$S'_j(e) = S_j, \quad T'_j(e) = T_j, \quad j=1, 2, \dots, r. \quad (4.3)$$

The identities (3.3) read

$$S'_j(g) + T'_j(g) = S_j + T_j, \quad j=1, 2, \dots, r. \quad (4.4)$$

Proposition 4.1. *The derivative generators of (S, T) satisfy the system of simultaneous Heisenberg-like equations*

$$v_k^n(g) \partial_n S'_j(g) = [T'_k(g), S'_j(g)], \quad (4.5a)$$

$$v_k^n(g) \partial_n T'_j(g) = [T'_j(g), S'_k(g)], \quad (4.5b)$$

$$j, k=1, 2, \dots, r,$$

Proof. As an example, let us check (4.5a). We have

$$\begin{aligned} v_k^n(g) \partial_n S'_j &= v_k^n(g) (\partial_n T_g \cdot S_j T_g^{-1} - T_g S_j T_g^{-1} \partial_n T_g \cdot T_g^{-1}) \\ &= T'_k(g) T_g S_j T_g^{-1} - T_g S_j T_g^{-1} T'_k(g) T_g T_g^{-1} \\ &= T'_k(g) S'_j(g) - S'_j(g) T'_k(g) \\ &= [T'_k(g), S'_j(g)]. \end{aligned}$$

Equations (4.5b) can be proved similarly. \square

Corollary 4.1. *The derivative generators of (S, T) satisfy the commutation relations (CR)*

$$[S'_j(g), T'_k(g)] = [T'_j(g), S'_k(g)], \quad j, k=1, \dots, r. \quad (4.6)$$

Proof. We get the desired commutation relations by adding (4.5a) and (4.5b), and then using (4.4). \square

Now, let us define the *structure functions* $c_{jk}^n(g)$ of G by

$$v_j^n(g) \partial_n v_k^i(g) - v_k^n(g) \partial_n v_j^i(g) = c_{jk}^n(g) v_n^i(g). \quad (4.7)$$

The direct computations show that $c_{jk}^i(e) = c_{jk}^i$.

Theorem 4.1. *The integrability conditions of the generalized Lie equations (3.1a, b) read as commutation relations*

$$[S'_j(g) S'_k(g)] = c_{jk}^n(g) S'_n(g) - 2[S'_j(g), T'_k(g)], \quad (4.8a)$$

$$[T_j(g)T_k(g)] = -c_{jk}^n(g)T_n(g) - 2[T_j(g), S_k(g)], \quad (4.8b)$$

$$j, k = 1, 2, \dots, r.$$

Proof. As an example, we prove (4.8a). Differentiating (4.1a) with respect to g^m , we obtain

$$\begin{aligned} v_k^m(g)\partial_m v_j^n(g)\partial_n S_g + v_k^m(g)v_j^n(g)\partial_m \partial_n S_g &= \\ &= v_k^m(g)\partial_m S_g S_j'(g) + v_k^m(g)S_g \partial_m S_j'(g) \\ &= S_g S_k'(g)S_j'(g) + S_g [T_k'(g), S_j'(g)]. \end{aligned}$$

Exchange now the indices j and k , and subtract the resulting equality from the original one. Reducing then the *second-order partial derivatives*, we get

$$\begin{aligned} c_{kj}^m(g)v_m^n(g)\partial_n S_g &= S_g [S_k'(g), S_j'(g)] + S_g [T_k'(g), S_j'(g)] - \\ &\quad - S_g [T_j'(g), S_k'(g)] = \\ &= S_g [S_k'(g), S_j'(g)] + S_g 2[T_k'(g), S_j'(g)]. \end{aligned}$$

We must use GLE once more on the left-hand side of this equality. Then, after the left-division by S_g and rearranging the terms, we obtain the required commutation relations (4.8a). Commutation relations (4.8b) can be stated similarly by starting from (4.1b). \square

Corollary 4.2. *The generators of (S, T) satisfy the commutation relations*

$$[S_j, S_k] = c_{jk}^n S_n - 2[S_j, T_k], \quad (4.9a)$$

$$[T_j, T_k] = -c_{jk}^n T_n - 2[T_j, S_k], \quad (4.9b)$$

$$j, k = 1, 2, \dots, r.$$

Remark. More general (nonlinear) version of CR (4.9a, b) has recently been established in another way [13, 14]. It is also easy to see that CR (4.6) can now be re-obtained from the identity $[S_j'(g), S_k'(g)] = -[S_k'(g), S_j'(g)]$.

5. Birepresentations of Mal'tsev algebras

The representation theory of the *Mal'tsev algebras* which follows the concept of *S. Eilenberg* [17] has been well elaborated [18, 19]; see also the reviews [20, 21] where the main results of this theory are outlined. In this section, the following two *non-Eilenbergian* but equivalent definitions of a *birepresentation* of the *Mal'tsev algebra* are formulated. These definitions seem also to be supremely natural from the point of view of the theory of *alternative algebras* [22, 23]. In Sec. 6, it will be shown how the integrability conditions of GLE are related to the non-Eilenbergian birepresentations of the Mal'tsev algebras.

Let M be a *Mal'tsev algebra* and let L be a *Lie algebra*.

Definition 5.1. *A pair (S, T) of linear maps $x \rightarrow Sx$, $x \rightarrow Tx$ of M into L is said to be a birepresentation of the Mal'tsev algebra M if for all $x, y, z \in M$ the following identities hold (in L):*

$$[Sx, Sy] = S[x, y] - 2[Sx, Ty], \quad (5.1a)$$

$$[Tx, Ty] = -T[x, y] - 2[Tx, Sy], \quad (5.1b)$$

$$6[F(x; y), Sz] = S[x, y, z], \quad (5.2a)$$

$$6[F(x; y), Tz] = T[x, y, z], \quad (5.2b)$$

where $F(x; y)$ is defined by

$$3F(x; y) := S[x, y] - T[x, y] - 3[Sx, Ty], \quad (5.3)$$

and

$$[x, y, z] := [x, [y, z]] - [y, [x, z]] + [[x, y], z] \quad (5.4)$$

is the Yamaguti triple product [24, 18] in M .

The identities (5.2a, b) are called the *reductivity conditions* of (S, T) . Definition 5.2. A pair (S, T) of linear maps $x \mapsto Sx$, $x \mapsto Tx$ of M into L is said to be a birepresentation of the Mal'tsev algebra M if the identities (5.1a, b) and

$$[[Sx, Sy], Sz] = S(x, y, z), \quad (5.5a)$$

$$[[Tx, Ty], Tz] = T(x, y, z) \quad (5.5b)$$

hold for all $x, y, z \in M$, where the Loos triple product (x, y, z) is defined [25] in M by

$$3(x, y, z) := [x, [y, z]] - [y, [x, z]] + 2[[x, y], z]. \quad (5.6)$$

These definitions can be motivated by the fact that the birepresentations in the above sense appear as *differentials of continuous birepresentations of the analytic Moufang loops* [13–15]. Also, it must be noted that the Jacobi identities in the Lie algebra L are guaranteed [13] by the identities of the *Lie* and *general Lie triple systems* of the Mal'tsev algebra M [18, 24, 25].

The connection of birepresentations with the *Eilenbergian* representations is not clear.

6. Derivative Mal'tsev algebras

For $x, y \in T_e(G)$, define their new product $[x, y]_a$ in $T_e(G)$ by

$$[x, y]_a^i := c_{jk}^i(a)x^jy^k = -[y, x]_a^i, \quad i = 1, \dots, r, \quad a \in G.$$

The tangent space $T_e(G)$ with such a multiplication is said to be the *derivative* of the tangent algebra Γ of G and is denoted by Γ'_a . For $g, h \in G$, define [1, 11, 12] their *derivative product* $(gh)'_a \in G$ as well:

$$(gh)'_a = (ga^{-1})(ah), \quad g, h, a \in G. \quad (6.1)$$

It turns out [2] that the derivative multiplication given by (6.1) satisfies all the axioms of a Moufang loop. The derived Moufang loop with the multiplication rule (6.1) is called the *derivative loop* of G and is denoted as G'_a . The identity element of G'_a is also e , and the inverse element of g in G'_a is g^{-1} .

Theorem 6.1. Let G'_a be a derivative loop of an analytic Moufang loop G , and let Γ be the tangent algebra of G . Then the derivative algebra Γ'_a of Γ is the tangent algebra of the derivative Moufang loop G'_a of G .

Idea of proof. One must in fact verify that the structure functions $c_{jk}^i(a)$ of G are the structure constants of G'_a . This can be done by direct computations. \square

Corollary 6.1. *The derivative algebra Γ'_a of Γ is a Mal'tsev algebra as well.*

Proof. Since G'_a is an analytic Moufang loop, its tangent algebra Γ'_a must inescapably be a Mal'tsev algebra [3]. \square

Based on the *Corollary 6.1*, it would be natural to call Γ'_a the *derivative Mal'tsev algebra* of Γ .

The following theorem states the prospective relationship between the integrability conditions of GLE and non-Eilenbergian birepresentations of Mal'tsev algebras.

Theorem 6.2. *Let (S, T) be a differentiable linear birepresentation of an analytic Moufang loop G , and let Γ be the tangent Mal'tsev algebra of G . Then, for each $a \in G$, the pair $*(S, T)_a$ of the maps*

$$x \rightarrow (Sx)_a := x^j S_j(a), \quad x \rightarrow (Tx)_a := x^j T_j(a) \quad (x \in \Gamma_a)$$

is a birepresentation of the derivative Mal'tsev algebra Γ'_a of Γ .

Idea of a proof. It can be proved [12] that if (S, T) is a birepresentation of G , then, for each $a \in G$, the pair $(S, T)_a$ of the maps

$$g \rightarrow (S_g)_a := T_a S_g T_a^{-1}, \quad g \rightarrow (T_g)_a := S_a^{-1} T_g S_a \quad (g \in G)$$

is a birepresentation of the derivative Moufang loop G'_a of G . But the differential $*(S, T)_a$ of the birepresentation $(S, T)_a$ of G'_a at the identity element $e \in G'_a$ must be a birepresentation of the tangent Mal'tsev algebra Γ'_a of G'_a [13, 14, 16]. \square

Final remark. It may seem surprising, but the *reductivity conditions* for $*(S, T)_a$ can be obtained (in a non-trivial way) from the commutation relations (4.8a, b) as well.

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MOUFANGI LUUBID JA ÜLDISTATUD LIE VÖRRANDID

On vaadeldud analüütilise Moufangi luubi lineaarsituste üldistatud Lie vörrandeid ja näidatud, kuidas nende vörrandite integreeruvustingimused on seotud Maltsevi algebrate biesitustega.

Эugen ПААЛ

ЛУПЫ МУФАНГ И ОБОБЩЕННЫЕ УРАВНЕНИЯ ЛИ

Предлагают и рассматривают обобщенные уравнения Ли для линейных бипредставлений аналитической лупы Муфанг. Показывают, каким образом условия интегрируемости этих уравнений связаны с бипредставлениями алгебр Мальцева.

For $x, y \in T_r(G)$, define their new product $[x, y]_r$ in $T_r(G)$ by

$$[x, y]_r := c_{\alpha} (a) x^r y^{\alpha} - [y, x]^r, \quad i=1, \dots, r, \quad a \in G.$$

The tangent space $T_r(G)$ with such a multiplication is said to be the derivative of the tangent algebra $T_r(G)$ and is denoted by T'_r . For $g, h \in G$, defining $(gh)^r$ their derivative product $(gh)^r = g^r$ as well,

$$[g, h]_r := c_{\alpha} (a) g^r h^{\alpha} - [h, g]^r, \quad i=1, \dots, r, \quad a \in G. \quad (6.1)$$

It turns out [1] that the last cannot happen unless G is a nilpotent Lie group. In this case all the axioms of a Lie algebra hold for T'_r . The multiplication rule of T'_r is $[g, h]_r = g^r h^{\alpha} - [h, g]^r$ and a Lie bracket $[g, h]_r$ is denoted as $[G]$. The identity $g^r h^{\alpha} - [h, g]^r = g^r h^{\alpha} + [g, h]^r$ is called the Jacobi identity. The commutator $[g, h]_r$ is the commutator of g in $T_r(G)$. The Lie bracket $[G]$ is the Lie bracket of G . The tangent algebra T'_r of $T_r(G)$ is the tangent algebra T'_r of G . The Lie bracket $[G]$ is the Lie bracket of G .