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## MULTIFRACTALITY OF ESTONIAN COASTLINE

(Presented by J. Engelbrecht)

It was in 1967 when B. Mandelbrot published his pioneering article in the theory of fractals «How Long is the Coast of Britain?» [1], where he indicated that the question raised in the title is not correct, and coastlines should rather be measured in meters on fractional powers. Nevertheless, up to now, only few people except for specialists in fractals are aware of the uselessness of giving a particular length in meters. So in [2] we find: «the length of Estonian coast is 3794 km, which is composed by 1242 km of continental part and 2552 km of the coast of islands». Thus, hopefully, it is not useless to discuss this almost classical problem once more.

In this paper we

- give a short historical review;
- speculate on the applicability of fractal and multifractal models for real coastlines;
- propose a generator-based algorithm which produces (coastline-like) multifractal curves.

It has been realized long ago that due to small-scale sinuosities the length of a coast measured on maps of increasing resolution, increase without limit. In 1961 Richardson suggested an empirical law [3]: the length of the coast  $L$  depends on the resolution  $\delta$  as

$$L \propto \delta^{1-D_h}, \quad D_h > 1. \quad (1)$$

Mandelbrot treated the exponent  $D_h$  as the fractal dimension of the coastline, implying statistical self-similarity of the latter (i. e., statistically identical on maps of different resolution) [1]. Fig. 1 explains his graphic way in introducing the fractal coast through generators. Also, he associated the exponent  $k$  in the Korchak's empirical number-area rule for islands,

$$N(S > A) \propto A^{-k}, \quad k = D_f/2, \quad (2)$$

with the fractal dimension  $D_f$  of the full set of coastlines (i. e. the coastline of continent together with the coastlines of islands) [4]. Here  $N(S > A)$  denotes the number of islands the area  $S$  of which exceeds the given value  $A$ .

When in some regions the coastline seems actually self-similar, in others this is not the case. For instance, Estonian coast does not resemble its northeastern part depicted in a more detailed map. It turns out that most often a *multifractal* model is more adequate: instead of one exponent  $D_h$  in (1), a sequence of exponents has to be used for a complete description of the structure [5].

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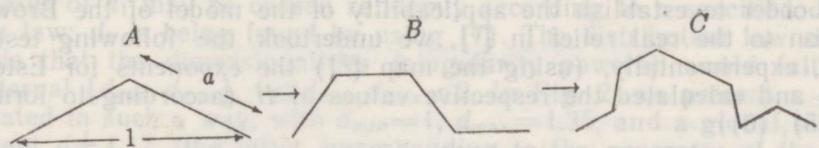


Fig. 1. *A* — the generator, *B* — the second generation, *C* — the third generation. In each next generation all the elements are replaced by the generator, properly reduced. In order to introduce irregularity, the orientation of generators is chosen randomly. The resulting fractal dimension is given by  $d = -\log_a(2)$ ,  $0.5 < a < 1/\sqrt{2}$ . This pattern corresponds to  $d = 1.19$ .

Perhaps the best way to explain the concept of multifractal sets is to introduce the exponents  $f(a)$  — fractal dimensions of the subsets of fixed order of singularity  $a$ , also called the *Lipschitz-Hölder exponents*. Not pretending to a stringent consideration, when applied to the coastlines the parameter  $a$  may be treated as the local value of fractal dimension. Indeed,  $\delta$  being the resolution of the map, the value  $a$  should be prescribed to a point on the coastline if the length of the curve, starting from that point scales as

$$L = \delta \left( \frac{l}{\delta} \right)^a, \quad (3)$$

where  $l$  is the separation from the point.

Multifractality is a form of scale invariance, more general than self-similarity. Thus, if a physical structure has no intrinsic space-scale within a large interval of scales (e.g., a turbulent flow with energy input at large scales, and dissipation at small ones) and is statistically isotropic, one can expect a multifractal pattern rather than a self-similar one.

Mandelbrot suggested to model real landscapes by the fractional Brownian function  $z = B_H(x, y)$  [4]. The index  $H$  characterizes the "mountyness" of the function:

$$\langle |B_H(\vec{r}) - B_H(\vec{r} + \vec{\varrho})| \rangle \propto |\vec{\varrho}|^H. \quad (4)$$

The average is taken here over an ensemble of landscapes. Omitting details let us note that the correlations of this function are believed to be Gaussian; particularly in the case of  $H = 1/2$ , all the vertical cross-sections  $z = f(r_\perp)$  are equivalent to the graphs plotting the displacement of a Brownian particle versus time. Such a not self-consistent artificial model obeys a simple self-similar structure.

Under these assumptions, Mandelbrot succeeded in finding an analytical expression of the fractal dimension of the full set of coastlines:

$$D_f = 2 - H. \quad (5)$$

The fractal dimension of a single coastline  $D_h$  appeared to be more difficult to find with no rigorous calculations to be known. Suggesting a rather questionable method of *virtual scale separation*, we derived [6]

$$D_h = 1 + \frac{7}{3}(1 - H). \quad (6)$$

Any simulations which allow us to check this formula are welcome.

In order to establish the applicability of the model of the Brownian function to the real relief in [6], we undertook the following test: we found, experimentally, (using the map [7]) the exponents for Estonian coast, and calculated the respective values of  $H$  (according to formulae (2), (5), (6)):

Table

measured	calculated	$H$
$D_h$ $1.17 \pm 0.02$	$H = (10 - 7D_h)/3$	$0.60 \pm 0.05$
$D_f$ $1.25 \pm 0.02$	$H = 2 - D_f$	$0.75 \pm 0.02$
$k$ $0.56 \pm 0.02$	$H = 2(1 - k)$	$0.88 \pm 0.04$

The dimensionality  $D_h$  was found by plotting the length of the continental coastline  $L$  versus the spread of the dividers  $\delta$ , the latter having been varied in the range from 0.8 km up to 25.6 km;  $D_f$  was assessed analogously. The assessment of the Korchak's exponent  $k$  is based on the data given in [2] concerning about 1350 Estonian islands.

The values of  $H$  in the Table, calculated in a different way, are quite different, and this circumstance may be explained as follows. The model of the Brownian relief assumes no influence of the sea on the relief. Actually, storms and ice drifts have a remarkable effect, especially on the coasts of smaller islands (here one should take also into account that the ground in Western Estonia is rising, hence these smoothing processes prevent small islands from arising and they are particularly responsible for the anomalously small value of  $k$ ), thus giving rise to the discrepancy mentioned above and, which is also likely, to multifractality. The latter conclusion may be confirmed by another experimental observation as well: local values of fractal dimension  $D_h$  vary in a rather wide interval, beginning from 1.02 for the northern part of Estonian coast between Aseri and Narva, and ending with 1.35 for the south-eastern coast of the island of Saaremaa. These numbers may be considered as limiting values of the Lipschitz-Hölder exponent  $a$ .

The multiplicative procedures based on the idea of energy flux conservation and yielding multifractal structures, are most often applied to scalar fields (see, e.g., [8]). However, the generating procedure presented in Fig. 1, may easily be modified to produce multifractal coastlines. Indeed, suppose we are not going to replace the elements of the chain which are shorter than the resolution scale  $\delta$ . Then a chain element on an intermediate step of generating, the length of which is  $l$ , would be ultimately replaced by the chain of total length  $L = \delta \left(\frac{l}{\delta}\right)^d$ ,  $d = -\log_a(2)$  ( $a$  being defined in Fig. 1). Now imagine that we replace this element by a generator of the next generation of the side length  $la$ , and prescribe to both sides of it (which are at the same time new chain elements) new dimensionalities  $d_1$  and  $d_2$ . In their turn, these new elements will be replaced by the generators of the new aspect ratios  $a_1$  and  $a_2$  instead of  $a$ ,  $a_{1,2} = 2^{-1/d_{1,2}}$ , and so on. There is an obvious additional condition of length conservation:

$$\left(\frac{la}{\delta}\right)^{d_1} + \left(\frac{la}{\delta}\right)^{d_2} = 2 \left(\frac{la}{\delta}\right)^d. \quad (7)$$

The value of  $d_1$  may be chosen randomly according to a specified distribution law;  $d_2$  is being found by using (7). The distribution law should provide that the dimensionalities (singularity powers) would fall into an interval  $[d_{\min}, d_{\max}]$ ,  $1 \leq d_{\min}, d_{\max} < 2$ . In Fig. 2, we present a curve generated in such a way, with  $d_{\min}=1$ ,  $d_{\max}=1.35$ , and a global scaling exponent  $d=1.17$  (the latter corresponding to the generator of the first generation).

## VIA DYNAMICAL SYSTEMS

(Presented by O. Isichenko)

ANNA SOR

This paper presents a new algorithm for generating coastlines. It is based on a nonlinear discrete-time system under which the output behaviour depends on the initial conditions. The algorithm is able to generate coastlines with fractal properties. The proposed algorithm is shown to be more efficient than other methods.



Fig. 2. A multifractal coastline obtained by the proposed generating algorithm. The lowest value of singularity powers is  $a_{\min}=1$ , the highest —  $a_{\max}=1.35$ , and global scaling exponent  $d=1.17$ .

The model matching problem (MMP), which consists in designing a compensation for a certain system so that the input-output map of the

Such a generating algorithm may be justified by speculations that more involved parts of the coast are to a lesser extent exposed to the influence of storms, so that the smaller structures are less smoothed.

As a conclusion, let us stress once more that real coastlines are in general multifractal, so that the quantities pretending to be fractal dimensions are rather simply scaling exponents. Also note that these exponents, besides their geographic and administrative importance (e.g., when finding the area of territorial waters), are applicable to several problems of physics [9].

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## ESTI RANNAJOONE MULTIFRAKTAALSUS

Lähtudes Richardsoni skeilingust rannajoone pikkuse  $L$  jaoks ( $L \propto \delta^{1-D}$ , kus  $\delta$  on mõõtmiseks kasutatud sirkli haarde vahe), on tuvastatud, et eri rannajoone lõikude puhul võib astmenäitajal  $D$  olla tunduvalt erinevaid väärtsusi. Järelikult on tegu multifraktaalse (mitte lihtsalt enesesarnase) objektiga. Samale järeldusele võib jõuda, kontrollides eksperimentaalselt leitud astendajate väärtsuste vastavust Mandelbroti esitatud maapinna reljeefi mudelist johtuvatele seostele; ilmneb oluline lahkunevus, mille peapõhjuseks on arvatavasti mere mõju kaldale. Mainitud kontrollarvudeks on parameeter  $D$ , arvutatuna Eesti mandriosa rannajoone jaoks; sama suurus, arvutatuna kogu rannajoone jaoks (koos saartega), ja saarte jaotumist suuruse järgi kirjeldav astmenäitaja  $k$ . On esitatud multifraktaalse rannajoone geomeetriline mudel.

Яан КАЛДА

## МУЛЬТИФРАКТАЛЬНОСТЬ БЕРЕГОВОЙ ЛИНИИ ЭСТОНИИ

Исходя из скейлинга Ричардсона для длины береговой линии  $L$  ( $L \propto \delta^{1-D}$ , где  $\delta$  — раствор циркуля, использованного для измерения), установлено, что для разных отрезков береговой линии показатель экспонента  $D$  может иметь существенно разные значения. Следовательно, рассматриваемый объект мультифрактален (а не просто самоподобен). К этому же выводу можно прийти, проверив насколько экспериментальные значения показателей степени соответствуют связям, вытекающим из модели Мандельброта для земного рельефа. Обнаруживается существенное разногласие между ними, вызванное, вероятно, воздействием моря на берег. Упомянутые проверочные числа — это параметр  $D$ , вычисленный для континентальной части берега Эстонии, та же величина, вычисленная для всего берега (т. е. включая острова), и показатель экспонента  $k$ , описывающий закон распределения островов по размерам. Представлена геометрическая модель мультифрактальной береговой линии.

Actually, storms and ice drifts have a remarkable effect especially on the coasts of smaller islands (here one should take also into account that the ground in Western Estonia is rather soft and has something protective, probably because the soil is not very deep). This may be responsible for the difference between the values of  $D$  corresponding to the different parts of the coastline. It is likely, to multifractality. The order condition may be confirmed by another experimental

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