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EVOLUTES OF HIGHER ORDER FOR A SUBMANIFOLD M^m IN E^n

(Presented by G. Vainikko)

Introduction. Let M^m be a smooth submanifold in a Euclidean space E^n with tangent bundle TM^m and normal bundle $T^\perp M^m$. The fibres of these bundles at a point $x \in M^m$ are, respectively, the tangent space $T_x M^m$ and the normal space $T^\perp_x M^m$ as vector spaces. Correspondingly, the tangential m-plane $[x, T_x M^m]$ and the normal $(n-m)$-plane $[x, T^\perp_x M^m]$ are defined at $x \in M^m$.

The evolutes, more exactly — the polar submanifolds (in the sense of the Definition 1 of this paper) of higher order of a submanifold M^m in E^n are investigated by R. Mullari [1], who specified the necessary and sufficient conditions for the existence of the sequence of polar submanifolds

$$\mathcal{E}^{(1)} \supset \mathcal{E}^{(2)} \supset \ldots \supset \mathcal{E}^{(\lambda)} \supset \ldots$$

of higher order. Here every $\mathcal{E}^{(\lambda)}$ consists of plane generatrices $S^{(\lambda)}_x$ as characteristics of order λ of the family $\{[x, T^\perp_x M^m] \mid x \in M^m\}$. The position of every plane $S^{(\lambda)}_x$ in E^n is also described. Precisely, it is shown that the osculating space $[x, O^{(\lambda)}_x M^m]$ of order λ and the plane generatrix $S^{(\lambda)}_x$ of $\mathcal{E}^{(\lambda)}$ at the point $x \in M^m$ are orthogonal to each other and the intersection $[x, O^{(\lambda)}_x M^m] \cap S^{(\lambda)}_x$ is the point. Here the vector space $O^{(\lambda)}_x M^m$ of the osculating plane $[x, O^{(\lambda)}_x M^m]$ of order λ is defined as follows:

$$O^{(\lambda)}_x M^m = T_x M^m \oplus \operatorname{span}\{h(X, Y), \overline{\nabla}_{Z_1} h(X, Y), \ldots, \overline{\nabla}_{Z_{\lambda-2}} \ldots \overline{\nabla}_{Z_1} h(X, Y)\},$$

where h is the second fundamental form [2] of M^m and $\overline{\nabla} h, \ldots, \overline{\nabla} \ldots \overline{\nabla} h$ are the covariant differentials of h with respect to the van der Waerden-Bortolotti connection by arbitrary vectors $X, Y, Z_1, \ldots, Z_{\lambda-2} \in T_x M^m$. There exists the last osculating plane $[x, O^{(p)}_x M^m]$, which coincides with E^n.

J. A. Shouten and D. J. Struik [3] define the evolute of order λ for a given line M^1 in E^n as the line $\widetilde{M}^1$ in E^n, whose osculating plane of order $\lambda - 1$ at the arbitrary point $y \in \widetilde{M}^1$ lies in a normal plane $[x, T^\perp_x M^1]$ and intersects the given line M^1 at the point $x \in M^1$. The next definition gives a generalization of this concept.

Definition 1. *The submanifold $\mathcal{E}^{(\lambda)}$ in E^n is called the* **evolute of order** λ *for a submanifold M^m in E^n, if there exists a submersion ϱ: $\mathcal{E}^{(\lambda)} \to M^m$, such that the osculating plane $[y, O^{(\lambda-1)}_y \mathcal{E}^{(\lambda)}]$ of order $\lambda - 1$ at the arbitrary point $y \in \mathcal{E}^{(\lambda)}$ lies in $[x, T^\perp_x M^m]$, where $x = \varrho(y)$, and it intersects the submanifold M^m at the point $x \in M^m$.*

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The existence of the submersion ϱ needs that $\dim \mathcal{E}^{(\lambda)} \geqslant m$.

In the special case, when the osculating plane $[y, O^{(\lambda-1)}_{y} \mathcal{E}^{(\lambda)}]$ of order $\lambda-1$ coincides with the normal plane $[x, T^{\perp}_{x} M^m]$, the evolute $\mathcal{E}^{(\lambda)}$ is the polar submanifold of order λ of the submanifold M^m.

For a line M^1 in E^n $(n \geqslant 3)$ it is known that the straight lines, tangent to the evolute $\widetilde{M}^1$ of first order and therefore normal to M^1, generate a tangentially degenerated surface (a torse) and thus the field of their unit direction vectors is parallel, in the normal connection of M^1. The evolute $\widetilde{M}^1$ is the envelope of the family of generatrices of this surface. Its point, corresponding to $x \in M^1$, lies in $S^{(1)}_{x}$ for M^1 at x.

The aim of this paper is to generalize these facts to the case of the evolute of higher order for a submanifold M^m in E^n. Hence it is necessary to introduce the next generalized concepts of higher order: parallelism of a normal r-field in normal connection, osculating degeneration and envelopment. They are given below by Definitions 2, 3, 4. In § 5 the concept of the generally located normal r-plane $[x, N^r(x)]$ for a fixed higher order in $[x, T^{\perp}_{x} M^m]$ is introduced. Finally, in § 6, the main Theorem is given which connects these generalizations and presents three criteria of the existence of the evolute of higher order for a general submanifold M^m in E^n. The results of this paper for a special case of Cartan normally flat submanifolds M^m in E^n, are announced in [4].

1. Preliminaries. If a submanifold M^m in E^n is given, the orthonormal frame bundle $O(E^n)$ can be reduced to $O(M^m; E^n)$; here $\{x, X_i, X_\alpha\} \in O(M^m; E^n)$ implies

$$X_i \in T_x M^m, \quad X_\alpha \in T^{\perp}_{x} M^m; \quad i = 1, \ldots, m; \quad \alpha = m+1, \ldots, n.$$

Identifying the point $x \in M^m$ with its radius vector, we have

$$dx = X_I \omega^I, \quad dX_J = X_K \omega^K_J, \quad \omega^K_I + \omega^I_K = 0, \tag{1.1}$$

where $I, J, K = 1, \ldots, n$, and the following differential system is satisfied:

$$\omega^\alpha = 0.$$

Therefore from structure equations (i. e. integrability conditions of (1.1))

$$d\omega^I = \omega^K \wedge \omega^I_K, \quad d\omega^K_I = \omega^J_I \wedge \omega^K_J$$

it follows, due to the Cartan lemma, that

$$\omega^\alpha_i = h^\alpha_{ij} \omega^j, \quad h^\alpha_{ij} = h^\alpha_{ji}.$$

Denoting $h_{ik} = h^\alpha_{ik} X_\alpha$, we get from (1.1)

$$dX_i = X_j \omega^j_i + h_{ik} \omega^k. \tag{1.2}$$

Here the vectors $h_{ik} = h^{(1)}_{ik}$ form at $x \in M^m$ the first principal normal vector space $N^{(1)}_{x} M^m$. Now, applying exterior differentiation and the same lemma for (1.2) we obtain

$$dh_{ik} - h_{jk}\omega^j_i - h_{ij}\omega^j_k + X_j \sum_{k<l} (h_{ik}h_{jl} - h_{il}h_{jk}) \omega^l =$$

$$= h_{ikl}\omega^l \equiv (h^{(0)}_{ikl} + h^{(1)}_{ikl} + h^{(2)}_{ikl}) \omega^l,$$

where h_{ikl} is decomposed so that the summands are mutually orthogonal and $h^{(0)}_{ikl} \in T_x M^m$, $h^{(1)}_{ikl} \in N^{(1)}_{x} M^m$. The vector space $N^{(2)}_{x} M^m$ spanned by the vectors $h^{(2)}_{ikl}$ is the second principal normal vector space at $x \in M^m$.

The last formula may be expressed differently in the following way —

$$dh_{ik}=h_{jk}\omega^j_i+h_{ij}\omega^j_k+(\bar{h}^{(0)}_{ikl}+h^{(1)}_{ikl}+h^{(2)}_{ikl})\,\omega^l, \tag{1.3}$$

where by $\bar{h}^{(0)}_{ikl}$ is denoted $h^{(0)}_{ikl}-X_j\sum\limits_{k<l}(h_{ik}h_{jl}-h_{il}h_{jk})$.

Analogous differential prolongation procedure proceeds to the following result —

$$d(\bar{h}^{(0)}_{ikl}+h^{(1)}_{ikl}+h^{(2)}_{ikl})=h_{jkl}\omega^j_i+h_{ijl}\omega^j_k+h_{ikj}\omega^j_l+$$
$$+(h^{(0)}_{ijkl}+\bar{h}^{(1)}_{ijkl}+h^{(2)}_{ijkl}+h^{(3)}_{ijkl})\,\omega^l, \tag{1.4}$$

where $h^{(0)}_{ijkl}\in T_xM^m$, $\bar{h}^{(1)}_{ijkl}\in N^{(1)}_xM^m$, $h^{(2)}_{ijkl}\in N^{(2)}_xM^m$ and the vectors $h^{(3)}_{ijkl}$, orthogonal to the preceding vectors, form the third principal normal-vector space $N^{(3)}_xM^m$ at $x\in M^m$.

In general we may write

$$d(\overset{(0)}{h}_{i_1\cdots i_\lambda}+\ldots+\overset{(\lambda-1)}{h}_{i_1\cdots i_\lambda})=\sum_{\mu=1}^{\lambda}h_{i_1\cdots i_{\mu-1}\,l\,i_{\mu+1}\cdots i_\lambda}\,\omega^l_{i_\mu}+$$
$$+(\overset{(0)}{h}_{i_1\cdots i_{\lambda+1}}+\ldots+\overset{(\lambda-2)}{\bar{h}}_{i_1\cdots i_{\lambda+1}}+\overset{(\lambda-1)}{h}_{i_1\cdots i_{\lambda+1}}+\overset{(\lambda)}{h}_{i_1\cdots i_{\lambda+1}})\,\omega^{i_{\lambda+1}}, \tag{1.5}$$

it can be verified by induction. Here the vectors $\overset{(\lambda)}{h}_{i_1\cdots i_{\lambda+1}}$ span the principal normal vector space $N^{(\lambda)}_xM^m$ of the order λ. The osculating vector space $O^{(\lambda)}_xM^m$ of the order λ can be presented in the following way —

$$O^{(\lambda)}_xM^m=T_xM^m\oplus N^{(1)}_xM^m\oplus\ldots\oplus N^{(\lambda)}_xM^m.$$

In (1.3)—(1.5) the upper index in round brackets denotes which order of the principal normal space the vectors belong to.

This differential prolongation procedure terminates if for some value $\lambda=p+1$ we get $h^{(p+1)}_{i_1\cdots i_{p+2}}=0$. Such a p exists because E^n has a finite dimension $n=m+m_1+\ldots+m_p$, where $\dim N^{(\lambda)}_xM^m$ is denoted by m_λ $(\lambda=1,\ldots,p)$. This quantity p is called the order of curvature of the submanifold M^m in E^n. In general, when all vectors h_{ik}, $h^{(2)}_{ikl}$, $\ldots$, $h^{(p)}_{i_1\cdots i_{p+1}}$ are linearly independent, every principal normal space has the possible maximum dimension $m_\lambda=\dfrac{1}{(\lambda+1)!}m(m+1)\ldots(m+\lambda)$, (except, perhaps, the last one).

R e m a r k 1. The linear system of equations of the plane generatrix $S^{(\lambda)}_x$ of the polar submanifold $\mathcal{E}^{(\lambda)}$, deduced in [5], can be presented in the following way —

$$\delta^{i_2}_{i_1}-h_{i_1i_2}\cdot(y-x)=0,$$
$$h_{i_1i_2i_3}\cdot(y-x)=0,$$
$$\partial^{(3-2)}_{i_4}[h_{i_1i_2i_3}\cdot(y-x)]=0$$

$$\partial_{i_{\lambda+1}}[h_{i_1i_2i_3}\cdot(y-x)]=0,$$

where dot $\cdot$ means the scalar product and for all values of $\lambda>2$ we have denoted

$$\partial^{(\lambda-2)}_{i_{\lambda+1}}[h_{i_1 i_2 i_3}\cdot(y-x)] = h_{i_1\cdots i_{\lambda+1}}\cdot(y-x) - \binom{\lambda-2}{1} h_{i_1\cdots i_\lambda}\cdot X_{i_{\lambda+1}} - \binom{\lambda-2}{2} h_{i_1\cdots i_{\lambda-1}}\cdot h_{i_\lambda i_{\lambda+1}} - \ldots - \binom{\lambda-2}{\lambda-3} h_{i_1\cdots i_4}\cdot h_{i_5\cdots i_{\lambda+1}} - h_{i_1 i_2 i_3}\cdot h_{i_4\cdots i_{\lambda+1}}.$$

2. Normal fields, parallel in the normal connection. Let Z be a smooth section of the normal bundle $T^{\perp}M^m$. This section is called the normal vector field on the M^m, parallel in the normal connection of M^m if $\overline{\nabla}Z=0$ [6].

Analogously is defined the parallelism of a r-field $\mathbf{N}^rM^m$ of normal r-directions $N^r(x)$, $x\in M^m$, where $\mathbf{N}^rM^m$ is obtained as a section of the Grassmann bundle of all the possible normal vector spaces $N^r(x)$ at the points $x\in M^m$. A field $\mathbf{N}^rM^m$ is called parallel in the normal connection of M^m if for every vector field Z on M^m with $Z_x\in N^r(x)$ we have $(\overline{\nabla}Z)_x\in N^r(x)$ [6].

We can describe the parallelism of the r-field analytically as follows. Let us give the orthogonal decomposition $T^{\perp}_{x}M^m = N^r(x)\oplus \overset{\perp}{N}{}^{n-m-r}(x)$ and the corresponding frame $\{\zeta_1,\ldots,\zeta_r;\ \xi_{r+1},\ldots,\xi_{n-m}\}$ i. e.

$$\zeta_a\in N^r(x);\quad \xi_u\in \overset{\perp}{N}{}^{n-m-r}(x);\quad a=1,\ldots,r;\quad u=r+1,\ldots,n-m.$$

Then we have

$$\begin{aligned} d\zeta_a &= \zeta_b\theta^b_a + X_k\theta^k_a + \xi_v\theta^v_a,\\ d\xi_u &= \zeta_b\theta^b_u + X_k\theta^k_u + \xi_v\theta^v_u,\\ dX_j &= \zeta_b\theta^b_j + X_k\omega^k_j + \xi_v\theta^v_u. \end{aligned} \tag{2.1}$$

It is obvious that $(\overline{\nabla}Z)_x\in \mathrm{N}^r(x)$ for arbitrary $Z=Z^a\zeta_a$ iff

$$\theta^v_a=0. \tag{2.2}$$

It is not complicated to check up that both fields $\mathbf{N}^rM^m$ and $\overset{\perp}{\mathbf{N}}{}^{n-m-r}M^m$ are parallel or not parallel in the normal connection of M^m simultaneously.

A generalization of this concept to the higher order can be presented in the following way —

Let us give an orthogonal decomposition

$$N^r(x) = N^{(1)}_x\oplus\ldots\oplus N^{(\lambda)}_x\oplus N^{r_\lambda}(x) \tag{2.3}$$

at every point $x\in M^m$. So the supplement field $\mathbf{N}^{r_\lambda}=\{N^{r_\lambda}(x)\,|\,x\in M^m\}$ is determined by the field $\mathbf{N}^rM^m$.

Definition 2. *A field* $\mathbf{N}^{r_\lambda}M^m$ *given by* (2.3) *is called* **parallel of order** λ *in the normal connection of* M^m *if for every vector field* Z *on* M^m *with* $Z_x\in N^{r_\lambda}(x)$ *we have*

$$dZ_x\in N^{(\lambda)}_xM^m\oplus N^{r_\lambda}(x)$$

at every $x\in M^m$.

Obviously, by $\lambda=0$ we get the above-mentioned concept of the parallelism of the field $\mathbf{N}^rM^m$.

Next we shall describe this concept analytically. For this purpose we use the frame $\{\zeta_a\}=\{\zeta_{a_1},\ldots,\zeta_{a_\lambda},\zeta_{a_{\lambda+1}}\}$ adapted to (2.3), i. e.

$$\zeta_{a_1} \in \underset{x}{N^{(1)}} M^m, \; \ldots, \; \zeta_{a_\lambda} \in \underset{x}{N^{(\lambda)}} M^m, \; \zeta_{a_{\lambda+1}} \in N^{r_\lambda}(x).$$

From (2.1), we get

$$\begin{aligned}
d\zeta_{a_{\lambda+1}} &= \zeta_{b_{\lambda+1}} \theta^{b_{\lambda+1}}_{a_{\lambda+1}} + \zeta_{b_\lambda} \theta^{b_\lambda}_{a_{\lambda+1}} + \xi_v \theta_{a_{\lambda+1}}, \\
d\zeta_{a_\lambda} &= \zeta_{b_{\lambda+1}} \theta^{b_{\lambda+1}}_{a_\lambda} + \zeta_{b_\lambda} \theta^{b_\lambda}_{a_\lambda} + \zeta_{b_{\lambda-1}} \theta^{b_{\lambda-1}}_{a_\lambda}, \\
&\cdots\cdots\cdots \\
d\zeta_{a_1} &= \zeta_{b_2} \theta^{b_2}_{a_1} + \zeta_{b_1} \theta^{b_1}_{a_1} + X_k \theta^k_{a_1}, \\
dX_i &= \zeta_{b_1} \theta_i{}^1 + X_k \omega^k_i, \\
d\xi_u &= \zeta_{b_{\lambda+1}} \theta_u{}^{\lambda+1} + \xi_v \theta^v_u;
\end{aligned} \tag{2.4}$$

here is considered that $\theta^{a_\mu}_{a_\lambda} = 0$ if $\mu \geqslant \lambda + 2$.

Now, the next lemma is valid:

Lemma 1. *A normal field $N^{r_\lambda} M^m$ given in* (2.3) *on M^m in E^n, is parallel of order λ in the normal connection of M^m iff*

$$\theta^v_{a_{\lambda+1}} = 0. \tag{2.5}$$

This result we obtain directly from the expression of the differential of the arbitrary $Z = Z^{a_{\lambda+1}} \zeta_{a_{\lambda+1}}$:

$$dZ = Z^{a_{\lambda+1}} \theta^v_{a_{\lambda+1}} \xi_v \; (\operatorname{mod} N^{r_\lambda}(x)).$$

3. Osculatingly degenerated submanifolds. Let M^{m+r} be a submanifold compiled from normal to M^m r-planes $[x, N^r(x)]$ in all points $x \in M^m$. This M^{m+r} is called tangentially degenerated of rank m if its tangent plane $[y, T_y M^{m+r}]$ is the same for all points y of a fixed generatrix $[x, N^r(x)]$ [7].

Identifying the point $y \in M^{m+r}$ with its radius vector we have (2.1) and

$$dy = \zeta_a \theta^a + X_k \theta^k + \xi_u \theta^u, \tag{3.1}$$

where now

$$\theta^u = 0.$$

So as in § 1 we get

$$\begin{aligned}
\theta^u_a &= h^u_{ab} \theta^b + h^u_{ak} \theta^k, \\
\theta^u_k &= h^u_{ak} \theta^a + h^u_{kl} \theta^l,
\end{aligned} \tag{3.2}$$

where h^u_{ab}, h^u_{kl} are symmetric with respect to lower indices. Now it follows that M^{m+r} is tangentially degenerated iff

$$h^u_{ab} = h^u_{al} = 0. \tag{3.3}$$

If we compare (2.2) with (3.2) and (3.3), we get the following result (see [8]).

A submanifold $M^{m+r} = \{y \mid y \in [x, N^r(x)], x \in M^m\}$ is tangentially degenerated iff the field $\mathbf{N}^r M^m = \{N^r(x) \mid x \in M^m\}$ is parallel in the normal connection of M^m.

The concept of tangentially degenerated submanifold can be generalizated in the following way —

Definition 3. *Let $M^{m+\rho}=\{y \mid y\in R^{\rho}(x),\ x\in M^m\}$ be a submanifold, where $R^{\rho}(x)$ is a ρ-dimensional plane in $[x, T^{\perp}_{x}M^m]$, perhaps not going through the point $x\in M^m$. We say that such $M^{m+\rho}$ is* **osculatingly degenerated of rank *m* and order $\lambda\geqslant 0$** *if*

1) *its osculating plane $[y, O^{(\lambda)}_{y}M^{m+\rho}]$ of order λ is the same for all points $y\in R^{\rho}(x)$ and for arbitrary fixed $x\in M^m$;*

2) *by $\lambda>0$ the osculating plane $[y, O^{(\lambda-1)}_{y}M^{m+\rho}]$ of order $\lambda-1$ lies in the normal plane $[x, T^{\perp}_{x}M^m]$ at the point x, where $y\in R^{\rho}(x)$.*

Note that if $\lambda=0$, we have the concept of tangentially degenerated submanifold M^{m+r}.

In the following we need an analytical criterion for the property of $M^{m+\rho}$ to be osculatingly degenerated for the special case when the linear space of $R^{\rho}(x)$ is the space $N^{r_\lambda}(x)$ given by (2.3). Then $\rho=r_\lambda$, $N^{r_\lambda}(x)=\operatorname{span}\{\zeta_{a_{\lambda+1}}\}$, $\{\zeta_{a_{\lambda+1}}, \zeta_{a_\lambda}\}\in T_yM^{m+r_\lambda}$; $\{\zeta_{a_{\lambda+1}}, \zeta_{a_\lambda}, \zeta_{a_{\lambda-1}}\}\in$ $\in O^{(1)}_{y}M^{m+r_\lambda}$; $\ldots$; $\{\zeta_{a_{\lambda+1}}, \ldots, \zeta_{a_1}, X_k\}\in O^{(\lambda)}_{y}M^{m+r_\lambda}$. From (3.1) it follows that

$$dy=\zeta_{a_{\lambda+1}}\theta^{a_{\lambda+1}}+\zeta_{a_\lambda}\theta^{a_\lambda}+\ldots+\zeta_{a_1}\theta^{a_1}+X_k\theta^k+\xi_u\theta^u,$$

where $\theta^u=\theta^k=\theta^{a_1}=\ldots=\theta^{a_{\lambda-1}}=0$. Now applying exterior differentiation and structure equations we obtain

$$\begin{aligned}
\theta^u_{a_{\lambda+1}}&=h^u_{a_{\lambda+1}b_{\lambda+1}}\theta^{b_{\lambda+1}},\\
\theta^u_k&=h^u_{kl}\theta^l, \qquad\qquad (3.4)\\
\theta^k_{a_1}&=h^k_{a_1b_1}\theta^{b_1},\ \theta^{c_1}_{a_2}=h^{c_1}_{a_2b_2}\theta^{b_2},\ \ldots,\ \theta^{\lambda-1}_{a_\lambda}=h^{c_{\lambda-1}}_{a_\lambda b_\lambda}\theta^{b_\lambda},
\end{aligned}$$

where all coefficients on the right sides are symmetric with respect to lower indices.

Now by the displacement of a point $y\in M^{m+r_\lambda}$ along a fixed r_λ-plane $R^{r_\lambda}(x)$, i. e. by $dy=\zeta_{a_{\lambda+1}}\theta^{a_{\lambda+1}}(\operatorname{mod}\theta^{a_\lambda})$, it follows from (2.4), due to the (3.4), that M^{m+r_λ} is osculatingly degenerated submanifold of order λ iff

$$h^u_{a_{\lambda+1}b_{\lambda+1}}=0. \qquad (3.5)$$

We get the following:

Proposition 1. *The submanifold $M^{m+r_\lambda}=\{y \mid y\in R^{r_\lambda}(x),\ x\in M^m\}$, where the linear space of $R^{r_\lambda}(x)$ is $N^{r_\lambda}(x)$ given by* (2.3) *for a submanifold M^m in E^n, is osculatingly degenerated of rank m and order λ iff the field $\mathbf{N}^{r_\lambda}M^m=\{N^{r_\lambda}(x) \mid x\in M^m\}$ is parallel of order λ in the normal connection of M^m.*

In fact, (3.5) is equivalent to (2.5).

4. Envelopes of higher order. Let $\mathcal{E}^{m+r}$ be a total space of the canonical bundle $\mathcal{E}(M^m, \mathbf{N}^rM^m)$, where $\mathbf{N}^rM^m$ is a subbundle of $T^{\perp}M^m$. Then $\mathcal{E}^{m+r}$ consists of pairs (x, y), where $x\in M^m$ and $y\in[x, N^r(x)]$, and it is a $(m+r)$-dimensional smooth manifold. We have the bundle projection

$$p : \mathcal{E}^{m+r} \longrightarrow M^m, \quad (x, y) \longmapsto x$$

and a smooth map

$$\pi : \mathcal{E}^{m+r} \longrightarrow E^n, \quad (x, y) \longmapsto y.$$

An image $\pi(\mathcal{E}^{m+r})$ consists of the points of normal to M^m r-planes which generate the m-parametrical family $\{[x, N^r(x)] \mid x \in M^m\}$. A set of the singular points of the map π (i. e. of the focal points; see [9]) is called in [10] the criminant, and the image of this criminant is called the discriminant.

The discriminant of π is called an envelope of the family $\{[x, N^r(x)] \mid x \in M^m\}$ (see [10]) if

1) the image of the criminant of π by p is an open set in M^m;

2) for every point (x, y) of the criminant there exists a neighbourhood $U_{(x,y)}$ in the criminant, so that the plane $[x, N^r(x)]$ in E^n is tangent plane to $\pi(U_{x,y})$ at the point $\pi(x, y) = y$.

The next definition gives a generalization of this concept.

Definition 4. *A submanifold V in E^n is called the* **envelope of order** λ *of the m-parametrical family* $\{[x, N^r(x)] \mid x \in M^m\}$ *of r-planes, normal to M^m if for arbitrary point $y \in V$ there exists a plane $[x, N^r(x)]$ which is an osculating plane of order $\lambda - 1$ for the V at this point $y \in V$.*

By $\lambda = 1$ we get the envelope in the sense of [10].

From definitions 1 and 4 we get immediately the following:

Remark 2. Every submanifold V in E^n appears to be an envelope of order λ of the family of its osculating planes of order $\lambda - 1$. If these planes are normal to a given M^m, then V is an evolute of order λ for this M^m, i. e. every evolute of order λ appears to be an envelope of the same order of the family of its tangent planes.

5. Generally located normal planes. For a line M^1 in E^3 it is known that not by any mutual position of the normal straight line $[x, Z]$ and polar straight line $\underset{x}{S^{(1)}} M^1$ in normal plane $[x, \underset{x}{T^\perp} M^1]$ there exists an envelope of the family $\{[x, Z] \mid x \in M^1\}$. Analogously for a M^m in E^n it is necessary to investigate what kind of mutual positions of r-plane $[x, N^r(x)]$ and plane generatrices $\underset{x}{S^{(1)}} M^m, \ldots, \underset{x}{S^{(\lambda)}} M^m$ of polar submanifolds in $[x, \underset{x}{T^\perp} M^m]$ are admitted for the existence of the evolutes of higher order.

Definition 5. *An r-plane $[x, N^r(x)]$ normal to a submanifold M^m in E^n, is said to be* **generally located of order** λ_0 *in $[x, \underset{x}{T^\perp} M^m]$ if it has with every one of plane generatrices $\underset{x}{S^{(1)}}, \ldots, \underset{x}{S^{(\lambda_0)}}$ a non-empty intersection of minimal possible dimension and empty intersections with all other planes $\underset{x}{S^{(\lambda_0+1)}}, \ldots, \underset{x}{S^{(p-1)}}$, where p is the order of curvature of M^m* in E^n.

The next proposition gives the necessary and sufficient conditions for finding out whether the plane $[x, N^r(x)]$ is generally located in $[x, \underset{x}{T^\perp} M^m]$ or not.

Proposition 2. *Let all the principal normal spaces of higher order of a submanifold M^m in E^n have maximal possible dimensions at a point $x \in M^m$. An r-plane $[x, N^r(x)]$ normal to M^m, is generally located of order λ_0 in $[x, \underset{x}{T^\perp} M^m]$ iff*

$$m_1 + \ldots + m_{\lambda_0} \leqslant r < m_1 + \ldots + m_{\lambda_0+1}$$

and

$$\overset{\perp}{N}{}^{n-m-r}(x) \cap \{\underset{x}{N^{(1)}} \oplus \ldots \oplus \underset{x}{N^{(\lambda_0)}}\} = \{0\},$$

where $m_\lambda = \dim \underset{x}{N^{(\lambda)}} M^m$, (see § 1).

P r o o f. Let $\lambda_0=1$. Due to (1.6), the linear system which determines the intersection $S^{(1)}_x \cap [x, N^r(x)]$ is

$$\delta^{i_2}_{i_1} - h_{i_1 i_2} \cdot (y-x) = 0,$$
$$\xi_v \cdot (y-x) = 0, \tag{5.1}$$

where we have used the frame $\{\zeta_a, \xi_u\}$ introduced in § 2. If the vectors $h_{i_1 i_2}$ are linearly independent and $[x, N^r(x)]$ is generally located of order 1, then the system (5.1) consists of $\frac{1}{2}m(m+1)+(n-m-r)$ linearly independent equations with $n-m$ unknowns $y^{m+1}, \ldots, y^n$. Thus the $(n-m)$-dimensional $T^{\perp}_x M^m$ contains $\frac{1}{2}m(m+1)+(n-m-r)$ linearly independent vectors $h_{i_1 i_2}$ and ξ_v, where the first ones span $N^{(1)}_x M^m$ and the second ones span $\overset{\perp}{N}{}^{n-m-r}(x)$. Therefore $\overset{\perp}{N}{}^{n-m-r}(x) \cap N^{(1)}_x M^m = \{0\}$ and $\frac{1}{2}m(m+1)+(n-m-r) \leqslant n-m$, which implies $m_1 = \frac{1}{2}m(m+1) \leqslant r$.

Now, the condition $[x, N^r(x)] \cap S^{(2)}_x = \varnothing$ yields $r < m_1 + m_2$.

Conversely, if $m_1 \leqslant r < m_1 + m_2$ and $\overset{\perp}{N}{}^{n-m-r}(x) \cap N^{(1)}_x M^m = \{0\}$, then the plane $[x, N^r(x)]$ has with $S^{(1)}_x$ a non-empty intersection of minimal possible dimension and $[x, N^r(x)] \cap S^{(2)}_x = \varnothing$, i. e. $[x, N^r(x)]$ is generally located of order 1.

Obviously, this scheme can be used also to prove the proposition in the general case $\lambda_0 > 1$.

6. Evolutes of higher order. The next Proposition shows us the way for finding an envelope of the family of r-planes normal to M^m.

P r o p o s i t i o n 3. *Let all principal normal spaces of higher order of a submanifold M^m in E^n have maximal possible dimensions. Let $\{[x, N^r(x)] \mid x \in M^m\}$ be the family of r-planes $[x, N^r(x)]$ normal to M^m and generally located of order λ_0. This family $\{[x, N^r(x)] \mid x \in M^m\}$ has an envelope*

$$\{y \mid y \in [x, N^r(x)] \cap S^{(\mu+1)}_x,\ x \in M^m\}$$

of order $\mu+1$ $(0 \leqslant \mu \leqslant \lambda_0)$ iff the field $\mathbf{N}^{\mu} \neq \{N^{r_\mu}(x) \mid x \in M^m\}$ (where $N^{r_\mu}(x)$ is the vector space of the r_μ-plane $S^{(\mu)}_x \cap [x, N^r(x)]$) is parallel of order μ in the normal connection of M^m.

P r o o f. Let $\mu=0$. A point y lies on the r-plane $[x, N^r(x)]$ iff

$$X_k \cdot (y-x) = 0,$$
$$\xi_u \cdot (y-x) = 0, \tag{6.1}$$

and is the point of the envelope $W^{(1)}$, the tangent (i. e. osculating of order 0) plane of which coincides with $[x, N^r(x)]$ iff by arbitrary $dx \in T_x M^m$ we have

$$dy \cdot X_k = 0, \quad dy \cdot \xi_u = 0. \tag{6.2}$$

By exterior differentiation we get from (6.1), due to (1.2), (2.1) and (6.2),

$$[\delta^{k_2}_{k_1} - h_{k_1 k_2} \cdot (y-x)]\omega^{k_2} = 0,$$

$$\theta^a_u \zeta_a \cdot (y-x) = 0,$$

where the last group of conditions is satisfied iff $\theta^a_u = 0$ (see (2.2)). Consequently, we obtain that $S^{(1)}_x \cap [x, N^r(x)]$ is a plane generatrix of $W^{(1)}$ iff the field $\mathbf{N}^r M^m$ of vector spaces $N^r(x)$ is parallel in the normal connection of M^m.

In general, when $0 < \mu < \lambda_0$, the proof is analogous. Let us suppose that the linear system

$$\begin{aligned}
&\delta^{i_2}_{i_1} - h_{i_1 i_2} \cdot (y-x) = 0, \\
&h_{i_1 i_2 i_3} \cdot (y-x) = 0, \\
&\partial^{(3-2)}_{i_4}[h_{i_1 i_2 i_3} \cdot (y-x)] = 0, \\
&\dots\dots\dots\dots \\
&\partial^{(\mu-2)}_{i_{\mu+1}}[h_{i_1 i_2 i_3} \cdot (y-x)] = 0, \\
&\xi_u \cdot (y-x) = 0
\end{aligned} \tag{6.3}$$

(see (1.6) by $\lambda = \mu$) determines the r_μ-plane $S^{(\mu)}_x \cap [x, N^r(x)]$ as the generatrix of the envelope $W^{(\mu)}$ of order μ of the family $\{[x, N^r(x)] \mid x \in \in M^m\}$. A point $y \in S^{(\mu)}_x \cap [x, N^r(x)]$ is the point of the envelope $W^{(\mu+1)}$ of this family iff, by arbitrary $dx \in T_x M^m$, we have

$$dy \cdot X_k = 0, \quad dy \cdot h_{i_1 i_2} = 0, \quad \dots, \quad dy \cdot h_{i_1 \cdots i_{\mu+1}} = 0, \quad dy \cdot \xi_u = 0. \tag{6.4}$$

By exterior differentiation, we conclude from (6.3) by means of (1.5), (2.4) and (6.4) that

$$\begin{aligned}
&\delta^{i_2}_{i_1} - h_{i_1 i_2} \cdot (y-x) = 0, \\
&h_{i_1 i_2 i_3} \cdot (y-x) = 0, \\
&\partial^{(3-2)}_{i_4}[h_{i_1 i_2 i_3} \cdot (y-x)] = 0, \\
&\dots\dots\dots\dots \\
&\partial^{(\mu+1-2)}_{i_{\mu+2}}[h_{i_1 i_2 i_3} \cdot (y-x)] = 0, \\
&\theta^{a_{\mu+1}}_u \zeta_{a_{\mu+1}} \cdot (y-x) = 0,
\end{aligned}$$

where the last group of conditions is satisfied iff (2.5) is valid. Thus $S^{(\mu+1)}_x \cap [x, N^r(x)]$ is a plane generatrix of $W^{(\mu+1)}$ iff the field $\mathbf{N}^{r^\mu} M^m = \{N^{r^\mu}(x) \mid x \in M^m\}$ is parallel of order μ in the normal connection of M^m. The proposition is proved.

For any M^m in E^n the next lemma is valid converse to Remark 2:

L e m m a 2. *Let us have a submanifold M^m in E^n and the family $\{[x, N^r(x)] \mid x \in M^m\}$, where r-planes $[x, N^r(x)]$ normal to M^m are generally located of order $\lambda_0 \geqslant \mu$ in $[x, T^\perp_x M^m]$. Then every envelope of order μ of this family is an evolute of order μ for M^m.*

P r o o f. Let $W^{(\mu)} = \{y \mid y \in S^{(\mu)}_x \cap [x, N^r(x)],\ x \in M^m\}$ be an envelope of order μ and let the vector space of the generatrix $S^{(\mu)}_x \cap [x, N^r(x)]$ be determined by the vectors $\zeta_{a_{\mu+1}}$. Then (2.4) yields that every osculating plane $[y, O^{(\mu-1)}_y, W^{(\mu)}]$, $y \in W^{(\mu)}$ is included in some $[x, T^\perp_x M^m]$. The Lemma is proved.

The assertions of Lemma 2, Proposition 1, Proposition 3 and Remark 2 can be summarized as follows.

Theorem. *Let all principal normal spaces of higher order of a submanifold M^m in E^n have the maximal possible dimensions. Suppose that on the submanifold M^m there is given a family $\{[x, N^r(x)] \mid x \in M^m\}$ $(m \leqslant r < n - m)$ of r-planes normal to M^m and generally located of order $\lambda_0 \leqslant p - 1$, in $[x, T_x^{\perp} M^m]$ (here p is the order of curvature of M^m). Then for every value of $\mu = 0, \ldots, \lambda_0 - 1$ the following assertions are equivalent:*

1) *for M^m there exists a r_μ-dimensional evolute $W^{(\mu+1)} = \{y \mid y \in [x, N^r(x)] \cap S_x^{(\mu+1)},\ x \in M^m\}$ of order $\mu + 1$;*

2) *the normal field $\mathbf{N}^{r_\mu}$ of the directions of the r_μ-planes $[x, N^r(x)] \cap S_x^{(\mu)}$ is parallel of the order μ in the normal connection of M^m;*

3) *the submanifold $\{y \mid y \in [x, N^r(x)] \cap S_x^{(\mu)},\ x \in M^m\}$ is osculatingly degenerated with rank m and order μ;*

4) *the family $\{[x, N^r(x)] \mid x \in M^m\}$ has a r_μ-dimensional envelope $\{y \mid y \in [x, N^r(x)] \cap S_x^{(\mu+1)},\ x \in M^m\}$ of the order $\mu + 1$.*

In this Theorem the suppositions 2), 3) and 4) can be considered as three mutually equivalent criteria for the existence of the evolute of higher order.

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Tõnis VIROVERE

ALAMMUUTKONNA M^m KÕRGEMAT JÄRKU EVOLUUDID RUUMIS E^n

Artiklis on üldistatud eukleidilise ruumi E^n alammuutkonna M^m kõrgema järgu tarvis mõisted: evoluut, normaalrihivälja paralleelsus M^m normaalseostuses, normaaltasandite parve kui alammuutkonna tangentsiaalne kidumine ja normaaltasandite parve mähispind. Nende mõistete abil on vaadeldava üldise M^m puhul tõestatud kõrgemat järku evoluudi olemasolu kolm võrdväärset kriteeriumi.

Тынис ВИРОВЕРЕ

ЭВОЛЮТЫ ВЫСШЕГО ПОРЯДКА ПОДМНОГООБРАЗИЯ M^m В E^n

В евклидовом пространстве E^n представляются для подмногообразия M^m высшего порядка обобщения следующих понятий: эволюта, параллельность нормального поля относительно нормальной связности M^m, тангенциальная вырожденность семейства нормальных плоскостей как подмногообразия и огибающая семейства нормальных плоскостей. С помощью этих понятий доказаны для рассматриваемого общего M^m в E^n три эквивалентных наличию его эволюты высшего порядка критерия.