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SOME REMARKS ON THE QUASI-LINEAR THEORY OF VISCOELASTICITY

(Presented by J. Engelbrecht)

Following the ideas of the principal quasi-linear theory of viscoelasticity [1, 2], the modified constitutive equation of the continuous non-linear viscoelastic medium is derived. The main feature of this equation is the similar structure of its instantaneous and regular parts which allows for an easier explaination of the physical meaning of numerous kernel functions describing viscous properties of the medium. In the case when viscous properties can be neglected, the equation coincides exactly with the constitutive equation of the non-linear elastic medium [3].

Non-linear constitutive equations of materials with memory have been derived by many authors [1, 2, 4–11, ...]. One can find the historical survey of the problem in [5–7], for example. What is typical is the fact that all these non-linear constitutive equations relate stress and strain by means of a large amount of various kernel functions. The essential inconvenience in the practical utilization of these equations is the difficulty in interpreting the physical meaning of kernel functions.

The main aim of the present paper is to show how to extend the idea formulated in [5, 7] for linear constitutive equations to the non-linear case and whether it might be possible to regard kernel functions as time-dependent Lamé coefficients for the stress-relaxation response to a sudden deformation at time $t=0$.

We consider initially isotropic and homogeneous medium with memory and assume that the applied deformations are small and finite. For simplicity in representation, we exploit the rectangular Lagrangian (X_I, $I=1, 2, 3$) and Eulerian (x_i, $i=1, 2, 3$) coordinates. The Lagrangian strain tensor E_{KL} is related to the displacement vector U_K by [3]

$$2E_{KL}=U_{K,L}+U_{L,K}+U_{M,L}U_{M,K}, \tag{1}$$

where a comma represent partial differentiation. All strain and stress tensors and displacement vectors are dependent on spacial coordinates X_i, and on time t.

We assume that the Kirchhoff pseudostress tensor T_{IJ} is the continuous and continuously differentiable functional in point X_I and in its neighbourhood, and it can be expanded in a Fréchet series. This expansion on the segment $(0, t)$ can be represented in the form [11]

$$\begin{aligned} T_{IJ}(t)= & \int_0^t G^{(1)}_{IJKL}(t,\tau)E_{KL}(\tau)\,d\tau+ \\ & +\int_0^t\int_0^t G^{(2)}_{IJKLMN}(t,\tau,\eta)E_{KL}(\tau)E_{MN}(\eta)\,d\tau\, d\eta+\dots \\ & \dots+\int_0^t\dots\int_0^t G^{(n)}_{IJKL\cdots PR}(t,\tau,\dots,\vartheta)E_{KL}(\tau)\dots E_{PR}(\vartheta)\,d\tau\dots d\vartheta, \end{aligned} \tag{2}$$

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where $\tau, \eta, \ldots, \vartheta$ are different time axes and $G^{(1)}_{IJKL}$, $G^{(2)}_{IJKLMN}$, ... are relaxation tensors.

Since the terms in expression (2) are linear in each of tensors E_{IJ}, we obtained a quasi-linear constitutive equation, where the assumption $E_{KL}(t)=0$ if $t<0$ is taken into account.

Considering small finite deformations, we can truncate the series (2) at some terms, and construct constitutive equations of the medium with different accuracy. In this paper we stick to the two first terms in (2).

Relaxation tensors $G^{(n)}_{IJKL\cdots PR}$ are proposed to be continuous and symmetric functions with respect to $\tau, \eta, \ldots, \vartheta$, and the expression (2) is proposed to be invariant with respect to the origin of the time coordinate. The latter leads to the following equality [2]

$$G^{(n)}_{IJKL\cdots PR}(t, \tau, \eta, \ldots, \vartheta) = G^{(n)}_{IJKL\cdots PR}(t-\tau, t-\eta, \ldots, t-\vartheta). \tag{3}$$

In the case of isotropic medium, relaxation tensors may be expressed in terms of scalar relaxation functions G_{VW} ($V=1,2$; $W=1, 2, 3, 4$) and the Kronecker delta δ_{IJ} [1,2]

$$G^{(1)}_{IJKL}(t,\tau) = G_{11}(t,\tau)\,\delta_{IJ}\delta_{KL} + G_{12}(t,\tau)\,(\delta_{IK}\delta_{JL}+\delta_{IL}\delta_{JK}), \tag{4}$$

$$\begin{aligned} G^{(2)}_{IJKLMN}(t,\tau,\eta) &= G_{21}(t,\tau,\eta)\,\delta_{IJ}\delta_{KL}\delta_{MN} + \\ &+ G_{22}(t,\tau,\eta)\,(\delta_{IJ}\delta_{KM}\delta_{LN}+\delta_{IJ}\delta_{KN}\delta_{LM}) + \\ &+ G_{23}(t,\tau,\eta)\,(\delta_{IK}\delta_{JL}\delta_{MN}+\delta_{IL}\delta_{JK}\delta_{MN}+\delta_{IM}\delta_{JN}\delta_{KL}+\delta_{IN}\delta_{JM}\delta_{KL}) + \\ &+ G_{24}(t,\tau,\eta)\,(\delta_{IK}\delta_{JK}\delta_{LN}+\delta_{IK}\delta_{IN}\delta_{LM}+\delta_{IL}\delta_{JM}\delta_{KN} + \\ &+ \delta_{IL}\delta_{JN}\delta_{KM}+\delta_{IM}\delta_{JK}\delta_{LN}+\delta_{IM}\delta_{JL}\delta_{KN}+\delta_{IN}\delta_{JK}\delta_{LM}+\delta_{IN}\delta_{JL}\delta_{KM}). \end{aligned} \tag{5}$$

Introducing (4) and (5) into the two first terms of series (2), the expression for stress tensor yields

$$T_{IJ}(t) = T^{(1)}_{IJ}(t) + T^{(2)}_{IJ}(t) + \ldots \tag{6}$$

$$T^{(1)}_{IJ}(t) = \delta_{IJ}\int_0^t G_{11}(t,\tau)\,E_{KK}(\tau)\,d\tau + 2\int_0^t G_{12}(t,\tau)\,E_{IJ}(\tau)\,d\tau,$$

$$\begin{aligned} T^{(2)}_{IJ}(t) &= \delta_{IJ}\int_0^t\!\!\int_0^t G_{21}(t,\tau,\eta)\,E_{KK}(\tau)\,E_{LL}(\eta)\,d\tau\,d\eta + \\ &+ 2\delta_{IJ}\int_0^t\!\!\int_0^t G_{22}(t,\tau,\eta)\,E_{KL}(\tau)\,E_{KL}(\eta)\,d\tau\,d\eta + \\ &+ 4\int_0^t\!\!\int_0^t G_{23}(t,\tau,\eta)\,E_{KK}(\tau)\,E_{IJ}(\eta)\,d\tau\,d\eta + \\ &+ 8\int_0^t\!\!\int_0^t G_{24}(t,\tau,\eta)\,E_{IK}(\tau)\,E_{KJ}(\eta)\,d\tau\,d\eta. \end{aligned}$$

Further we try to construct the constitutive equation of the viscoelastic medium so that an essential part of the deformation of the medium occurs instantaneously after the force has acted upon it, which is followed by the long-term time-dependent behaviour of the medium. It leads to the separation of singular and regular parts of the relaxation functions [2]

$$G_{1V}(t,\tau) = \delta(t-\tau)\,G^{1}_{1V}(t-\tau) - G^{2}_{1V}(t-\tau), \tag{7}$$

$$\begin{aligned} D_{2W}(t,\tau,\eta) &= -\delta(\tau-\eta)\,[G^{1}_{2W}(t-\tau)+G^{1}_{2W}(t-\tau)] - \\ &- \delta(t-\tau)\,G^{2}_{2W}(t-\eta) - \delta(t-\eta)\,G^{2}_{2W}(t-\tau) + \\ &+ G^{3}_{2W}(t-\tau, t-\eta)\,[\delta(t-\tau)\,\delta(t-\eta)+\delta(t-\eta)\,\delta(\eta-\tau)] + \\ &+ G^{4}_{2W}(t-\tau, t-\eta)\,\delta(t-\tau)\,\delta(t-\eta) - G^{5}_{2W}(t-\tau, t-\eta). \end{aligned} \tag{8}$$

Here the relaxation function G_{2W} remains symmetric with respect to τ and η.

Introducing (7) and (8) into equation (6), we get an expression for the components of the constitutive equation

$$T^{(1)}_{IJ}(t)=\delta_{IJ}G^1_{11}(0)E_{KK}(t)+2G^1_{12}(0)E_{IJ}(t)-$$
$$-\int_0^t \delta_{IJ}G^2_{11}(t-\tau)E_{KK}(\tau)+2G^2_{12}(t-\tau)E_{IJ}(\tau)]\,d\tau, \qquad (9)$$

$$\begin{aligned}T^{(2)}_{IJ}(t)=&2\delta_{IJ}[2G^3_{21}(0,0)+G^4_{21}(0,0)]E_{KK}(t)E_{LL}(t)+\\
&+4\delta_{IJ}[2G^3_{22}(0,0)+G^4_{22}(0,0)]E_{KL}(t)E_{KL}(t)+\\
&+8[2G^3_{23}(0,0)+G^4_{23}(0,0)]E_{KK}(t)E_{IJ}(t)+\\
&+16[2G^3_{24}(0,0)+G^4_{24}(0,0)]E_{IK}(t)E_{KJ}(t)-\\
&-\int_0^t[4\delta_{IJ}G^1_{21}(t-\tau)E_{KK}(\tau)E_{LL}(\tau)+\\
&+8\delta_{IJ}G^1_{22}(t-\tau)E_{KL}(\tau)E_{KL}(\tau)+\\
&+16G^1_{23}(t-\tau)E_{KK}(\tau)E_{IJ}(\tau)+\\
&+32G^1_{24}(t-\tau)E_{IK}(\tau)E_{KJ}(\tau)]\,d\tau-\\
&-\int_0^t[2\delta_{IJ}G^2_{21}(t-\eta)E_{KK}(t)E_{LL}(\eta)+\\
&+4\delta_{IJ}G^2_{22}(t-\eta)E_{KL}(t)E_{KL}(\eta)+\\
&+8G^2_{23}(t-\eta)E_{KK}(t)E_{IJ}(\eta)+\\
&+16G^2_{24}(t-\eta)E_{IK}(t)E_{KJ}(\eta)]\,d\eta-\\
&-\int_0^t[2\delta_{IJ}G^2_{21}(t-\tau)E_{KK}(\tau)E_{LL}(t)+\\
&+4\delta_{IJ}G^2_{22}(t-\tau)E_{KL}(\tau)E_{KL}(t)+\\
&+8G^2_{23}(t-\tau)E_{KK}(\tau)E_{IJ}(t)+\\
&+16G^2_{24}(t-\tau)E_{IK}(\tau)E_{KJ}(t)]\,d\tau-\\
&-\int_0^t\int_0^t[\delta_{IJ}G^5_{21}(t-\tau,t-\eta)E_{KK}(\tau)E_{LL}(\eta)+\\
&+2\delta_{IJ}G^5_{22}(t-\tau,t-\eta)E_{KL}(\tau)E_{KL}(\eta)+\\
&+4G^5_{23}(t-\tau,t-\eta)E_{KK}(\tau)E_{IJ}(\eta)+\\
&+8G^5_{24}(t-\tau,t-\eta)E_{IK}(\tau)E_{KJ}(\eta)]\,d\tau\,d\eta.\end{aligned} \qquad (10)$$

Here the equality of integrals [2]

$$\int_0^t\int_0^t G_{IJKLMN}(t,\tau,\eta)E_{KL}(\tau)E_{MN}(\eta)\,d\tau\,d\eta=$$
$$=2\int_0^t E_{KL}(\tau)\,d\tau\int_0^\tau G_{IJKLMN}(t,\tau,\eta)E_{MN}(\eta)\,d\eta, \qquad (11)$$

is taken into account.

We utilize the principle of material invariance for isotropic medium, i. e. the equations must remain form-invariant with respect to the group of coordinate transformations. As the consequence of this principle, a symmetry with respect to the couple of indexes IJ, KL, MN in $G_{IJKLMN}(t, \tau, \eta)$ exists, and the following equation holds

$$G_{IJKLMN}(t, \tau, \eta) = G_{KLIJMN}(t, \tau, \eta). \tag{12}$$

It leads to the equality

$$G^{S}_{22}(t, \tau, \eta) = G^{S}_{23}(t, \tau, \eta), \quad S = 1, 2, \ldots 5, \tag{13}$$

that enables us to simplify expression (10), and thus also the whole constitutive equation.

Let us consider the instantaneous part of the simplified constitutive equation and use the notation

$$\begin{aligned} &\lambda(0) = G^{1}_{11}(0), \quad \mu(0) = G^{1}_{12}(0), \\ &3\nu_1(0) = 2[2G^{3}_{21}(0, 0) + G^{4}_{21}(0, 0)], \\ &\nu_2(0) = 4[2G^{3}_{22}(0, 0) + G^{4}_{22}(0, 0)], \\ &3\nu_3(0) = 16[2G^{3}_{24}(0, 0) + G^{4}_{24}(0, 0)]. \end{aligned} \tag{14}$$

Introducing expressions (1) and (14) into the instantaneous part of equations (9) and (10), we obtain the following constitutive equation

$$\begin{aligned} T_{KL}(t) = {} & \lambda(0)\,\delta_{KL}\left(U_{I,I} + \frac{1}{2}U_{M,N}U_{M,N}\right) + \mu(0)\,(U_{K,L} + U_{L,K} + U_{M,K}U_{M,L}) + \\ & + 3\nu_1(0)\,\delta_{KL}U_{I,I}U_{M,M} + \\ & + \nu_2(0)\left[\frac{1}{2}\delta_{KL}(U_{M,N}U_{M,N} + U_{M,N}U_{N,M}) + U_{I,I}U_{K,L} + U_{I,I}U_{L,K}\right] + \\ & + \frac{3}{4}\nu_3(0)\,(U_{M,K}U_{M,L} + U_{K,M}U_{M,L} + U_{K,M}U_{L,M} + U_{M,K}U_{L,M}). \end{aligned} \tag{15}$$

Equation (15) coincides exactly with the constitutive equation of the non-linear five-constant elastic medium [3]. Making use of new constants (14), we reduce their number from eight to five, and now each constant obtains physical meaning well known in the theory of non-linear elasticity. Here $\lambda(0)$ and $\mu(0)$ denote the Lamé constants and $\nu_1(0)$, $\nu_2(0)$ and $\nu_3(0)$ denote constants of elasticity of the third order.

It should be noticed that the structure of the instantaneous and regular parts of equation (10) is quite similar. It leads to the idea of using the modified relaxation kernel functions

$$\begin{aligned} &\lambda_1(t) = G^{2}_{11}(t), && \mu_1(t) = G^{2}_{12}(t), \\ &3\nu^{1}_{1}(t) = 4G^{1}_{21}(t), && 3\nu^{2}_{1}(t) = 2G^{2}_{21}(t), \\ &\nu^{1}_{2}(t) = 8G^{1}_{22}(t), && \nu^{2}_{2}(t) = 4G^{2}_{22}(t), \\ &3\nu^{1}_{3}(t) = 32G^{1}_{24}(t), && 3\nu^{2}_{3}(t) = 16G^{2}_{24}(t), \\ &3\nu^{3}_{1}(t) = G^{5}_{21}(t), && \\ &\nu^{3}_{2}(t) = 2G^{5}_{22}(t), && \\ &3\nu^{3}_{3}(t) = 8G^{5}_{24}(t). && \end{aligned} \tag{16}$$

Now the constitutive equation of the viscoelastic medium reads

$$
\begin{aligned}
T_{IJ}(t) = {} & \lambda(0)\delta_{IJ}E_{KK}(t)+2\mu(0)E_{IJ}(t)+\\
& +3\nu_1(0)\delta_{IJ}E_{KK}(t)E_{LL}(t)+\\
& +\nu_2(0)\left[\delta_{IJ}E_{KL}(t)E_{KL}(t)+2E_{KK}(t)E_{IJ}(t)\right]+\\
& +3\nu_3(0)E_{IK}(t)E_{KJ}(t)-\\
& -\int_0^t \{\lambda_1(t-\tau)\delta_{IJ}E_{KK}(\tau)+2\mu_1(t-\tau)E_{IJ}(\tau)+\\
& +3\nu_1^1(t-\tau)\delta_{IJ}E_{KK}(\tau)E_{LL}(\tau)+\\
& +\nu_2^1(t-\tau)\left[\delta_{IJ}E_{KL}(\tau)E_{KL}(\tau)+2E_{KK}(\tau)E_{IJ}(\tau)\right]+\\
& +3\nu_3^1(t-\tau)E_{IK}(\tau)E_{KJ}(\tau)\}\,d\tau-\\
& -\int_0^t \{3\nu_1^2(t-\eta)\delta_{IJ}E_{KK}(t)E_{LL}(\eta)+\\
& +\nu_2^2(t-\eta)\left[\delta_{IJ}E_{KL}(t)E_{KL}(\eta)+2E_{KK}(t)E_{IJ}(\eta)\right]+\\
& +3\nu_3^2(t-\eta)E_{IK}(t)E_{KJ}(\eta)\}\,d\eta-\\
& -\int_0^t \{3\nu_1^2(t-\tau)\delta_{IJ}E_{KK}(\tau)E_{LL}(t)+\\
& +\nu_2^2(t-\tau)\left[\delta_{IJ}E_{KL}(\tau)E_{KL}(t)+2E_{KK}(\tau)E_{IJ}(t)\right]+\\
& +3\nu_3^2(t-\tau)E_{IK}(\tau)E_{KJ}(t)\}\,d\tau-\\
& -\int_0^t\int_0^t 3\nu_1^3(t-\tau,\, t-\eta)\delta_{IJ}E_{KK}(\tau)E_{LL}(\eta)+\\
& +\nu_2^3(t-\tau,\, t-\eta)\left[\delta_{IJ}E_{KL}(\tau)E_{KL}(\eta)+2E_{KK}(\tau)E_{IJ}(\eta)\right]+\\
& +3\nu_3^3(t-\tau,\, t-\eta)E_{IK}(\tau)E_{KJ}(\eta)\}\,d\tau\, d\eta.
\end{aligned}
\tag{17}
$$

Relaxation kernel functions with the same lower index in (17) bind stress and strain in a similar way. The difference is only in their dependence on time. This permits us to transfer the physical meaning of the well-known elastic constants that characterize the instantaneous deformation of the medium, to the relaxation kernel functions with the same lower index.

Equation (17) is the modified representation of the constitutive equation of the quasi-linear theory of viscoelasticity proposed in [1,2].

Relaxation kernel functions in (17) are to be established for real media on the basis of experdimental data, and there may be cases when some of them are equal to zero. In the case when $\nu_1^3(t)=\nu_2^3(t)=\nu_3^3(t)=0$, or if the kernel function G_{2W}^5 in (8) can be neglected, the constitutive equation becomes essentially simpler. We get the constitutive equation of the principal quasi-linear theory of viscoelasticity [2]. Further simplication ($\nu_1^2(t)=\nu_2^2(t)=\nu_3^2(t)=0$) leads to the quasi-linear theory with five kernel functions.

It should be noted that in the linear case ($\nu_K(0)=0$, $\nu_K^L(t)=0$, $K, L=1, 2, 3$) equation (17) takes the form proposed by L. Boltzmann in [12] if relaxation kernel functions have the following representation:

$$\lambda_1(t)=-\Lambda_{,t}(t), \qquad \mu_1(t)=-\mathrm{M}_{,t}(t). \tag{18}$$

Provided that

$$\lambda(0)=\Lambda(0), \qquad \mu(0)=\mathrm{M}(0) \tag{19}$$

after introducing (18) into (17) and integrating (17) by parts, the linear constitutive equation takes the form

$$T_{KL}(t)=\delta_{IJ}\int_0^t \Lambda(t-\tau)E_{KK,\tau}(\tau)\,d\tau+2\int_0^t \mathrm{M}(t-\tau)E_{IJ,\tau}(\tau)\,d\tau. \tag{20}$$

This stress tensor may be divided into a pressure (T_{KK}) and a deviation part (S_{IJ})

$$T_{KK}(t)=3\int_0^t\left[\Lambda(t-\tau)+\frac{2}{3}\mathrm{M}(t-\tau)\right]E_{KK,\tau}(\tau)\,d\tau,$$
$$S_{IJ}(t)=2\int_0^t \mathrm{M}(t-\tau)E_{IJ,\tau}(\tau)\,d\tau, \tag{21}$$

and, similarily to the theory of elasticity, functions $\mathrm{M}(t)$ and $\Lambda(t)+ +2\mathrm{M}(t)/3$ may be regarded here as relaxation functions of simple shear and volume expansion, respectively.

One may derive similar expressions for a pressure and a deviation part of tensor $T_{IJ}(t)$ expressed by (17)

$$T_{KK}(t)=3\left\{\left[\lambda(0)+\frac{2}{3}\mu(0)\right]E_{KK}(t)-\right.$$
$$\left.-\int_0^t\left[\lambda_1(t-\tau)+\frac{2}{3}\mu_1(t-\tau)\right]E_{KK}(\tau)\,d\tau\right\},$$
$$S_{IJ}(t)=2\left[\mu(0)E_{IJ}(t)-\int_0^t\mu(t-\tau)E_{IJ}(\tau)\,d\tau\right]. \tag{22}$$

Let us consider the principal quasi-linear theory of viscoelasticity, and propose that relaxation kernel functions of the non-linear part of the stress tensor may be represented in the form

$$\nu_K^L(t)=-N_{K,t}^L(t),\quad K=1,2,3;\quad L=1,2. \tag{23}$$

Introducing (23) into the non-linear part of expression (17), and integrating (17) by parts, we have

$$\begin{aligned}T_{IJ}^{(2)}(t)=\int_0^t\{&3N_1^1(t-\tau)\delta_{IJ}[E_{KK}(\tau)E_{LL}(\tau)]_{,\tau}+\\&+N_2^1(t-\tau)[\delta_{IJ}(E_{KL}(\tau)E_{KL}(\tau))_{,\tau}+2(E_{KK}(\tau)E_{IJ}(\tau))_{,\tau}]+\\&+3N_3^1(t-\tau)[E_{IK}(\tau)E_{KJ}(\tau)]_{,\tau}+\\&+6N_1^2(t-\tau)\delta_{IJ}E_{KK}(t)E_{LL,\tau}(\tau)+\\&+2N_2^2(t-\tau)[\delta_{IJ}E_{KL}(t)E_{KL,\tau}(\tau)+E_{KK}(t)E_{IJ,\tau}(\tau)+\\&+E_{IJ}(t)E_{KK,\tau}(\tau)]+\\&+3N_3^2(t-\tau)[E_{IK}(t)E_{KJ,\tau}(\tau)+E_{KJ}(t)E_{IK,\tau}(\tau)]\}\,d\tau.\end{aligned} \tag{24}$$

Equation (24) is valid if the following equalities are satisfied

$$\nu_1(0)=N_1^1(0)+2N_1^2(0),$$
$$\nu_2(0)=N_2^1(0)+2N_2^2(0), \tag{25}$$
$$\nu_3(0)=N_3^1(0)+2N_3^2(0).$$

The principal quasi-linear theory of viscoelasticity is based on the fundamental postulates of mechanics of continuous media and is quite general. Many theories of viscoelasticity proposed by various authors [7, 9, 10, 13, 14] may be regarded as special cases of this theory. Here, we are interested in the relation between this theory and the theory of non-linear viscoelasticity presented in [10] and applied to the problems of wave propagation [10, 15].

Viscoelastic medium is defined in [10] for an one-dimensional problem by a constitutive equation

$$\sigma_{11}(t)=E\Big\{U_{1,1}(t)+\frac{1}{2}k_1U^2_{1,1}(t)-$$
$$-\int_0^t R(t-\tau)\Big[U_{1,1}(\tau)+\frac{1}{2}k_1U^2_{1,1}(\tau)\Big]d\tau\Big\}, \tag{26}$$

where

$$E=\lambda^*(0)+2\mu^*(0),$$
$$k_1=E^{-1}\{\lambda^*(0)+2\mu^*(0)+6[\nu^*_1(0)+\nu^*_2(0)+\nu^*_3(0)]\},$$

$\sigma_{11}(t)$ denotes the Lagrange (Piola-Kirchhoff) pseudostress tensor and $R(t)$ is the relaxation kernel function.

The Lagrange pseudostress tensor is related to the Kirchhoff pseudostress tensor by the formula [3]

$$\sigma_{11}(t)=x_{1,1}T_{11}(t). \tag{27}$$

Introducing expression (1) into (17) and assuming that $\nu^J_I(t)=0$ if $J=2, 3$, the formula for σ_{11} yields

$$\sigma_{11}(t)=[\lambda(0)+2\mu(0)]U_{1,1}(t)+$$
$$+3\Big[\frac{1}{2}\lambda(0)+\mu(0)+\nu_1(0)+\nu_2(0)+\nu_3(0)\Big]U^2_{1,1}(t)-$$
$$-\int_0^t\Big\{[\lambda_1(t-\tau)+2\mu_1(t-\tau)]U_{1,1}(\tau)+$$
$$+3\Big[\frac{1}{2}\lambda_1(t-\tau)+\mu_1(t-\tau)+\nu^1_1(t-\tau)+\nu^1_2(t-\tau)+$$
$$+\nu^1_3(t-\tau)\Big]U^2_{1,1}(\tau)\Big\}d\tau. \tag{28}$$

Comparison of (28) and (26) leads to the conclusion that the constitutive equation (26) may be regarded as a special case of the constitutive equation of quasi-linear theory of viscoelasticity (17) if the following conditions are satisfied:

$$\lambda^*(0)=\lambda(0), \qquad \mu^*(0)=\mu(0),$$
$$\nu^*_1(0)+\nu^*_2(0)+\nu^*_3(0)=\frac{1}{3}[\lambda(0)+2\mu(0)], \tag{29}$$
$$R(t)=[\lambda(0)+2\mu(0)]^{-1}[\lambda_1(t)+2\mu_1(t)].$$

Conditions (29) are valid if additional simplifications $\nu_I(0)=0$, $\nu^1_I(t)=0$, $I=1, 2, 3$ in (28) are made.

It is interesting to note that now the nonlinear coefficient k_1 in (26) can have one constant value $k_1=3$, only.

To sum up, in this paper a modified constitutive equation of the viscoelastic medium has been derived, taking into account the two first terms in series (2). Consideration of three first terms enables us to derive a more complicated constitutive equation with an instantaneous part that coincides exactly with the constitutive equation of the nine-constant nonlinear theory of elasticity. The regular part has once again the same structure as the instantaneous one.

Here the modified expression for the Kirchhoff pseudostress tensor is presented. Similarly it is possible to derive the corresponding expression for the strain tensor, making use of the basic equations presented in [2].

On the basis of the modified constitutive equation further research will be carried out aiming at the unified theory of acoustoelasticity. Such a theory is needed for interpreting the results of the nondestructive testing and for raising its effectiveness [16].

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Kvaasilineaarsest viskoelastsusteooriast lähtudes on tuletatud mittelineaarse viskoelastse pideva keskkonna olekuvõrrand. Võrrandi iseärasuseks on tema hetkelise ja regulaarse osa struktuuri sarnasus. Võrrandi hetkeline osa ühtib täpselt viiekonstantse elastsusteooria olekuvõrrandiga. Regulaarses osas relaksatsioonituumade funktsioonid seovad pinge- ja deformatsioonitensori komponente analoogselt sellega, kuidas elastsuskonstandid seovad neid elastsusteoorias. Erinevus seisneb vaid selle seose sõltuvuses ajast. Viimane tähelepanek lihtsustab oluliselt arvukate relaksatsioonituumade funktsioonide füüsikalise sisu mõistmist.

Арви РАВАСОО

НЕКОТОРЫЕ ЗАМЕЧАНИЯ О КВАЗИЛИНЕЙНОЙ ТЕОРИИ ВЯЗКОУПРУГОСТИ

На основе квазилинейной теории вязкоупругости выведено модифицированное уравнение состояния нелинейной вязкоупругой сплошной среды. Особенностью уравнения является структура его мгновенной и регулярной частей. Мгновенная часть уравнения точно совпадает с уравнением состояния пятиконстантной теории упругости. Регулярная часть имеет схожую с мгновенной частью структуру. Функции ядер релаксации связывают в ней компоненты тензоров напряжения и деформации подобно тому, как модули упругости связывают их в теории упругости. Различие состоит только в разной зависимости этой связи от времени. В итоге облегчается раскрытие физического смысла многочисленных функций ядер релаксации.