Proc. Estonian Acad. Sci. Phys. Math., 1991, 40, N 2, 105—111
https://doi.org/10.3176/phys.math.1991.2.06

УДК 512.548.7; 530.1 : 51—72

Eugen PAAL *

BIREPRESENTATIONS OF DERIVATIVE MOUFANG LOOPS

(Presented by G. Liidja)

Following the idea of V. Belousov of a *derivative loop*, the *derivatives of birepresentations* of the Moufang loops are introduced. The properties of the derivatives of birepresentations are proved to imitate the corresponding properties of derivative (Moufang) loops.

1. Moufang loops

A *Moufang loop* [1,2] is a set G with a binary operation (multiplication) denoted by juxtaposition so that the following axioms are satisfied: 1) in the equation $gh=k$, the knowledge of any two of $g, h, k \in G$ specifies the third one uniquely; 2) there is a distinguished element e of G with the property $eg=ge=g$ for all $g \in G$ 3) the *Moufang identity*

$$(ag)(ha)=a(gh)a \tag{1.1}$$

holds in G. If only the axioms 1) and 2) are satisfied, the set G is called a *loop*. An element e is called the *identity* element of G.

The most familiar kind of loops are those with the *associative law* $(ag)h=a(gh)$, and they are called *groups*. A (Moufang) loop is said to be *commutative* if $gh=hg$ for all $g, h \in G$, and only commutative associative (Moufang) loops are said to be *Abelian*.

The most remarkable property of Moufang loops is their *diassociativity*: the subloop generated by every two elements in a Moufang loop is *associative* (group). Hence, for all g, h in a Moufang loop G one has

$$g\cdot gh=g^2h \quad , \quad hg\cdot g=hg^2 \quad , \quad gh\cdot g=g\cdot hg. \tag{1.2}$$

As in the case of groups, the notion of the inverse element can be defined. The unique solution of the equation $gx=e$ ($xg=e$) is called the *right* (*left*) *inverse element* of $g \in G$ and it is denoted as g_R^{-1} (g_L^{-1}). It follows from the diassociativity of the Moufang loop G that

$$g_R^{-1}=g_L^{-1}=:g^{-1}, \tag{1.3}$$

$$g^{-1}(gh)=(hg)g^{-1}=h, \tag{1.4}$$

$$(g^{-1})^{-1}=g, \tag{1.5}$$

$$(gh)^{-1}=h^{-1}g^{-1}, \quad \forall g, h \in G. \tag{1.6}$$

2. Derivative Moufang loops

Let a be a fixed element of the Moufang loop G. The *left* (L), *right* (R) and *medial* (M) *derivatives* of gh are defined as

---

* Tallinna Tehnikaülikool (Tallinn Technical University). 200 108 Tallinn, Akadeemia tee 1. Estonia.

$$(gh)'_{aL} := (g \cdot h\hat{a})\, a, \tag{2.1a}$$

$$(gh)'_{aR} := \hat{a}(ag \cdot h), \tag{2.1b}$$

$$(gh)'_{aM} := (g\hat{a})\,(ah), \quad g, h, a \in G, \tag{2.1c}$$

where $\hat{a}$ denotes the inverse element of a in G: $\hat{a}=a^{-1}$. The left and right derivatives of gh can also be defined by [1]

$$g \cdot h\hat{a} = (gh)'_{aL}\, \hat{a}, \tag{2.2a}$$

$$ag \cdot h = a(gh)'_{aR}, \quad g, h, a \in G. \tag{2.2b}$$

The Moufang identity (1.1) can be rewritten in *three* ways:

$$(g \cdot h\hat{a})\, a = \hat{a}\,(ag \cdot h) = (g\hat{a})\,(ah). \tag{2.3}$$

One can therefore ignore distinction between three kinds of derivatives in the Moufang loop G and define

$$(gh)'_a := (gh)'_{aL} = (gh)'_{aR} = (gh)'_{aM}, \quad g, h, a \in G. \tag{2.4}$$

It turns out [1] that the derivative multiplication of the Moufang loop satisfies all the axioms of the Moufang loop. The derived Moufang loop with the multiplication rule (2.4) is called the *derivative* Moufang loop of G, and denoted as G'_a. The identity element of G'_a is e, and the inverse element of g in G'_a is $\hat{g}$: $(g\hat{g})'_a = (\hat{g}g)'_a = e$. By denoting $g \circ h := (gh)'_a$, the Moufang identity for G'_a reads

$$(b \circ g) \circ (h \circ b) = b \circ (g \circ h) \circ b, \quad g, h, b \in G. \tag{2.5}$$

The product $(gh)'_g$ is called the first derivative of gh. Now, let a and b be two fixed elements of G. The *second derivative* $(gh)''_{a,b}$ of gh is natural to be defined as

$$(gh)''_{a,b} := ((gh)'_a)'_b. \tag{2.6}$$

However, due to the identity [1]

$$((gh)'_a)'_b = (gh)'_{ab}, \tag{2.7}$$

the second derivative (and, hence, all the higher derivatives as well) can be reduced to the first one. This property can symbolically be expressed as

$$G''_{a,b} := (G'_a)'_b = G'_{ab}. \tag{2.8}$$

3. Birepresentations

Let $\mathfrak{X}$ be a set and let $\mathfrak{T}(\mathfrak{X})$ be the *transformation group* of $\mathfrak{X}$, i. e the group of bijective mappings of $\mathfrak{X}$ onto $\mathfrak{X}$. The elements of $\mathfrak{T}(\mathfrak{X})$ are called *transformations* of $\mathfrak{X}$. The multiplication in $\mathfrak{T}(\mathfrak{X})$ is defined as the *composition* of transformations and the identity element of $\mathfrak{T}(\mathfrak{X})$ coincides with the *identity transformation* E of $\mathfrak{X}$.

A pair (S, T) of the mappings $g \to S_g$, $g \to T_g$ of a Moufang loop G into the group $\mathfrak{T}(\mathfrak{X})$ is said [3, 4] to be an *action* of G on $\mathfrak{X}$ if

$$S_e = T_e = E \tag{3.1}$$

and

$$S_g T_g S_h = S_{gh} T_g, \tag{3.2a}$$

$$S_g T_g T_h = T_{hg} S_g \tag{3.2b}$$

are satisfied for all g, h in G. The pair (S, T) is called also a *birepresentation* of G (in $\mathfrak{T}(\mathfrak{X})$). The transformations S_g, $T_g \in \mathfrak{T}(\mathfrak{X})$ $(g \in G)$ are called *G-transformations* (*Moufang transformations*) of $\mathfrak{X}$.

By means of (3.1, 2a—b), one can easily check that

$$S_g T_h = T_g S_g \quad , \quad S_g^{-1} = S_{\hat{g}} \quad , \quad T_g^{-1} = T_{\hat{g}} \quad \forall g \in G. \tag{3.3}$$

4. Triality

For $g \in G$ define $P_g \in \mathfrak{T}(\mathfrak{X})$ by

$$S_g T_g P_g = E. \tag{4.1}$$

It follows from the definition of P_g that $P_e = E$ and $P_{\hat{g}} P_g = P_g P_{\hat{g}} = E$. Hence, $P_g^{-1} = P_{\hat{g}}$ for all $g \in G$. The triple of the mappings $g \to S_g$, $g \to T_g$, $g \to P_g$ of G into $\mathfrak{T}(\mathfrak{X})$ with the property (4.1) will be denoted as (S, T, P).

The following *proposition* is proved in [3].

Let (S, T) be a birepresentation of a Moufang loop G in $\mathfrak{T}(\mathfrak{X})$. Then the pairs of the mappings

$$\begin{aligned}
(T^{-1}, S^{-1}) &: \quad g \to T_g^{-1} \quad , \quad g \to S_g^{-1} \; ; \\
(T, P) &: \quad g \to T_g \quad , \quad g \to P_g \; ; \\
(P^{-1}, T^{-1}) &: \quad g \to P_g^{-1} \quad , \quad g \to T_g^{-1} \; ; \\
(P, S) &: \quad g \to P_g \quad , \quad g \to S_g \; ; \\
(S^{-1}, P^{-1}) &: \quad g \to S_g^{-1} \quad , \quad g \to P_g^{-1}
\end{aligned}$$

of G into $\mathfrak{T}(\mathfrak{X})$ are birepresentations of G in $\mathfrak{T}(\mathfrak{X})$ as well.

The defining identities of the birepresentations from the triple (S, T, P) can conveniently be found through the table

(S, T)	(T^{-1}, S^{-1})	(T, P)	(P^{-1}, T^{-1})	(P, S)	(S^{-1}, P^{-1})
(4.2a)	(4.2b)	(4.2b)	(4.2c)	(4.2c)	(4.2a)
(4.3b)	(4.3a)	(4.3c)	(4.3b)	(4.3a)	(4.3c)

with

$$S_{\hat{gh}} \overset{(a)}{=} P_g S_h T_g \quad , \quad T_{\hat{gh}} \overset{(b)}{=} S_g T_h P_g \quad , \quad P_{\hat{gh}} \overset{(c)}{=} T_g P_h S_g. \tag{4.2}$$

$$S_{h\hat{g}} \overset{(a)}{=} T_g S_h P_g \quad , \quad T_{h\hat{g}} \overset{(b)}{=} P_g T_h S_g \quad , \quad P_{h\hat{g}} \overset{(c)}{=} S_g P_h T_g. \tag{4.3}$$

It follows from the above proposition that the defining identities of the birepresentation (S, T) are invariant under the substitutions

$$\alpha := (S \to T \to P \to S), \tag{4.4a}$$

$$\beta := (S \to T^{-1} \to S)\,(P \to P^{-1}), \tag{4.4b}$$

called the *triality substitutions* of (S, T). Hence, all the algebraic consequences of the defining identities (3.1, 2a-b) of (S, T) have also to be *triality invariant*, i. e. invariant under the triality substitutions. This property of the Moufang transformations is said to be the *triality*.

5. First derivatives

Let (S, T) be a birepresentation of a Moufang loop G. The action of $g \in G$ on a point $x \in \mathfrak{X}$ can be written multiplicatively as $gx := S_g x$, $xg := T_g x$. The defining identities of (S, T) read

$$ex = xe = x, \tag{5.1}$$

$$(ag)(xa) = a(gx)a, \tag{5.2a}$$

$$(ax)(ga) = a(xg)a; \quad a, g \in G, \quad x \in \mathfrak{X}. \tag{5.2b}$$

Let a be a fixed element of G. The *left* (L), *right* (R) and *medial* (M) *derivatives* of gx and xg $(g \in G;\ x \in \mathfrak{X})$ are defined as follows:

$$(gx)'_{aL} := (g \cdot x\hat{a})a \quad , \quad (xg)'_{aL} := (x \cdot g\hat{a})a; \tag{5.3}$$

$$(gx)'_{aR} := \hat{a}(ag \cdot x) \quad , \quad (xg)'_{aR} := \hat{a}(ax \cdot g); \tag{5.4}$$

$$(gx)'_{aM} := (g\hat{a})(ax) \quad , \quad (xg)'_{aM} := (x\hat{a})(ag). \tag{5.5}$$

The left and right derivatives can also be defined by

$$g \cdot x\hat{a} = (gx)'_{aL}\hat{a} \quad , \quad x \cdot g\hat{a} = (xg)'_{aL}\hat{a}; \tag{5.6}$$

$$ag \cdot x = a(gx)'_{aR} \quad , \quad ax \cdot g = a(xg)'_{aR}. \tag{5.7}$$

The defining identities (5.2a) and (5.2b) of (S, T) can be rewritten, respectively, in *three* ways:

$$(g \cdot x\hat{a})a = \hat{a}(ag \cdot x) = (g\hat{a})(ax), \tag{5.8}$$

$$(x \cdot g\hat{a})a = \hat{a}(ax \cdot g) = (x\hat{a})(ag). \tag{5.9}$$

As an example, let us check the first equality in (5.8). It reads

$$T_a S_g T_{\hat{a}} x = S_{\hat{a}} S_{ag} x \quad \forall x \in \mathfrak{X},$$

from which follows the defining identity of (S, T): $S_a T_a S_g = S_{ag} T_a$. The remaining equalities in (5.8, 9) can be checked similarly.

One can thus ignore distinction between three kinds of derivatives and define

$$(gx)'_a := (gx)'_{aL} = (gx)'_{aR} = (gx)'_{aM}, \tag{5.10}$$

$$(xg)'_a := (xg)'_{aL} = (xg)'_{aR} = (xg)'_{aM}; \tag{5.11}$$

$$a, g \in G, \quad x \in \mathfrak{X}.$$

6. Derivatives of birepresentations

The (*first*) *derivative* $(S, T)'_a$ of (S, T) is defined as the pair of the mappings

$$g \to (S_g)'_a := T_a S_g T_{\hat{a}} \quad , \quad g \to (T_g)'_a := S_{\hat{a}} T_g S_a$$

of G into $\mathfrak{T}(\mathfrak{X})$. One can check that

$$(gx)'_a = (S_g)'_a x \quad , \quad (xg)'_a = (T_g)'_a x. \tag{6.1}$$

Following the triality, let at first compute

$$(S_g)'_a\ (T_g)'_a = T_a S_g T_{\hat{a}}\ S_{\hat{a}} T_g S_a$$
$$= T_a S_g\ P_a\ T_g S_{\hat{a}}$$
$$= T_a\ P_{a\hat{g}}\ S_a$$
$$= P_{\hat{a}\cdot ag}$$
$$= P_{\hat{g}}$$
$$= S_g T_g.$$

Similarly one can check that $(T_g)'_a\ (S_g)'_a = P_{\hat{g}} = T_g S_g$ as well. Hence, defining $(P_g)'_a \in \mathfrak{T}(\mathfrak{X})$ by

$$(S_g)'_a\ (T_g)'_a\ (P_g)'_a = E, \tag{6.2}$$

one has

$$(P_g)'_a = P_g \quad ; \quad \forall g, a \in G. \tag{6.3}$$

Now, the following *proposition* can be proved.

Let (S, T) be a birepresentation of a Moufang loop G in $\mathfrak{T}(\mathfrak{X})$ and let a be fixed element of G. Then the derivative $(S, T)'_a$ of (S, T) is a birepresentation of the derivative Moufang loop G'_a of G in $\mathfrak{T}(\mathfrak{X})$.

The defining identities of $(S, T)'_a$ read

$$(S_e)'_a = (T_e)'_a = E, \tag{6.4}$$

$$(S_g)'_a\ (T_g)'_a\ (S_h)'_g = (S_{(gh)'_a})'_a\ (T_g)'_a, \tag{6.5a}$$

$$(S_g)'_a\ (T_g)'_a\ (T_h)'_g = (T_{(hg)'_a})'_a\ (S_g)'_a; \quad a, g, h \in G, \tag{6.5b}$$

The validity of (6.4) is evident. The other equalities can be rewritten as

$$P_{\hat{g}}\ T_a S_h T_{\hat{a}} = T_a\ S_{(gh)'_a}\ T_{\hat{a}}\ S_{\hat{a}} T_g S_a, \tag{6.6a}$$

$$P_{\hat{g}}\ S_{\hat{a}} T_h S_a = S_{\hat{a}}\ T_{(hg)'_a}\ S_a\ T_a S_g T_{\hat{a}}. \tag{6.6b}$$

It follows from (6.6a) that

$$S_{(gh)'_a} = T_{\hat{a}} P_{\hat{g}}\ T_a S_h P_a\ T_{\hat{g}} P_{\hat{a}}$$
$$= T_{\hat{a}}\ P_{\hat{g}} S_{h\hat{a}} T_{\hat{g}}\ P_{\hat{a}}$$
$$= T_{\hat{a}}\ S_{g\cdot h\hat{a}}\ P_{\hat{a}}$$
$$= S_{(gh)'_{\hat{a}L}},$$

which proves (6.5a). In a similar way (6.5b) can be checked.

By denoting $x \circ g := (xg)'_a$, $g \circ x := (gx)'_a$ and $g \circ h := (gh)'_a$, the defining identities of $(S, T)'_a$ read

$$e \circ x = x \circ e = x, \tag{6.7}$$

$$(b \circ g) \circ (x \circ b) = b \circ (g \circ x) \circ b, \tag{6.8a}$$

$$(b \circ x) \circ (g \circ b) = b \circ (x \circ g) \circ b; \quad g, a, b \in G, \quad x \in \mathfrak{T}. \tag{6.8b}$$

7. Second derivatives

Now, let a and b be two fixed elements of G. The second derivatives $(gx)''_{a,b}$ and $(xg)''_{a,b}$ of gx and xg $(x \in \mathfrak{X};\ g, a, b \in G)$ are natural to define as

$$(gx)''_{a,b} := ((gx)'_a)'_b \quad , \quad (xg)''_{a,b} := ((xg)'_a)'_b . \tag{7.1}$$

The second derivative $(S, T)''_{a,b}$ of (S, T) is defined as the pair of the mappings

$$g \to (S_g)''_{a,b} := ((S_g)'_a)'_b = (T_b)'_a \ (S_g)'_a \ (T_{\hat{b}})'_a$$

$$g \to (T_g)''_{a,b} := ((T_g)'_a)'_b = (S_{\hat{b}})'_a \ (T_g)'_a \ (S_b)'_a$$

of G into $\mathfrak{T}(\mathfrak{X})$. One can easily see that

$$(gx)''_{a,b} = (S_g)''_{a,b} x \quad , \quad (xg)''_{a,b} = (T_g)''_{a,b} x. \tag{7.2}$$

Really,

$$((gx)'_a)'_b = ((S_g)'_a x)'_b = ((S_g)'_a)'_b x, \tag{7.3a}$$

$$((xg)'_a)'_b = ((T_g)'_a x)'_b = ((T_g)'_a)'_b x. \tag{7.3b}$$

Thus one can symbolically define $(S, T)'_{a,b} := ((S, T)'_a)'_b$ It follows from the results of Sec. 6 that $((S, T)'_a)'_b$ is a birepresentation of the Moufang loop $(G'_a)'_b = G'_{ab}$. On the other hand, $(S, T)'_{ab}$ has to be a birepresentation of G'_{ab} as well. In view of this, it is not surprising that

$$((S_g)'_a)'_b = (S_g)'_{ab} \quad \text{and} \quad ((T_g)'_a)'_b = (T_g)'_{ab} . \tag{7.4}$$

This property can symbolically be expressed as

$$((S, T)'_a)'_b = (S, T)'_{ab} , \tag{7.5}$$

and formulated as follows: every birepresentation of G is closed under the double derivation.

As an example, let us check the first equality in (7.4):

$$\begin{aligned} ((S_g)'_a)'_b &= (T_b)'_a \ (S_g)'_a \ (T_{\hat{b}})'_a \\ &= S_{\hat{a}} T_b S_a \ T_a S_g T_{\hat{a}} \ S_{\hat{a}} T_{\hat{b}} S_a \\ &= S_{\hat{a}} T_b P_{\hat{a}} \ S_g \ P_a T_{\hat{b}} S_a \\ &= T_{ab} \ S_g \ T_{\hat{b}\hat{a}} \\ &= T_{ab} \ S_g \ T_{ab} \\ &= (S_g)'_{ab} . \end{aligned}$$

The second equality in (7.4) can be proved similarly. In the multiplicative form, the closure property of (S, T) reads

$$((gx)'_a)'_b = (gx)'_{ab} \quad , \quad ((xg)'_a)'_a = (xg)'_{ab}, \tag{7.6}$$

$$g, a, b \in G, \ x \in \mathfrak{X}.$$

8. Derivative birepresentations

Now let $(\bar{S}, \bar{T})$ be a birepresentation of the derivative Moufang loop G'_a of G $(a \in G)$. Then $(\bar{S}, \bar{T})'_{\hat{a}}$ is a birepresentation of the Moufang loop $(G'_a)'_{\hat{a}} = G'_{a\hat{a}} = G'_e = G$. By defining $(S, T) := (\bar{S}, \bar{T})'_{\hat{a}}$, one has

$$(S, T)'_a = ((\bar{S}, \bar{T})'_{\hat{a}})'_a = (\bar{S}, \bar{T})'_{\hat{a}a} = (\bar{S}, \bar{T})'_e = (\bar{S}, \bar{T}).$$

Thus the following *proposition* can be proposed.

Every birepresentation of the derivative Moufang loop G'_a of G $(a \in G)$ has to be the derivative of some birepresentation of the Moufang loop G.

In view of the results of Sec. 6 and 8, the derivatives of birepresentations of the Moufang loops are natural to call their *derivative birepresentations*. Properties of *derivative Moufang transformations* imitate thus the corresponding properties of derivative Moufang loops [1].

REFERENCES

1. *Белоусов В. Д.* 1967. Основы теории квазигрупп и луп. Москва, Наука.
2. *Bruck, R. H.* 1971. A Survey of Binary Systems. Berlin; Heidelberg; New York, Springer.
3. *Паал Э.* 1987. Тр. Ин-та физ. АН ЭССР, 62, 142—158.
4. *Паал Э.* 1989. Тр. Ин-та физ. АН Эстонии, 64, 104—124.

Received
Nov. 12, 1990

Eugen PAAL

MOUFANGI TULETISLUUPIDE BIESITUSED

Järgides V. Beloussovi tuletisluubi ideed, on toodud sisse Moufangi luupide biesituste tuletised, uuritud nende omadusi ja leitud, et viimased jäljendavad tuletisluupide vastavaid omadusi.

Эуген ПААЛ

БИПРЕДСТАВЛЕНИЯ ПРОИЗВОДНЫХ ЛУП МУФАНГ

Следуя идее В. Д. Белоусова о производной лупе, вводятся производные бипредставления луп Муфанг. Изучаются их свойства и доказывается, что последние имитируют соответствующие свойства производных луп.