EESTI NSV TEADUSTE AKADEEMIA TOIMETISED.
FÜÜSIKA * MATEMAATIKA
ИЗВЕСТИЯ АКАДЕМИИ НАУК ЭСТОНСКОЙ ССР. ФИЗИКА * МАТЕМАТИКА
PROCEEDINGS OF THE ACADEMY OF SCIENCES OF THE ESTONIAN SSR.
PHYSICS * MATHEMATICS

1989, 38, 2

УДК 539.12; 519.46

K. KIIRANEN and V. ROSENHAUS

ON INVARIANCE GROUPS IN GAUGE THEORIES

(Presented by V. Hizhnyakov)

The symmetries of the Maxwell and Yang-Mills fields are discussed. The generalized gauge transformations are shown to correspond to the Lie-Bäcklund tangent transformations.

1. Introduction

The present paper deals with the invariance properties of the simplest gauge theories, the Maxwell and the Yang-Mills equations. The paper is organized as follows: in Part 2 some known results concerning the Lie point symmetry groups of the systems under investigation are reviewed. The tensor $P^{\pm}_{\mu\nu\rho\sigma}$ is introduced and some of its properties are discussed.

In Part 3, the considered gauge systems are shown to possess also the algebra of the Lie-Bäcklund tangent transformations. The latter correspond to the gauge transformations in the extended (to the potentials and their derivatives) space.

2. Gauge field equations and Lie point symmetry groups

The simplest gauge theory is electrodynamics with the Maxwell equation

$$\partial_\mu F_{\mu\nu}=0,$$
$$F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu} \quad (\mu, \nu=1, 2, 3, 4). \tag{1}$$

(All the expressions are given for the Euclidean space, but the obtained results are valid also for the Minkowski space.)

For the non-Abelian gauge theory with an arbitrary simple gauge group G we have the Yang-Mills equation

$$D_\mu G^a_{\mu\nu}=0,$$
$$D_\mu G^a_{\mu\nu}=\partial_\mu G^a_{\mu\nu}+gf_{abc}A^b_\mu G^c_{\mu\nu},$$
$$G^a_{\mu\nu}=A^a_{\nu,\mu}-A^a_{\mu,\nu}+gf_{abc}A^b_\mu A^c_\nu, \tag{2}$$
$$a, b, c=1, 2, \ldots, N,$$

where N is the dimension of the group G, f_{abc} are the structure constants. The tensor f_{abc} is completely antisymmetric (due to the choosing of the invariant inner product of the basis generators orthonormal (in the adjoint representation)

$$(L_a, L_b)=k\,\mathrm{Tr}\,(L_a, L_b)=\delta_{ab}).$$

In the simplest case $G=SU(2)$, $N=3$, $f_{abc}=\varepsilon_{abc}$. Consider also the self-dual (anti-dual) Yang-Mills fields

$$G^a_{\mu\nu}=\pm\tilde{G}^a_{\mu\nu}, \tag{3}$$

where

$$\tilde{G}^a_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}G^a_{\rho\sigma}.$$

It is known that each solution of Eq. (3) automatically satisfies Eq. (2). (Indeed, it is sufficient to apply to Eq. (3) the operator D_μ and to take into account the Bianchi identity $D_\mu\tilde{G}^a_{\mu\nu}\equiv 0$.) Let us write Eq. (3) in the form

$$P^{\pm}_{\mu\nu\rho\sigma}G^a_{\rho\sigma}=0, \tag{4}$$

where

$$P^{\pm}_{\mu\nu\rho\sigma}=\pm\varepsilon_{\mu\nu\rho\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho}-\delta_{\mu\rho}\delta_{\nu\sigma}. \tag{5}$$

The tensor $P^{\pm}$ has algebraic properties similar to those of the Riemann-Christoffel tensor

$$P^{\pm}_{\mu\nu\rho\sigma}=-P^{\pm}_{\mu\nu\sigma\rho}=-P^{\pm}_{\nu\mu\rho\sigma},$$
$$P^{\pm}_{\mu\nu\rho\sigma}=P^{\pm}_{\rho\sigma\mu\nu}, \tag{6}$$

but, instead of the identity

$$R_{\mu\nu\rho\sigma}+R_{\mu\sigma\nu\rho}+R_{\mu\rho\sigma\nu}=0$$

we have

$$P^{\pm}_{\mu\nu\rho\sigma}+P^{\pm}_{\mu\sigma\nu\rho}+P^{\pm}_{\mu\rho\sigma\nu}=\pm 3\varepsilon_{\mu\nu\rho\sigma}.$$

The following relations are also satisfied:

$$P^{\pm}_{\mu\nu\rho\sigma}=\mp\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}P^{\pm}_{\alpha\beta\rho\sigma},$$
$$P^{\pm}_{\mu\nu\rho\sigma}P^{\pm}_{\rho\sigma\alpha\beta}=-4P^{\pm}_{\mu\nu\alpha\beta}, \tag{7}$$
$$P^{+}_{\mu\nu\rho\sigma}P^{-}_{\rho\sigma\alpha\beta}=0.$$

It can be easily checked that

$$P^{-}_{\mu\nu 4a}=\eta^a_{\mu\nu},$$
$$P^{+}_{\mu\nu 4a}=\bar{\eta}^a_{\mu\nu}, \tag{8}$$

where $\eta^a_{\mu\nu}$ $(\bar{\eta}^a_{\mu\nu})$ are the t'Hooft symbols [1]:

$$\eta^a_{mn}=\eta^a_{mn}=\varepsilon_{amn},\qquad \eta^a_{\mu 4}=-\bar{\eta}^a_{\mu 4}=\delta_{a\mu},$$
$$\eta^a_{\mu\nu}=-\eta^a_{\nu\mu},\qquad \bar{\eta}^a_{\mu\nu}=-\bar{\eta}^a_{\nu\mu}.$$

The non-zero components of $P^{\pm}_{\nu\mu\rho\sigma}$ are only

$$P^{\pm}_{1234}=\pm 1$$

and

$$P^{\pm}_{abba}=1$$

(and, certainly, the components with the corresponding permutation of the indices).

In terms of $P^{\pm}_{\mu\nu\rho\sigma}$ Yang-Mills equation (2) can be written as

$$D^{ab}_{\nu} G^{b}_{\nu\mu} = \frac{1}{2} D^{ab}_{\nu} (P^{\pm}_{\mu\nu\rho\sigma} G^{b}_{\rho\sigma}) = 0, \tag{9}$$

where the Bianchi identity is used and

$$D^{ab}_{\nu} = \delta_{ab}\partial_{\nu} + g f_{abc} A^{c}_{\nu}$$

is the covariant differential operator.

Let us briefly discuss the situation with the groups of point transformations for the Maxwell and the Yang-Mills equations. As usual [2,3], the infinitesimal operator corresponding to the invariance group of the system of differential equations (of the second order) $\omega_p = 0$ ($p = 1, \ldots$ $\ldots, M$) is of the form

$$X = \xi_{\mu}(x, A) \frac{\partial}{\partial x_{\mu}} + \eta_{\mu}(x, A) \frac{\partial}{\partial A_{\mu}}, \tag{10}$$

whereby

$$\overset{\approx}{X} \omega_p \Big|_{\omega_p = 0} = 0$$

($\overset{\approx}{X}$ is the corresponding twice-prolonged operator).

The symmetry properties of the Maxwell equation have been already investigated by H. Bateman [4]. A detailed analysis of the symmetries (including nonlocal ones) of the Maxwell equation in the different forms has been made in [5]. Thus, for Eq. (1) we have

$$\begin{aligned} &\xi_{\mu} = k_{\mu} + g x_{\mu} + a_{\mu\nu} x_{\nu} + 2c_{\nu} x_{\nu} x_{\mu} - c_{\mu} x_{\nu} x_{\nu}, \\ &\eta_{\alpha} = \partial_{\alpha}\varphi(x) + bA_{\alpha} + a_{\alpha\beta} A_{\beta} + 2(c_{\beta} x_{\alpha} - c_{\alpha} x_{\beta}) A_{\beta} - 2c_{\beta} x_{\beta} A_{\alpha}, \end{aligned} \tag{11}$$

where $\varphi(x)$ is an arbitrary function (the corresponding transformation is the known $U(1)$-gauge (gradient) transformation), and g, b, k_{μ}, c_{μ}, $a_{\mu\nu} = -a_{\nu\mu}$ are arbitrary constants. The respective generators lead to the conformal group in 4-space (the given 16-parameter group includes two dilatation operators). Taking the system of the Maxwell equations in the form

$$\begin{aligned} &\partial_{\mu} F_{\mu\nu} = 0, \\ &\partial_{\mu} \tilde{F}_{\mu\nu} = 0, \\ &\tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}, \end{aligned}$$

we get the 17-parameter group with infinitesimal operator [6]:

$$X = \xi_{\mu} \frac{\partial}{\partial x_{\mu}} + \sum_{\alpha<\beta} \left(F_{\alpha\gamma} \frac{\partial \xi_{\gamma}}{\partial x_{\beta}} - F_{\beta\gamma} \frac{\partial \xi_{\gamma}}{\partial x_{\alpha}} \right) \frac{\partial}{\partial F_{\alpha\beta}} +$$

$$+ A \sum_{\alpha<\beta} F_{\alpha\beta} \frac{\partial}{\partial F_{\alpha\beta}} + B \sum_{\alpha<\beta} \tilde{F}_{\alpha\beta} \frac{\partial}{\partial F_{\alpha\beta}}, \tag{13}$$

where ξ_{μ} are defined in (11), and A, B are arbitrary constants.

The 17th generator $\left(\sum_{\alpha<\beta} \tilde{F}_{\alpha\beta} \frac{\partial}{\partial F_{\alpha\beta}} \right)$ is, obviously, nonlocal in terms of A_{α}, and corresponds to the change to dual quantities.

The non-Abelian gauge field theory is also conformally invariant [7]. The corresponding infinitesimal operator of the group of Lie point transformation for the Yang-Mills equation is [8,9]

$$X=\xi_\mu\frac{\partial}{\partial x_\mu}+\eta^a_\mu\frac{\partial}{\partial A^a_\mu},$$
$$\eta^a_\alpha=D^{ab}_\alpha\theta^b(x)-gA^a_\alpha+a_{\alpha\beta}A^a_\beta+2(c_\beta x_\alpha-c_\alpha x_\beta)A^a_\beta-2c_\beta x_\beta A^a_\alpha \tag{14}$$

with ξ_μ from (11).

Note that, compared with the Abelian case group, (14) consists only of dilatation operator. It is shown in [8] that the Lie point symmetry group for the self-dual Eq. (3) coincides with that of the Yang-Mills equation. In the Lorentz gauge $\partial_\mu A^a_\mu=0$, the conformal symmetriy of the system turns out to be broken (up to the Weyl group [9]), and the gauge group $G\equiv SU(2)$ is reduced to global rotations in the isotopic space.

Note also that assuming the infinitesimal operator X in the canonical form

$$X=f^a_\mu\frac{\partial}{\partial A^a_\mu}+\ldots,$$

we obtain for the Yang-Mills equation

$$f^a_\mu=A^a_{\mu,\nu}\xi_\nu+A^a_\nu\,\xi_{\nu,\mu}=\xi_\nu G^a_{\nu\mu}+D^{ab}_\mu(\xi_\nu A^b_\nu) \tag{15}$$

with ξ_ν from (11).

3. Gauge transformations as Lie-Bäcklund transformations

In this chapter, it is shown that the gauge transformations defined in the extended (to potentials and their derivatives) space correspond to the algebra of the Lie-Bäcklund operators.

Consider the gauge (gradient) transformation for the Maxwell equation

$$A_\mu\to A_\mu+\partial_\mu\varphi(x). \tag{16}$$

In the extended space $(x_\mu, A_\alpha, A_{\alpha,\mu}, A_{\alpha,\mu\nu},\ldots)$, $A_{\alpha,\beta}\equiv\frac{\partial A_\alpha}{\partial x_\beta}$ one can write the transformation corresponding to (16)

$$A_\mu\to A_\mu+\mathrm{d}_\mu F(x_\nu, A_\alpha, A_{\alpha,\beta}, A_{\alpha,\beta\gamma},\ldots), \tag{17}$$

where d_μ is the total derivative

$$\mathrm{d}_\mu\equiv\frac{\partial}{\partial x_\mu}+A_{\alpha,\mu}\frac{\partial}{\partial A_\alpha}+A_{\alpha,\beta\mu}\frac{\partial}{\partial A^\alpha_{,\beta}}+\ldots$$

(transformation (17) as well as (16) leave $F_{\mu\nu}$ invariant).

Now let us recall that the system of differential equations

$$\omega_p(x_\nu, A_\alpha, A_{\alpha,\beta}, A_{\alpha,\beta\gamma})=0$$
$$(p=1,\ldots,M)$$

(x_ν and A_α are the arguments and functions, respectively) admits a tangent transformation group generated by a Lie-Bäcklund operator

$$X=f_\alpha\frac{\partial}{\partial A^\alpha}+\mathrm{d}_\nu f_\alpha\frac{\partial}{\partial A^\alpha_{,\nu}}+\mathrm{d}_\mu\,\mathrm{d}_\nu f_\alpha\frac{\partial}{\partial A^\alpha_{,\mu\nu}}+\ldots,$$
$$f_\alpha=f_\alpha(x_\nu, A_\beta, A_{\beta,\nu}, A_{\beta,\mu\nu},\ldots) \tag{18}$$

(we write the corresponding tangent vector field of the group in the canonical form) iff

$$X\omega_p = \Big|_{\substack{\omega_p=0 \\ d_\mu \omega_p=0 \\ d_\mu d_\nu \omega_p=0 \\ \cdots\cdots}} = 0$$

(where $d_\mu\omega_p=0$, $d_\mu d_\nu\omega_p=0$, ... are the differential consequences of the initial system). For details about the theory of the Lie-Bäcklund transformations, see [10, 11].

Thus, Maxwell equation (1) admits the group with the Lie-Bäcklund operator (18) with

$$f_\mu = d_\mu \varphi, \tag{19}$$

where $\varphi=\varphi(x_\nu, A_\alpha, A_{\alpha,\nu}, \ldots)$ is an abritrary function. If $\varphi=\varphi(x)$, then (19) defines the usual gauge transformation (16). Any $U(I)$-gauge invariant system (depending on $F_{\mu\nu}$ only) evidently possesses the same property. The commutator of two Lie-Bäcklund operators (18) (with f_μ from (19)) is

$$[X_\varphi, X_\psi] = X_\theta,$$

where

$$\theta = X_\varphi \psi - X_\psi \varphi.$$

Now let us proceed to non-Abelian theories and consider Yang-Mills equation (2). Using the invariance of Eq. (2) under gauge transformations (in the infinitesimal and finite form)

$$\begin{gathered} A^a_\mu \to A^a_\mu - \frac{1}{g}\partial_\mu \omega^a + f_{abc}\omega^b A^c_\mu, \\ A_\mu \to U A_\mu U^{-1} + \frac{i}{g} U \partial_\mu U^{-1}, \\ A_\mu = A^a_\mu L_a, \quad U = e^{-i\omega^a L_a}, \quad \omega^a = \omega^a(x), \end{gathered} \tag{20}$$

we introduce the dependence of the group parameters on all variables of the extended space $(x_\nu, A^a_\mu, A^a_{\mu,\nu}, \ldots)$ (analogously to the Abelian case)

Then Eqs (20) change to

$$\begin{gathered} A^a_\mu \to A^a_\mu - \frac{1}{g} d_\mu \omega^a + f_{abc}\omega^b A^c_\mu, \\ A_\mu \to U A_\mu U^{-1} + \frac{i}{g} U\, d_\mu U^{-1}, \\ \omega^a = \omega^a(x_\nu, A^b_\mu, A^b_{\mu,\nu}, \ldots). \end{gathered} \tag{21}$$

(The invariance of Eqs (2) under transformations (21) can be easily checked.) Thus, the Yang-Mills equation admits a group of Lie-Bäcklund transformations with the operator

$$\begin{gathered} X_\varphi = f^a_\mu(\varphi)\frac{\partial}{\partial A^a_\mu} + d_\alpha f^a_\mu(\varphi)\frac{\partial}{\partial A^a_{\mu,\alpha}} + d_\alpha d_\beta f^a_\mu(\varphi)\frac{\partial}{\partial A^a_{\mu,\alpha\beta}} + \ldots, \\ f^a_\mu(\varphi) = -\frac{1}{g} d_\mu \varphi^a + f_{abc}\varphi^b A^c_\mu, \\ \varphi^a = \varphi^a(x_\nu, A^b_\mu, A^b_{\mu,\nu}, \ldots). \end{gathered} \tag{22}$$

Note that for the given f^a_μ with the aid of the tensor $P^\pm_{\mu\nu\rho\sigma}$ one can immediately conclude that the defining equation for self-dual fields is identically satisfied

$$P^{\pm}_{\mu\nu\rho\sigma} D^{ab}_{\rho} f^{b}_{\sigma} \Big|_{(4)} = 0.$$

The commutation relations in the algebra of the Lie-Bäcklund operators are

$$[X_{\varphi}, X_{\psi}] = X_{\theta},$$
$$\theta^{a} = X_{\varphi}\psi^{a} - X_{\psi}\varphi^{a} + f_{abc}\psi^{b}\varphi^{c}. \tag{23}$$

Here the Jacobi identity

$$f_{abn}f_{ncd} + f_{bcn}f_{nad} + f_{can}f_{nbd} = 0$$

has been used.

It is reasonable to note also that the Lie-Bäcklund algebra (22) for Eq. (2) is nontrivial: it cannot be obtained from the Lie point symmetry group of the Yang-Mills equation by a simple prolongation (see e. g. [3]). Really, besides local gauge invariance (20), the Yang-Mills equation possesses only the group of conformal transformations. Therefore, by a simple prolongation the group of point transformations of Eq. (2), naturally, cannot lead to the group with operator (22). The same conclusion is valid also for the Maxwell equation.

Similar Lie-Bäcklund operators can be constructed also for the gauge theories with interaction between different fields. For example, let us consider a gauge invariant (with an arbitrary simple group G of the dimension N) system of the Yang-Mills fields coupled with the multiplet of the N scalar particles Φ^{a}:

$$L = -\frac{1}{4} G^{a}_{\mu\nu} G^{a}_{\mu\nu} + \frac{1}{2} (D_{\mu}\Phi^{a})(D_{\mu}\Phi^{a}) - V(\Phi^{2}), \tag{24}$$

where the adjoint representation is chosen,

$$D_{\mu}\Phi^{a} = \partial_{\mu}\Phi^{a} + g f_{abc} A^{b}_{\mu} \Phi^{c},$$
$$a, b, c, \ldots = 1, \ldots, N,$$

and $V(\Phi^{2})$ is the G-invariant polynomial with respect to Φ^{a} (in the Higgs model $V(\Phi^{2})$ is the polynomial of the 4th order with the minimum at $\Phi = v$ (e. g. [12]).

The local gauge transformations, leaving Lagrangian (24) invariant, are given by expressions (20) with $(\omega^{a} = \omega^{a}(x))$

$$\Phi^{a} \to \Phi^{a} + f_{abc}\omega^{b}\Phi^{c},$$
$$\Phi \to U\Phi.$$

As earlier, let us allow for the dependence of the group parameters ω^{a} on all the variables of the extended space:

$$\omega^{a} = \omega^{a}(x_{\nu}, A^{b}_{\mu}, A^{b}_{\mu,\nu}, \ldots, \Phi^{b}, \Phi^{b}_{,\nu}, \ldots) \tag{25}$$

(the invariance of Lagrangian (24) holds as well).

The Lie-Bäcklund operator for System (25) is of the form

$$X_{\omega} = f^{a}_{\mu}(\omega) \frac{\partial}{\partial A^{a}_{\mu}} + d_{\alpha} f^{a}_{\mu}(\omega) \frac{\partial}{\partial A^{a}_{\mu,\alpha}} + \ldots$$
$$\ldots + \bar{f}^{a}(\omega) \frac{\partial}{\partial \Phi_{a}} + d_{\alpha} \bar{f}^{a}(\omega) \frac{\partial}{\partial \Phi^{a}_{,\alpha}} + \ldots, \tag{26}$$

where

$$f^a_\mu(\omega) = -\frac{1}{g} d_\mu\omega^a + f_{abc}\omega^b A^c_\mu,$$

$$\bar{f}^a(\omega) = f_{abc}\omega^b\Phi^c,$$

and ω^a is an arbitrary function (25). As in the case of the pure Yang-Mills theory, the commutator of two Lie-Bäcklund fields has the same form (23) in space (25).

The authors are grateful to M. Kõiv for valuable discussions.

REFERENCES

1. *t'Hooft, G.* // Phys. Rev., 1976, **D14**, № 12, 3432—3450.
2. *Овсянников Л. В.* Лекции по теории групповых свойств дифференциальных уравнений. Новосибирск, Наука, 1966.
3. *Овсянников Л. В.* Групповой анализ дифференциальных уравнений. М., Наука, 1978. (*Ovsyannikov, L. V.* Group Analysis of Differential Equations. New York, Academic, 1982.)
4. *Bateman, H.* // Proc. London Math. Soc., 1909, **8**, 223—264.
5. *Фущич В. И., Никитин А. Г.* Симметрия уравнения Максвелла. Киев, Наукова думка. 1983.
6. *Ибрагимов Н. Х.* Групповые свойства некоторых дифференциальных уравнений. Новосибирск, Наука, 1967.
7. *Mack, G., Salam, A.* // Ann. Phys., 1969, **53**, № 1, 174—202.
8. *Schwarz, F.* // Lett. Math. Phys., 1982, **6**, № 5, 355—359.
9. *Rosenhaus, V., Kiiranen, K.* // Proc. Acad. Sci. ESSR. Phys. Math., 1982, **31**, № 3, 304—313.
10. *Anderson, R. L., Ibragimov, N. H.* Lie-Bäcklund Transformations in Applications. Philadelphia, SIAM, 1979.
11. *Ибрагимов Н. Х.* Группы преобразований в математической физике. М., Наука, 1983. (*Ibragimov, N. H.* Transformation Groups Applied to Mathematical Physics. Dordrecht, Reidel, 1985.)
12. *Abers, E. S., Lee, B. W.* // Phys. Rep., 1973, 9c. 1—141.

Academy of Sciences of the Estonian SSR,
Institute of Physics

Received
Feb. 22, 1988

K. KIIRANEN, V. ROSENHAUS

INVARIANTSUSE RÜHMADEST KALIBRATSIOONI TEOORIATES

On uuritud Maxwelli ja Yang-Millsi väljade sümmeetriat ning näidatud, et üldistatud kalibratsiooniteisendused vastavad Lie-Bäcklundi puutujateisendustele.

К. КИЙРАНЕН. В. РОЗЕНГАУЗ

О ГРУППАХ ИНВАРИАНТНОСТИ В КАЛИБРОВОЧНЫХ ТЕОРИЯХ

Обсуждаются симметрии полей Максвелла и Янга—Миллса. Показано, что обобщенные калибровочные преобразования соответствуют касательным преобразованиям Ли—Бэклунда.