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NORMALLY FLAT SUBMANIFOLDS WITH PARALLEL THIRD FUNDAMENTAL FORM

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1. Introduction. Let M^m be a submanifold in Euclidean space E^n, and h its second fundamental form. Then $\bar{\nabla}h$ is a trilinear symmetric form on M^m with values in the normal bundle $T^{\perp}M^m$ and it is called the third fundamental form. Here $\bar{\nabla}$ is the van der Waerden-Bortolotti connection [1]. If $\bar{\nabla}\bar{\nabla}h=0$, then the third fundamental form is said to be parallel.

The class of submanifolds M^m with parallel third fundamental form contains all the submanifolds M^m with parallel second fundamental form h, i.e. with $\bar{\nabla}h=0$. Each of the latter is a symmetric submanifold, i.e. it admits symmetry in E^n with respect to its arbitrary normal $(n-m)$-plane (D. Ferus [2,3]). The class of all M^m with $\bar{\nabla}\bar{\nabla}h=0$ itself is a subclass of the class of M^m with $\bar{\nabla}_{[X}\bar{\nabla}_{Y]}h=0$, or, equivalently, with $\bar{R}(X, Y)h=0$, where $\bar{R}$ is the curvature operator of $\bar{\nabla}$. A submanifold M^m with this property is called semi-symmetric ([4–7], or semi-parallel [8,9]).

A well-known theorem (R. Walden [10]) states that a submanifold M^m with flat normal connection $\nabla^{\perp}$ and parallel second fundamental form h in E^n is a product of planes or spheres (particularly, straight lines or circles) or a part of such product.

All complete lines $(m=1)$ and surfaces $(m=2)$ with parallel $\bar{\nabla}h$ are found in [11]. Except for lines and surfaces with $\bar{\nabla}h=0$ they are: 1) the plane clothoid, 2) the spherical clothoid, 3) the product of two of them or of one of them with straight line or circle, 4) the spherical regulus in $S^3(r)$, generated by binormal great circles of a twisted clothoid in $S^3(r)$ with spherical natural equations $k_S=as$, $\varkappa_S=\pm\frac{1}{r}$. All lines and surfaces from this list have flat normal connection.

The next theorem gives a full description of submanifolds M^m with flat $\nabla^{\perp}$ and parallel $\bar{\nabla}h$ in E^n.

Theorem. *Let M^m be a submanifold with flat normal connection $\nabla^{\perp}$ and parallel third fundamental form $\bar{\nabla}h$ in E^n. Then it is a part of a complete submanifold, which is either plane or sphere (included straight line and circle) or one of lines and surfaces with parallel $\bar{\nabla}h\neq 0$, listed above, or a product of several of them.*

It is seen that if we keep the flatness of the normal connection in the theorem of [10], but in the parallelity condition replace the second fundamental form h by the third fundamental form $\bar{\nabla}h$, then only two lines and one surface from the previous list are to be added as the new components of the products.

Another consequence is, that M^m in E^n with flat $\nabla^\perp$ and parallel $\overline{\nabla}h$ lies essentially in $E^{3m}\subset E^n$ (if M^m is a product of m spherical clothoids and $n\geqslant 3m$) or in $E^p\subset E^n$ with $p<3m$ (in other cases).

2. Apparatus. If we have a M^m in E^n, then the orthonormal frame bundle $O(E^n)$ can be reduced to $O(M^m, E^n)$, where $\{x, e_1, \dots, e_n\}\in O(M^m, E^n)$ implies $e_i\in T_xM^m$, $e_\alpha\in T_x^\perp M^m$; here and in the following $1\leqslant i, j, k, \dots\leqslant m$; $m+1\leqslant\alpha, \beta, \gamma, \dots\leqslant n$; $1\leqslant I, J, K, \dots\leqslant n$. Identifying the point $x\in E^n$ with its radius vector, we have for $O(E^n)$ the infinitesimal displacement formulae

$$dx=\omega^I e_I,\quad de_I=\omega_I^K e_K,\quad \omega_I^K+\omega_K^I=0 \tag{2.1}$$

and the structure equations

$$d\omega^I=\omega^K\wedge\omega_K^I,\quad d\omega_I^K=\omega_I^L\wedge\omega_L^K. \tag{2.2}$$

Restricting these to $O(M^m, E^n)$, we have $\omega^\alpha=0$ and so $0=d\omega^\alpha=\omega^i\wedge\omega_i^\alpha$. It follows, due to Cartan's lemma, that

$$\omega^\alpha=0,\quad \omega_i^\alpha=h_{ij}^\alpha\omega^j,\quad h_{ij}^\alpha=h_{ji}^\alpha. \tag{2.3}$$

Now by exterior differentiation, using (2.2), we get, due to the same lemma, that

$$\overline{\nabla}h_{ij}^\alpha=h_{ijk}^\alpha\omega^k,\quad h_{ikj}^\alpha=h_{ijk}^\alpha(=\overline{\nabla}_k h_{ij}^\alpha), \tag{2.4}$$

where $\overline{\nabla}$ is the covariant differential operator of the van der Waerden-Bortolotti connection; in particular

$$\overline{\nabla}h_{ij}^\alpha=dh_{ij}^\alpha-h_{kj}^\alpha\omega_i^k-h_{ik}^\alpha\omega_j^k+h_{ij}^\beta\omega_\beta^\alpha. \tag{2.5}$$

In the same manner, (2.4) yield

$$-\overline{\nabla}h_{ijk}^\alpha\wedge\omega^k=h_{kj}^\alpha\Omega_i^k+h_{ik}^\alpha\Omega_j^k-h_{ij}^\beta\Omega_\beta^\alpha, \tag{2.6}$$

where

$$\Omega_i^j=d\omega_i^j-\omega_i^k\wedge\omega_k^j=-\sum_\alpha h_{i[k}^\alpha h_{l]j}^\alpha\,\omega^k\wedge\omega^l, \tag{2.7}$$

$$\Omega_\alpha^\beta=d\omega_\alpha^\beta-\omega_\alpha^\gamma\wedge\omega_\gamma^\beta=-\sum_i h_{i[k}^\alpha h_{l]i}^\beta\,\omega^k\wedge\omega^l \tag{2.8}$$

are the curvature 2-forms of the van der Waerden-Bortolotti connection $\overline{\nabla}$.

The second and third fundamental forms are symmetric $T^\perp M^m$-valued forms, respectively

$$h:(X, Y)\mapsto h_{ij}^\alpha X^iY^je_\alpha,$$

$$\overline{\nabla}h:(X, Y, Z)\mapsto h_{ijk}^\alpha X^iY^jZ^ke_\alpha,$$

where $X=X^ie_i$, $Y=Y^je_j$, $Z=Z^ke_k$. The second (resp. third) fundamental form is said to be parallel if $\overline{\nabla}h=0$ or $\overline{\nabla}h_{ij}^\alpha=0$ (resp. $\overline{\nabla}\overline{\nabla}h=0$ or $\overline{\nabla}h_{ijk}^\alpha=0$).

From (2.6) it follows that if M^m has parallel $\overline{\nabla}h$, then

$$h_{kj}^\alpha\Omega_i^k+h_{ik}^\alpha\Omega_j^k-h_{ij}^\beta\Omega_\beta^\alpha=0. \tag{2.9}$$

A submanifold M^m satisfying (2.9) is called a semi-symmetric submanifold ([4–7]) or a semi-parallel submanifold ([8, 9]), i.e a M^m with parallel $\overline{\nabla}h$ is semi-symmetric.

The normal connection $\nabla^\perp$ of M^m in E^n is said to be flat if $\Omega^\beta_\alpha=0$. Then the matrices $\|h^\alpha_{ij}\|$ and $\|h^\beta_{ij}\|$ commute due to (2.8), and are diagonalizable simultaneously by choosing a suitable orthonormal frame in T_xM^m. In this frame, which is called the principal frame, we have $h^\alpha_{ij}=$ $=k^\alpha_i\delta_{ij}$; its basic directions are called the principal directions and the normal vectors $k_i=k^\alpha_i e_\alpha$ are called the principal curvature vectors of the M^m with flat $\nabla^\perp$ in E^n.

From (2.1), (2.3) and (2.4) it follows for M^m with flat $\nabla^\perp$ that in the principal frame bundle

$$dx=\sum_i \omega^i e_i, \quad de_i=\sum_j \omega^j_i e_j+k_i\omega^i, \tag{2.10}$$

$$dk_i=-\sum_l \langle k_i, k_l\rangle e_l\omega^l+K_i\omega^i+\sum_{j\neq i} L_{ij}\omega^j, \tag{2.11}$$

$$(k_i-k_j)\,\omega^j_i=L_{ij}\omega^i+L_{ji}\omega^j+\sum_{l\neq i}^{l\neq j} E_{ijl}\omega^l, \quad i\neq j, \tag{2.12}$$

where $K_i=h^\alpha_{iii}e_\alpha$, $L_{ij}=h^\alpha_{iij}e_\alpha$ $(i\neq j)$, $E_{ijl}=h^\alpha_{ijl}e_\alpha$ $(i, j, l$ have three distinct values) and summing by Latin indices is denoted only by the sign $\sum$ with necessary hints.

3. Some lemmas. In the case of semi-symmetric M^m with flat $\nabla^\perp$ there are some simplifications in formulae (2.10)—(2.12).

Lemma 1 (see [6]). *Every two principal curvature vectors of a semi-symmetric M^m with flat $\nabla^\perp$ are either equal or orthogonal.*

Proof. In the principal frame bundle (2.7) reduces to

$$\Omega^j_i=-\langle k_i, k_j\rangle\omega^i\wedge\omega^j$$

and, consequently, (2.9) reduces to

$$(k_i-k_j)\langle k_i, k_j\rangle=0.$$

Lemma 2. *If M^m with flat $\nabla^\perp$ in E^n is semi-symmetric, then in* (2.12) *we have $E_{ijl}=0$. If this M^m has r distinct principal curvature vectors $k_{(1)}, \ldots, k_{(r)}$; $1\leqslant r\leqslant m$, and $k_{(\rho)}$ corresponds to principal directions of e_{i_ρ}, then*

$$L_{i_\rho j_\rho}=0, \quad L_{i_\rho j_\sigma}=\lambda_{(\rho)j_\sigma}(k_{(\rho)}-k_{(\sigma)}), \quad \omega^{j_\sigma}_{i_\rho}=\lambda_{(\rho)j_\sigma}\omega^{i_\rho}-\lambda_{(\sigma)i_\rho}\omega^{j_\sigma}, \tag{3.1}$$

where $\varrho, \sigma=1, \ldots, r$; $\varrho\neq\sigma$. If $k_{(\rho)}$ is nonsimple, then $K_{i_\rho}=0$.

Proof. Formulae (2.12) can contain E_{ijl} only if $m>2$, and give immediately that $E_{i_\rho j_\rho l}=0$. Due to symmetry, E_{ijl} is zero, if some two of i, j, l lead to the same $k_{(\rho)}$. It follows that if $r=1$ or $r=2$, then all E_{ijl} are zero. If $r>2$, it remains to consider $E_{i_\rho j_\sigma l_\tau}$ with three distinct ϱ, σ, τ. Now (2.12) give that $k_{(\rho)}-k_{(\sigma)}\| E_{i_\rho j_\sigma l_\tau}$ and, similarly, $k_{(\rho)}-k_{(\tau)}\| E_{i_\rho l_\tau j_\sigma}=$ $=E_{i_\rho j_\sigma l_\tau}$. Thus $E_{i_\rho l_\sigma j_\tau}=0$ due to Lemma 1; so all E_{ijl} are zero.

Let $k_{(\rho)}$ be simple. Then i_ρ takes only one value, which we can denote also (ϱ). If $k_{(\rho)}$ is nonsimple, then $k_{i_\rho}=k_{j_\rho}=k_{(\rho)}$, $i_\rho\neq j_\rho$, and from (2.11) it follows that $K_{i_\rho}=K_{j_\rho}=0$ and $L_{i_\rho l}=L_{j_\rho l}$. Now (2.12) reduces to (3.1).

Next we go to the special case of semi-symmetricity — to the case of parallel $\overline{\nabla}h$.

Lemma 3. *If M^m with flat $\nabla^\perp$ in E^n has parallel third fundamental form $\overline{\nabla}h$, then in principal frame bundle in addition to* (2.10)—(2.12), *where $E_{ijl}=0$, we have*

$$dK_i = -\sum_l \langle K_i, k_l \rangle e_l \omega^l + 3 \sum_{j \neq i} L_{ij} \omega^j_i, \tag{3.2}$$

$$dL_{ij} = -\sum_l \langle L_{ij}, k_l \rangle e_l \omega^l + (2L_{ji} - K_i)\omega^j_i + \sum_{l \neq i}^{l \neq j} L_{il} \omega^l_j; \quad i \neq j, \tag{3.3}$$

and if $r \geqslant 3$, then the coefficients in (3.1) *satisfy*

$$\lambda_{(\rho) j_\sigma} \lambda_{(\rho) l_\tau} = 0, \qquad \lambda_{(\rho) j_\sigma} \lambda_{(\sigma) l_\tau} = \lambda_{(\rho) l_\tau} \lambda_{(\tau) j_\sigma} \tag{3.4}$$

for every three distinct ϱ, σ and τ.

Proof. The parallelity condition of $\overline{\nabla} h$ is $\overline{\nabla} h^\alpha_{ijl} = 0$ and gives immediately (3.2) and (3.3). Taking in this condition $i = i_\rho$, $j = j_\sigma$, $l = l_\tau$ with three distinct ϱ, σ and τ we have, due to $E_{ijl} = 0$, that

$$(L_{i_\rho l_\tau} - L_{j_\sigma l_\tau}) \omega^{j_\sigma}_{i_\rho} + (L_{j_\sigma i_\rho} - L_{l_\tau i_\rho}) \omega^{l_\tau}_{j_\sigma} + (L_{l_\tau j_\sigma} - L_{i_\rho j_\sigma}) \omega^{i_\rho}_{l_\tau} = 0,$$

and substituting here (3.1), we get (3.4).

In the following also the next lemma is useful.

Lemma 4. *Let M^m in E^n be a product of submanifolds M^{m_φ} in E^{n_φ}, i. e. $M^m = M^{m_1} \times \ldots \times M^{m_s}$, $E^n = E^{n_1} \times \ldots \times E^{n_s}$ and every two distinct E^{m_φ} and E^{m_ψ} are totally orthogonal. Then M^m has flat $\nabla^\perp$ (resp. is semi-symmetric, has parallel $\overline{\nabla} h$) iff every M^{m_φ} has flat $\nabla^\perp$ (resp. is semi-symmetric, has parallel $\overline{\nabla} h$).*

This lemma is proved in [5], but follows also immediately from the previous formulae.

4. Proof of the Theorem: the first step. At first we consider the submanifold M^m in Theorem (see Introduction) supposing that one of its principal curvature vectors is zero.

Proposition 1. *If a submanifold M^m with flat $\nabla^\perp$ and parallel $\overline{\nabla} h$ in E^n has zero principal curvature vector with multiplicity m_1, $0 < m_1 < m$, then M^m is a part of a product $M^{m-m_1} \times E^{m_1}$, where M^{m-m_1} has flat $\nabla^\perp$ and parallel $\overline{\nabla} h$, too, and only nonzero principal curvature vectors.*

Proof. Let $k_{(1)} = 0$. Then $K_{i_1} = 0$, $L_{i_1 j} = 0$ due to (2.11) and now (3.2) and (3.3) give $\omega^{j_\sigma}_{i_1} = -\lambda_{(\sigma) i_1} \omega^{j_\sigma}$ and $2L_{j_\sigma i_1} \omega^{\sigma}_{i_1} = 0$, $\sigma \neq 1$. So $\lambda_{(\sigma) i_1} k_{(\sigma)} (-\lambda_{(\sigma) i_1} \omega^{j_\sigma}) = 0$ and thus $\omega^{j_\sigma}_{i_1} = 0$.

Therefore the system $\omega^{j_\sigma} = 0$, where $\sigma = 2, \ldots, r$, is totally integrable, i. e. gives a foliation on M^m and from (2.10) it follows for its leaf that

$$dx = \omega^{i_1} e_{i_1}, \qquad de_{i_1} = \sum_{j_1 \neq i_1} \omega^{j_1}_{i_1} e_{j_1}.$$

Thus the leaf is a plane E^{m_1} or its part.

Similarly, the system $\omega^{i_1} = 0$ is totally integrable. It is easy to see that the corresponding leaf M^{m-m_1} has properties stated in the proposition. Moreover M^m is the product $M^{m-m_1} \times E^{m_1}$.

This proposition reduces the investigation of the arbitrary submanifold M^m with flat $\nabla^\perp$ and parallel $\overline{\nabla} h$ in E^n to the submanifold with the same properties having nonzero principal curvature vectors only.

5. Proof of the Theorem: the second step. Let at least one of principal curvature vectors of the latter submanifold be nonsimple, i. e. have multiplicity > 1.

Proposition 2. *If a submanifold M^m with flat $\nabla^\perp$ and parallel $\overline{\nabla} h$ in E^n has nonzero principal curvature vectors only, among which $k_{(1)}, \ldots, k_{(s)}$ are nonsimple (see denotations in Lemma 2) with multiplicities $m_1, \ldots, m_s$, respectively $(1 \leqslant s \leqslant r;\ \sum_{\sigma=1}^{s} m_\sigma = p \leqslant m)$, then M^m is a part of a product $S^{m_1} \times \ldots \times S^{m_s} \times M^{m-p}$, where M^{m-p} has flat $\nabla^\perp$ and parallel $\overline{\nabla} h$, too, and only simple nonzero curvature vectors.*

Proof. Let $1 \leqslant \varrho \leqslant s$, so that $k_{(\rho)}$ be nonsimple. Lemma 2 gives that $K_{i_\rho} = 0$ and (3.1) hold. Substituting it into (3.2), we have

$$0 = 3 \sum_{j_\sigma}^{\sigma \neq \rho} L_{i_\rho j_\sigma} \omega_{i_\rho}^{j_\sigma} = 3 \sum_{j_\sigma}^{\sigma \neq \rho} \lambda_{(\rho) j_\sigma} (k_{(\rho)} - k_{(\sigma)}) (\lambda_{(\rho) j_\sigma} \omega^{i_\rho} - \lambda_{(\sigma) i_\rho} \omega^{j_\sigma}),$$

thus $\lambda_{(\rho) j_\sigma} = 0$, i. e. $L_{i_\rho j_\sigma} = 0$, if $1 \leqslant \varrho \leqslant s$ and $\sigma \neq \varrho$. Now substitution into (3.3) gives

$$0 = 2 L_{j_\sigma i_\rho} \omega_{i_\rho}^{j_\sigma} = 2 \lambda_{(\sigma) i_\rho} (k_{(\rho)} - k_{(\sigma)}) \lambda_{(\sigma) i_\rho} \omega^{j_\sigma},$$

thus $\lambda_{(\sigma) i_\rho} = 0$ and, consequently, $\omega_{i_\rho}^{j_\sigma} = 0$ if $1 \leqslant \varrho \leqslant s$ and $\sigma \neq \varrho$.

From this it follows that the system $\omega^{i_1} = 0, \ldots, \omega^{i_s} = 0$, as well as the complementary system $\omega^{i_{s+1}} = 0, \ldots, \omega^{i_r} = 0$ both are totally integrable. The submanifold M^m is a product of leaves of corresponding foliations. To see it we have to consider formulae (2.10), (2.11), (3.1), (3.2) and (3,3) on these leaves using Lemma 1, i. e. $\langle k_{(\rho)}, k_{(\sigma)} \rangle = 0$ and its differential consequences $\langle k_{(\rho)}, K_{i_\sigma} \rangle = 0$; $1 \leqslant \varrho \leqslant s < \sigma \leqslant r$ (see [12]).

The leaf of the first foliation is a M^{m-p} with flat $\nabla^\perp$, parallel $\overline{\nabla} h$ and only simple nonzero principal curvature vectors, as is seen from the considered formulae, reduced to this leaf.

The leaf of the second foliation is a M^p with parallel h, due to $E_{ijl} = 0$, $L_{i_\rho j_\rho} = 0$ (see Lemma 2 and (3.1)) and above received $K_{i_\rho} = 0$, $L_{i_\rho j_\sigma} = 0$, where $1 \leqslant \varrho, \sigma \leqslant s$, $\sigma \neq \varrho$. From [10] it follows that this M^p is the product $S^{m_1} \times \ldots \times S^{m_s}$. We can also see it immediately, using the formulae considered above.

This proposition together with the previous one reduces the investigation of the submanifold M^m with flat $\nabla^\perp$ and parallel $\overline{\nabla} h$ in E^n to the submanifold with the same properties having nonzero simple principal curvature vectors only.

6. Proof of the Theorem: the third step. For the last type submanifold we have the next proposition.

Proposition 3. *A submanifold M^m with flat $\nabla^\perp$, parallel $\overline{\nabla} h$ and nonzero simple principal curvature vectors $k_1, \ldots, k_m$ is a line ($m=1$), a surface ($m=2$) or a product of surfaces and a line with the same properties.*

Proof. Let us denote $\|k_i\| = \varkappa_i$. Lemma 1 gives that now $\langle k_i, k_j \rangle = 0$ if $i \neq j$. Using here (2.11) and (3.1) which take the form

$$dk_i = (-\varkappa_i^2 e_i + K_i) \omega^i + \sum_{j \neq i} L_{ij} \omega^j, \tag{6.1}$$

$$L_{ij} = \lambda_{ij} (k_i - k_j), \quad \omega_i^j = \lambda_{ij} \omega^i - \lambda_{ji} \omega^j, \tag{6.2}$$

respectively, we get

$$\langle K_i, k_j \rangle = \varkappa_i^2 \lambda_{ji}. \tag{6.3}$$

By exterior differentiation from (6.1) we obtain, using (2.10), (2.2), (3.2) and (3.3), that

$$0=[-2\langle k_i, \sum_{j\neq i} L_{ij}\omega^j\rangle e_i - \varkappa_i^2 \sum_{j\neq i} \omega_i^j e_j - \sum_{j\neq i} \langle K_i, k_j\rangle e_j\omega^j + 3\sum_{j\neq i} L_{ij}\omega_i^j] \wedge \omega^i +$$

$$+(-\varkappa_i^2 e_i + K_i)\sum_{j\neq i}\omega^j \wedge \omega_j^i +$$

$$+\sum_{j\neq i}[-\sum_{l\neq j}\langle L_{ij}, k_l\rangle \omega^l e_l + (2L_{ji} - K_i)\omega_i^j + \sum_{l\neq i}^{l\neq j} L_{il}\omega_j^l] \wedge \omega^j +$$

$$+\sum_{j\neq i} L_{ij}\sum_{l\neq j}\omega^l \wedge \omega_l^j.$$

Substituting here (6.2) and (6.3), we see that all tangential terms cancel, as well as normal terms with $\omega^i \wedge \omega^j$. Thus we have

$$0=\sum_{j\neq i}^{l\neq i}[\lambda_{il}\lambda_{lj}(k_i - k_l) + \lambda_{ij}\lambda_{jl}(k_i - k_j)]\omega^j \wedge \omega^l,$$

and so in addition to (3.4), in which the first relations take the form $\lambda_{ij}\lambda_{il}=0$, we obtain $\lambda_{ij}\lambda_{jl}=0$, where i, j, l have arbitrary three distinct values; remark that second relations in (3.4) turn to identity.

Let us fix some three distinct values i, j, k and consider the matrix

$$\begin{Vmatrix} 0 & \lambda_{ij} & \lambda_{ik} \\ \lambda_{ji} & 0 & \lambda_{jk} \\ \lambda_{ki} & \lambda_{kj} & 0 \end{Vmatrix}.$$

The last equalities say that in every row and in every side of Sarrus «+»-triangle there can be only one nonzero element. Up to permutations there are only two possibilities: nonzero can be either 1) only λ_{jk} and λ_{kj}, or 2) only λ_{jk} and λ_{ik}. In the second case $\lambda_{ij}=\lambda_{ji}=\lambda_{ki}=$ $=\lambda_{kj}=0$ and therefore, in particular, $L_{ij}=L_{ji}=0$. Now (3.3) gives due to (6.2) that

$$\sum_{l\neq i}^{l\neq j}\lambda_{il}(k_i - k_l)(\lambda_{jl}\omega^j - \lambda_{lj}\omega^l)=0,$$

therefore $\lambda_{ik}\lambda_{jk}=0$.

As a final result we get that in every principal (3×3)-matrix of the $(m\times m)$-matrix

$$\begin{Vmatrix} 0 & \lambda_{12} & \lambda_{13} & \dots & \lambda_{1m} \\ \lambda_{21} & 0 & \lambda_{23} & & \lambda_{2m} \\ \lambda_{31} & \lambda_{32} & 0 & \dots & \lambda_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_{m1} & \lambda_{m2} & \lambda_{m3} & \dots & 0 \end{Vmatrix}$$

nonzero elements can be only in one pair of elements symmetric with respect to principal diagonal. It follows that indices in two such pairs cannot have a common value. Without loss of generality we can take that these pairs are $(\lambda_{12}, \lambda_{21})$, $(\lambda_{34}, \lambda_{43})$, $(\lambda_{56}, \lambda_{65})$, etc. Thus among ω_i^j only ω_1^2, ω_3^4, ω_5^6, etc. can be nonzero; similarly among L_{ij} only L_{12}, L_{21}; L_{34}, L_{43}; L_{56}, L_{65}, etc. and among $\langle K_i, k_j\rangle$, $i\neq j$, only $\langle K_1, k_2\rangle$, $\langle K_2, k_1\rangle$; $\langle K_3, k_4\rangle$, $\langle K_4, k_3\rangle$; $\langle K_5, k_6\rangle$, $\langle K_6, k_5\rangle$, etc. can be nonzero.

Let $2q\leqslant m\leqslant 2q+1$, and let $\langle K_i, k_j\rangle$, $i\neq j$, be not contained in one of these last pairs $\langle K_{2p-1}, k_{2p}\rangle$, $\langle K_{2p}, k_{2p-1}\rangle$, $p=1, \dots, q$, i.e. $\langle K_i, k_j\rangle=0$. By means of (2.11) and (3.2) we have

$$\langle 3\sum_{l\neq i} L_{il}\omega_i^l, k_j\rangle + \langle K_i, K_j\omega^j + \sum_{l\neq j} L_{jl}\omega^l\rangle = 0, \quad i\neq j.$$

The first scalar product after substitution $L_{il}=\lambda_{il}(k_i-k_l)$ reduces to $-3\lambda_{ij}\varkappa_j^2\omega_i^j$, which is zero. Due to lineary independence of $\omega^1, \dots, \omega^m$, we get $\langle K_i, K_j\rangle=0$, if (i, j) is not a $(2p-1, 2p)$ or $(2p, 2p-1)$.

Now the linear hulls $[e_{2p-1}, e_{2p}, k_{2p-1}, k_{2p}, K_{2p-1}, K_{2p}]$ for every two distinct values of p are totally orthogonal; if $m=2q+1$, then the last hull is to be replaced here by corresponding $[e_{2q+1}, k_{2q+1}, K_{2q+1}]$. For a point $x\in M^m$ the plane determined by x and one of these hulls, except the last one, contains the integral surface of the differential system $\omega^1=\dots=\omega^{2p-2}=\omega^{2p+1}=\dots=\omega^m=0$, which is totally integrable, because for $s\neq p$ we have

$$d\omega^{2s-1}=\omega^{2s}\wedge\omega_{2s}^{2s+1}, \quad d\omega^{2s}=\omega^{2s-1}\wedge\omega_{2s-1}^{2s}.$$

In fact, from (2.10), (2.11) and (3.2) it follows for this integral surface that

$$dx=e_{2p-1}\omega^{2p-1}+e_{2p}\omega^{2p},$$

$$de_{2p-1}=e_{2p}\omega_{2p-1}^{2p}+k_{2p-1}\omega^{2p-1},$$

$$de_{2p}=e_{2p-1}\omega_{2p}^{2p-1}+k_{2p}\omega^{2p},$$

$$dk_{2p-1}=-\varkappa_{2p-1}^2e_{2p-1}\omega^{2p-1}+K_{2p-1}\omega^{2p-1}+\lambda_{2p-1,2p}(k_{2p-1}-k_{2p})\omega^{2p},$$

$$dk_{2p}=-\varkappa_{2p}^2e_{2p}\omega^{2p}+K_{2p}\omega^{2p}+\lambda_{2p,2p-1}(k_{2p}-k_{2p-1})\omega^{2p-1},$$

$$dK_{2p-1}=-\langle K_{2p-1}, k_{2p-1}\rangle e_{2p-1}\omega^{2p-1}-\langle K_{2p-1}, k_{2p}\rangle e_{2p}\omega^{2p}+ +3\lambda_{2p-1,2p}(k_{2p-1}-k_{2p})\omega_{2p-1}^{2p},$$

$$dK_{2p}=-\langle K_{2p}, k_{2p-1}\rangle e_{2p-1}\omega^{2p-1}-\langle K_{2p}, k_{2p}\rangle e_{2p}\omega^{2p}+ +3\lambda_{2p,2p-1}(k_{2p}-k_{2p-1})\omega_{2p}^{2p-1}.$$

Analogically, for a point $x\in M^m$, the plane determined by x and the last hull, if $m=2q+1$, contains the integral line of the system $\omega^1=\dots=\omega^{2q}=0$, as is easy to see.

Thus the submanifold M^m is the product of these integral surfaces (surfaces and line if m is odd). The proposition is proved.

7. Proof of the Theorem: the last step. The propositions 1, 2 and 3 reduce the classification problem of submanifolds M^m with flat $\nabla^\perp$ and parallel $\overline{\nabla}h$ in E^n to the problem to describe all lines and surfaces in E^n with properties assumed in Proposition 3. This last problem is solved in [11] and leads to circles (special cases of spheres) and to lines and surfaces listed in Introduction.

For the sake of completeness we give here short new proofs, using the formulae deduced above.

Let us have $m=1$ in Proposition 3. Then (2.10), (2.11) and (3.1) give

$$dx=e_1\,ds, \quad de_1=k_1\,ds, \quad dk_1=-\varkappa_1^2e_1\,ds+K_1\,ds, \quad dK_1=-\langle K_1, k_1\rangle e_1\,ds.$$

Comparing this with the formulae

$$dx=t_1\,ds, \quad dt_1=\varkappa_1t_2\,ds, \quad d(\varkappa_1t_2)=-\varkappa_1^2t_1+\dot\varkappa_1t_2+\varkappa_1\varkappa_2t_3,$$

$$d(\dot\varkappa_1t_2+\varkappa_1\varkappa_2t_3)=-\dot\varkappa_1\varkappa_1t_1+(\ddot\varkappa_1-\varkappa_1\varkappa_2^2)t_2+ +[(\varkappa_1\varkappa_2)^{\cdot}+\dot\varkappa_1\varkappa_2]t_3+\varkappa_1\varkappa_2\varkappa_3t_4,$$

which follows from the Frenet formulae of a line, we see that

$$\ddot\varkappa_1-\varkappa_1\varkappa_2^2=0, \quad (\varkappa_1\varkappa_2)^{\cdot}+\dot\varkappa_1\varkappa_2=0, \quad \varkappa_1\varkappa_2\varkappa_3=0.$$

If here $\varkappa_2=0$ then $\ddot{\varkappa}_1=0$ and $\varkappa_1=as+b$; the case $a=0$ gives a circle, the case $a\neq 0$ can be reduced to $\varkappa_1=as$ and gives a plane clothoid.

If $\varkappa_2\neq 0$, then $\varkappa_3=0$ and $(\ln\varkappa_2)^{\cdot}=-2(\ln\varkappa_1)^{\cdot}$, thus $\varkappa_2=c\varkappa_1^{-2}$, and now $\ddot{\varkappa}_1=c^2\varkappa_1^{-3}$. Hence

$$\varkappa_1=\sqrt{As^2+2Bs+C},\quad AC-B^2=c^2$$

and, after choosing suitable origin for s, we get

$$\varkappa_1=\sqrt{As^2+D},\quad \varkappa_2=\frac{\sqrt{AD}}{As^2+D}.$$

This line lies on a sphere $S^2(r)$, where $r=D^{-1}$, and has the geodesic curvature $\varkappa_g=\sqrt{A}\,s$ (see [11]), i.e. the line is a spherical clothoid.

Let us have now $m=2$ in Proposition 3. So, due to (2.10)

$$dx=e_1\omega^1+e_2\omega^2,\quad de_1=e_2\omega_1^2+k_1\omega^1,\quad de_2=-e_1\omega_1^2+k_2\omega^2,\tag{7.1}$$

where $\langle k_1,k_2\rangle=0$, $\|k_i\|=\varkappa_i\neq 0$, and

$$\omega_1^2=\lambda_{12}\omega^1-\lambda_{21}\omega^2.\tag{7.2}$$

Futher, due to (6.1),

$$dk_1=(-\varkappa^2e_1+K_1)\,\omega^1+L_{12}\omega^2,\tag{7.3}$$

$$dk_2=(-\varkappa_2^2e_2+K_2)\,\omega^2+L_{21}\omega^1,\tag{7.4}$$

where $L_{12}=\lambda_{12}(k_1-k_2)$, $L_{21}=\lambda_{21}(k_2-k_1)$ and therefore (3.2) give

$$dK_1=-\langle K_1,k_1\rangle e_1\omega^1-\varkappa_1^2\lambda_{21}e_2\omega^2+3\lambda_{12}(k_1-k_2)\,(\lambda_{12}\omega^1-\lambda_{21}\omega^2),\tag{7.5}$$

$$dK_2=-\varkappa_2^2\lambda_{12}e_1\omega^1-\langle K_2,k_2\rangle e_2\omega^2+3\lambda_{21}(k_2-k_1)\,(\lambda_{21}\omega^2-\lambda_{12}\omega^1),\tag{7.6}$$

where we have used (6.3); finally, from (3.3) it follows that

$$dL_{12}=-\lambda_{12}(\varkappa_1^2e_1\omega^1-\varkappa_2^2e_2\omega^2)+[2\lambda_{21}(k_2-k_1)-K_1]\,(\lambda_{12}\omega^1-\lambda_{21}\omega^2),\tag{7.7}$$

$$dL_{21}=\lambda_{21}(\varkappa_1^2e_1\omega^1-\varkappa_2^2e_2\omega^2)+[2\lambda_{12}(k_1-k_2)-K_2]\,(\lambda_{21}\omega^2-\lambda_{12}\omega^1).\tag{7.8}$$

If we denote $\lambda_{12}=\gamma_1$, $\lambda_{21}=\gamma_2$, then (7.1) gives, by the differential prolongation, that

$$d\gamma_1=\gamma_{11}\omega^1+\gamma_{12}\omega^2,$$

$$d\gamma_2=\gamma_{21}\omega^1+\gamma_{22}\omega^2,$$

where $\gamma_{12}+\gamma_{21}=\gamma_1^2+\gamma_2^2$. Substituting now $L_{12}=\gamma_1(k_1-k_2)$ and $L_{21}=\gamma_2(k_2-k_1)$ into (7.7) and (7.8), we get

$$2\gamma_1K_1=-(\gamma_{11}+3\gamma_1\gamma_2)\,(k_1-k_2),\tag{7.9}$$

$$2\gamma_2K_2=-(\gamma_{22}+3\gamma_1\gamma_2)\,(k_2-k_1),\tag{7.10}$$

$$\gamma_2K_1+\gamma_1K_2=(\gamma_{12}+\gamma_1^2-2\gamma_2^2)\,(k_1-k_2),$$

$$\gamma_2K_1+\gamma_1K_2=(\gamma_{21}+\gamma_2^2-2\gamma_1^2)\,(k_2-k_1).$$

If $\gamma_1=\gamma_2=0$, then $\omega_1^2=0$, $L_{12}=L_{21}=0$ and M^2 is a product of two lines considered above.

If $\gamma_2=0$, then $\gamma_1K_1=\gamma_{11}(k_1-k_2)$, $\gamma_1K_2=2\gamma_1^2(k_1-k_2)$ and now (6.3) gives $0=\gamma_{11}\varkappa_2^2$, $\gamma_1^2\varkappa_2^2=2\gamma_1^2\varkappa_1^2$. Thus, $\gamma_{11}=0$, $\gamma_1^2(2\varkappa_1^2-\varkappa_2^2)=0$. Here $\gamma_1=0$ leads to the previous case, so let $\varkappa_2^2=2\varkappa_1^2$, $K_1=0$, $K_2=2\gamma_1(k_1-k_2)$. From (7.5), now $3\gamma_1^2(k_1-k_2)\omega^1=0$, and we go to the previous case, too.

So we can take $\gamma_1\gamma_2\neq 0$. The formulae (7.9) and (7.10) together with (6.3) lead to

$$K_1=-\frac{\varkappa_1^2}{\varkappa_2^2}\gamma_2(k_1-k_2),\quad K_2=-\frac{\varkappa_2^2}{\varkappa_1^2}\gamma_1(k_2-k_1). \tag{7.11}$$

It is seen that M^2 lies in a E^4 which is spanned on $x\in M^2$ and vectors e_1, e_2, k_1, k_2. Let us take the orthonormal base $\{e_3, e_4\}$ in the normal space $T_x^\perp M^2$ so that $e_3\| k_1-k_2$. Then

$$k_1=Ae_3+\beta e_4,\quad k_2=Be_3+\beta e_4,\quad AB+\beta^2=0,$$
$$\varkappa_1^2=A^2+\beta^2,\qquad \varkappa_2^2=B^2+\beta^2.$$

Substituting it into (7.3), using (7.11), we get

$$d\beta=0,\quad \omega_3^4=0,\quad dA=(A^2+\beta^2)\left(-\frac{\gamma_2}{B}\omega^1+\frac{\gamma_1}{A}\omega^2\right).$$

From (7.11) it follows that

$$K_1=\frac{A^2+\beta^2}{B}\gamma_2 e_3,\quad K_2=\frac{B^2+\beta^2}{A}\gamma_1 e_3,$$

and now (7.5) and (7.6) give

$$d\gamma_1=(2\alpha-3)\gamma_1\gamma_2\omega^1+[(1+3\alpha^{-1})\gamma_1^2-3\alpha\gamma_2^2]\omega^2,$$
$$d\gamma_2=[(3\alpha+1)\gamma_2^2-3\alpha^{-1}\gamma_1^2]\omega^1+(2\alpha^{-1}-3)\gamma_1\gamma_2\omega^2,$$

where $\alpha=A^2/\beta^2$.

Finally, if we substitute $L_{12}=\frac{A^2+\beta^2}{A}\gamma_1 e_3$ and $L_{21}=-\frac{A^2+\beta^2}{A}\gamma_2 e_3$ into (7.7) and (7.8), we obtain

$$\gamma_1^2:A^2=\gamma_2^2:\beta^2,$$

so that

$$\omega_1^2=\psi(\varepsilon A\omega^1+\beta\omega^2),\quad dA=\varepsilon\frac{A^2+\beta^2}{\beta}\omega_1^2,\quad \varepsilon=\pm 1,$$

$$d\psi=\varepsilon\psi\left(\frac{2\beta}{A}-\frac{3A}{\beta}\right)\omega_1^2.$$

We have got the Pfaff system for M^2 deduced in [11, 12] in another way.

The point with radius-vector $x+\frac{1}{\beta}e_4$ is a fixed one, thus M^2 lies in a sphere $S^3(\beta^{-1})$. Turning $\{e_1, e_2\}$ in T_xM^2 so that $e_{1'}$ is tangent to asymptotic line of M^2 in spherical geometry, we have

$$de_{1'}=-\beta^2x^*\omega^{1'}+\beta e_3\omega^{2'},$$
$$de_{2'}=-\beta^2x^*\omega^{2'}+(\beta\omega^{1'}+2H\omega^{2'}),$$

where $x^*=-\beta^{-1}e_4$ is the radius vector of $x\in M^2$ with origin in the centre of $S^3(\beta^{-1})$ and H is the mean curvature in spherical geometry; $dH=c\omega^{2'}$, $c=\text{const}$. These asymptotic lines are great circles. For their orthogonal trajectory we get the spherical curvature $k_S=cs$ and torsion $\varkappa_S=\pm\beta$, and the asymptotic great circles are its spherical binormals (for details see [11]). So the considered M^2 is the spherical regulus described in the Introduction.

This finishes the proof of the Theorem.

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Ü. LUMISTE

PARALLEELSE KOLMANDA FUNDAMENTAALVORMIGA NORMAALTASASED ALAMMUUTKONNAD

On klassifitseeritud kõik nimetatud alammuutkonnad M^m eukleidilises ruumis E^n ning antud nende täielik geomeetriline kirjeldus. Iga niisugune alammuutkond on järgmiste faktorite korrutis: sfäärid S^{m_κ}, tasand E^{m_0}, tasandilised või sfäärilised klotoidid ning teatavad kahemõõtmelised pinnad sfääridel $S^3(r) \subset E^4$.

Ю. ЛУМИСТЕ

НОРМАЛЬНО ПЛОСКИЕ ПОДМНОГООБРАЗИЯ С ПАРАЛЛЕЛЬНОЙ ТРЕТЬЕЙ ФУНДАМЕНТАЛЬНОЙ ФОРМОЙ

Пусть M^m является подмногообразием в E^n, h его второй фундаментальной формой, ∇ и $\nabla^\perp$ его связностью Леви—Чивита и нормальной связностью, а $\bar{\nabla} = \nabla \oplus \nabla^\perp$ его связностью ван дер Вардена—Бортолотти. Тогда симметрическая трилинейная $T^\perp M^m$-значная форма $\bar{\nabla} h$ называется третьей фундаментальной формой. Если $\bar{\nabla}\bar{\nabla} h = 0$, то говорят о M^m с параллельной $\bar{\nabla} h$. Такое M^m входит в класс полусимметрических M^m, его частным случаем является симметрическое M^m (т. е. M^m с $\bar{\nabla} h = 0$).

Известно [10], что симметрическое M^m с плоской $\nabla^\perp$ в E^n является произведением $S^{m_1} \times \ldots \times S^{m_\kappa} \times E^{m_{\kappa+1}}$. Известно [11], что при $m \leqslant 2$ единственными неприводимыми несимметрическими M^m с плоской $\nabla^\perp$ и параллельной $\bar{\nabla} h$ являются: клотоида, сферическая клотоида, поверхность бинормальных больших окружностей линии на $S^3(r) \subset E^4$ со сферическими натуральными уравнениями $k_s = as$, $\varkappa_s = \pm \frac{1}{r}$.

Теорема. *Подмногообразие M^m с плоской $\nabla^\perp$ и параллельной $\bar{\nabla} h$ в E^n является произведением линий и поверхностей предыдущего списка, сфер и плоскости при произвольном их выборе.*

Следствие. Такое M^m располагается в $E^p \subset E^n$, где $p \leqslant 3m$, причем $p = 3m$ ровно тогда, когда M^m есть произведение m сферических клотоид.