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THE INVARIANT AND THE GENERAL SOLUTIONS OF THE MONGE-AMPÈRE EQUATION

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В. РОЗЕНГАУЗ. ИНВАРИАНТНЫЕ И ОБЩЕЕ РЕШЕНИЯ УРАВНЕНИЯ МОНЖА—АМПЕРА

(Presented by H. Keres)

In the present paper, the Monge-Ampère equation is studied by the methods of the theory of group properties of differential equations [1–2]. It is shown that the set of invariant solutions of the considered equation leads to its general solution, i.e. all solutions of the Monge-Ampère equation are invariant; this fact is the manifestation of the uniqueness of the determination of the given equation by its symmetry group [3–4].

So the Monge-Ampère equation for the surface $u=u(x,y)$ with the zero Gaussian curvature (developable surface)

$$u_{xx}u_{yy}-u_{xy}^2=0. \tag{1}$$

Equation (1) is the simplest example of the fully symmetric system with respect to all variables x, y, u. The invariance group of equation (1) is the 15-parameter group of projective transformations of a 3-dimensional space (x,y,u) [5] (the group of linear fractional transformations). The corresponding Lie algebra L_{15} is formed by all kinds of infinitesimal generators of the form

$$\frac{\partial}{\partial a},\quad a\frac{\partial}{\partial a},\quad a\frac{\partial}{\partial b},\quad aD;$$

$$\forall\, a,b\in(x,y,u),\qquad D\equiv x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+u\frac{\partial}{\partial u}, \tag{2}$$

where the first 12 generators correspond to the group of affine transformations (non-homogeneous linear group). It has been shown in [3, 4] that group (2) and some of its subgroups determine equation (1) uniquely. Let us choose the minimal subalgebras L_6 possessing this property in the form

$$\frac{\partial}{\partial x},\quad \frac{\partial}{\partial y},\quad \frac{\partial}{\partial u},\quad x\frac{\partial}{\partial x},\quad x\frac{\partial}{\partial y},\quad x\frac{\partial}{\partial u}. \tag{3}$$

(We can certainly choose, instead of (3), the corresponding system of basic operators with $x\leftrightarrow y$ change.)

Let us find now the invariant solutions of equation (1) with respect to the subgroup H_1 corresponding to the subalgebra of (3) generated by the infinitesimal operators

$$\frac{\partial}{\partial x},\quad \frac{\partial}{\partial y},\quad \frac{\partial}{\partial u}. \tag{4}$$

The group H_1, however, has no invariant manifolds. Really, if

$$X_\alpha = \xi^i_\alpha \frac{\partial}{\partial x^i} + \eta^k_\alpha \frac{\partial}{\partial u^k}, \quad (i=1, \ldots, n; \ k=1, \ldots, m)$$

are the basic operators of the group H, the form of the invariant H-solution reads [1,2]

$$\Phi^k(I^1, I^2, \ldots, I^t) = 0,$$

where $\{I^i\}$ are independent invariants of the group H: $X_\alpha I^i = 0 \ \forall \alpha, i$; $t = N - R$, $N = n + m$, $R = \mathrm{rank}\, \| \xi^i_\alpha, \eta^k_\alpha \|$. For $H_1: R=3$, $N=3$ and H_1 is the transitive group.

Let us consider now the differential invariants of H_1, i.e. the invariants of the extended (to the derivatives) group H_1. In that case $N=5$, $R=3$ and $t=2$; $\tilde{X}_\alpha = X_\alpha$

$$I^1 = p$$
$$I^2 = q \qquad (p \equiv u_x, \ q \equiv u_y). \tag{5}$$

Thus the invariant $\tilde{H}_1$-solution takes the form

$$\Phi(p, q) = 0, \tag{6}$$

or

$$p = \varphi(q) \tag{6'}$$

with some function Φ, φ.
For the other subgroup of (3):

$$\mathrm{H}_2: \quad x\frac{\partial}{\partial x}, \ x\frac{\partial}{\partial y}, \ x\frac{\partial}{\partial u}, \tag{7}$$

the corresponding system of the extended operators

$$\tilde{X}_1 = x\frac{\partial}{\partial x} - p\frac{\partial}{\partial p}, \quad \tilde{X}_2 = x\frac{\partial}{\partial y} - q\frac{\partial}{\partial p}, \quad \tilde{X}_3 = x\frac{\partial}{\partial u} + \frac{\partial}{\partial p},$$

from which we find the differential invariants

$$I^1 = u - xp - yq, \tag{8}$$
$$I^2 = q,$$

and the invariant $\tilde{H}_2$-solution is of the form

$$\Phi(u - xp - yq, \ q) = 0, \tag{9}$$

or

$$q = \varrho(u - xp - yq). \tag{9'}$$

Putting (6′) and (9′) ((6) and (9)) into equation (1) we can see that equality is satisfied for the arbitrary functions φ and ϱ. However, according to the classical result of Monge [6] the general solution (the first integral) of equation (1) can be represented in the form (6′) or (9′) with arbitrary functions φ and ϱ. So, all the solutions of the Monge-Ampère equation are invariant. Further we shall use the solution of equation (1) in the form

$$\Psi(u_x, u_y, u - xu_x - yu_y) = 0 \tag{10}$$

with an arbitrary function Ψ. (Expression (10) is of Clairaut type equation.)

Up to now we have examined the group of point transformations. Let us make some remarks on the group of tangent (contact) transformations, the infinitesimal operator of which in the canonical form is the following:

$$X = f(x, y, u, p, q) \frac{\partial}{\partial u}. \tag{11}$$

In the case of the sufficiently nontrivial function f the condition $f=0$ $(X=0)$ leads to some relations between the variables of the manifold which must be connected with the form of the solution of the system studied: $f=0$ on the manifold solution. Really, for the Monge-Ampère equation the form of the function f is determined by the form of $\Psi(I^1, I^2, I^3)$ in (10). It is not difficult to show that on the solutions manifold of equation (1) Ψ satisfies the equations

$$u_{yy} D_x \Psi - u_{xy} D_y \Psi = 0,$$

and

$$(u_{yy} D_x^2 - 2u_{xy} D_x D_y + u_{xx} D_y^2)\Psi = 0. \tag{12}$$

(D_i is the total derivative.) On the other hand, applying the twice-extended operator (11) to equation (1), we get for f the same equation (12). It is obvious that besides $\Psi_1(I^1, I^2, I^3)$ the solutions of equation (12) are $x\Psi_2(I^1, I^2, I^3)$ and $y\Psi_3(I^1, I^2, I^3)$. So,

$$f = \Psi_1 + x\Psi_2 + y\Psi_3 \tag{13}$$

with the arbitrary functions $\Psi_i(u_x, u_y, u - xu_x - yu_y)$, which is the well-known result (see e. g. [7]).

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