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## ON MULTIPLIERS OF CONVERGENCE OF SOME CLASSES OF FOURIER SERIES

(Presented by A. Humal)

If  $X$  and  $Y$  are two classes of  $2\pi$ -periodic integrable functions, we say that a two-way infinite sequence of complex numbers  $\lambda = \{\lambda_k\}$  is a multiplier sequence from  $X$  into  $Y$ , and we write  $\lambda \in (X, Y)$  if whenever

$$\sum_{k=-\infty}^{\infty} c_k e^{ikh} \quad (1)$$

is the Fourier series of a function  $f$  in  $X$ , the series

$$\sum_{k=-\infty}^{\infty} \lambda_k c_k e^{ikh} \quad (2)$$

is the Fourier series of a function  $f_\lambda$  in  $Y$ . Let  $C$  denote the class of  $2\pi$ -periodic continuous functions and  $L$  the class of  $2\pi$ -periodic integrable functions. Let  $C_F$  and  $L_F$  denote the subclasses of those functions whose Fourier series converge in the metrics of the corresponding functional space. The class  $(C, C_F)$  was described by J. Karamata [1], and the class  $(L, L_F)$  by G. Goes [2]. They showed that a necessary and sufficient condition for  $\lambda$  to be in those classes is that

$$\|s_n A\|_L = O(1), \quad (n \rightarrow \infty) \quad (3)$$

where

$$s_n A(x) = \sum_{k=-n}^n \lambda_k e^{ikh}.$$

If we restrict ourselves to a subclass of  $C$  or  $L$ , the condition (3) may be relaxed. Multipliers from a class of functions with a given modulus of continuity into the class of functions with convergent Fourier series, have been widely discussed (see e.g. [3-6]).

In the present paper we shall study multipliers from a class of functions whose Fourier series have a convergent subsequence of partial sums, into a class of functions that have another convergent subsequence of partial sums. As far as we know, the first result in this direction was obtained by G. S. Yanakov [7], who considered the case when the first subsequence of partial sums is lacunary, and the other is dense.

We shall prove our results only for integral metrics, since the proof of analogous statements for continuous metrics would differ only in details.

Let

$$s_n f(x) = \sum_{k=-n}^n c_k e^{ikh}$$

denote the  $n$ -th partial sum of the series (1) and let  $\mathfrak{N} = \{n_k\}$  be a strictly

increasing sequence of positive integers. We shall say that  $f$  belongs to the class  $L_F(\mathfrak{N})$  if the sequence  $\{s_{n_k}f\}$  converges in integral metrics, i. e.  $\|f - s_{n_k}f\|_L = o(1)$  ( $k \rightarrow \infty$ ). Given another increasing sequence of positive integers  $\mathfrak{M}$ , we may consider the class of multipliers  $(L_F(\mathfrak{N}), L_F(\mathfrak{M}))$ .

Throughout this paper we shall assume that  $\lambda$  is of the class  $S$  of Fourier-Stieltjes sequences (which in particular characterizes the class  $(L, L)$ ).

Let  $v_{n,m}f$  denote the de la Vallée Poussin means of the series (1)

$$v_{n,m}f(x) = \frac{1}{m} (s_n f(x) + s_{n+1} f(x) + \dots + s_{n+m-1} f(x))$$

and let  $v_{n,m}(x)$  be the corresponding kernel

$$v_{n,m}f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) v_{n,m}(x-t) dt.$$

Let

$$K_n(x) = \sum_{k=-n}^n \left(1 - \frac{|k|}{n+1}\right) e^{ikx} = 2 \left(\frac{1}{2} + \sum_{k=1}^n \left(1 - \frac{k}{n+1}\right) \cos kx\right)$$

be the symmetrical Fejér kernel. Then we may write  $v_{n,n}(x) = 2K_{2n-1}(x) - K_{n-1}(x)$ , and as  $\|K_n\|_L \leq 1$  we shall have  $\|v_{n,n}\|_L \leq 3$ . Writing  $s_n \Delta$  and  $v_{n,m} \Delta$  respectively for the partial sums and the de la Vallée Poussin means of the series

$$\sum_{k=-\infty}^{\infty} \lambda_k e^{ikx}, \quad (4)$$

the condition  $\lambda \in S$  is equivalent to the condition

$$\|v_{n,n} \Delta\|_L = O(1) \quad (5)$$

(see [8], p. 137).

Let  $F_n$  denote the set of all trigonometric polynomials of an order not higher than  $n$ , and  $F_n^\perp$  the set of all trigonometric polynomials orthogonal to  $F_n$ . We shall use the notations  $E_n(f)_L = \inf_{P_n \in F_n} \|f - P_n\|_L$ , and  $E_n^\perp(f)_L = \inf_{P_n^\perp \in F_n^\perp} \|f - P_n^\perp\|_L$  for the best approximations of the function  $f$  with the respective sets.

Given two sequences of positive integers  $\mathfrak{N}$  and  $\mathfrak{M}$ , we may, for every  $m_j$ , consider the constants

$$R_j(\lambda) = \min \{E_{n_{k-1}}(s_{m_j} \Delta)_L, E_{n_k}(s_{n_k} \Delta - s_{m_j} \Delta)_L\},$$

where  $n_{k-1} \leq m_j \leq n_k$ .

**Theorem 1.** *Let  $\lambda \in S$ . Then a necessary and sufficient condition for the sequence  $\lambda$  to be in  $(L_F(\mathfrak{N}), L_F(\mathfrak{M}))$  is that*

$$R_j(\lambda) = O(1) \quad (j \rightarrow \infty). \quad (6)$$

**Proof.** First we shall prove that the condition is sufficient. As  $\lambda \in S$ , we have  $f_\lambda \in L$ , and since that implies

$$\|f_\lambda - v_{n,n} f_\lambda\|_L = o(1) \quad (n \rightarrow \infty)$$

we obtain the estimate

$$\|f_\lambda - s_{n_k} f_\lambda\|_L \leq \|f_\lambda - v_{n_k, n_k} f_\lambda\|_L + \|v_{n_k, n_k} f_\lambda - s_{n_k} f_\lambda\|_L =$$

$$=o(1) + \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} \{f(t) - s_{n_k} f(t)\} \cdot v_{n_k, n_k}(\cdot - t) dt \right\|_L \leq \quad (7)$$

$$\leq o(1) + \|f - s_{n_k} f\|_L \cdot \|v_{n_k, n_k}\|_L.$$

Thus if  $\lambda \in S$  and  $f \in L_F(\mathfrak{R})$ , it follows that

$$\|f_\lambda - s_{n_k} f_\lambda\|_L = o(1) \quad (k \rightarrow \infty). \quad (8)$$

Therefore it is sufficient to consider the difference  $s_{n_k} f_\lambda(x) - s_{m_j} f_\lambda(x)$ . We have

$$s_{n_k} f_\lambda(x) - s_{m_j} f_\lambda(x) =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \{s_{n_k} f(t) - s_{n_{k-1}} f(t)\} \{s_{n_k} \Lambda(x-t) - s_{m_j} \Lambda(x-t)\} dt.$$

Observing that the first factor of the integrand is a polynomial of order  $n_k$  orthogonal to  $F_{n_{k-1}}$ , we see that for arbitrary  $P_{n_k}^\perp \in F_{n_k}^\perp$  and  $T_{n_{k-1}} \in F_{n_{k-1}}$

$$s_{n_k} f_\lambda(x) - s_{m_j} f_\lambda(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \{s_{n_k} f(t) - s_{n_{k-1}} f(t)\} \{P_{n_k}^\perp(x-t) +$$

$$+ s_{n_k} \Lambda(x-t) - s_{m_j} \Lambda(x-t) + T_{n_{k-1}}(x-t)\} dt. \quad (9)$$

Applying Young's inequality, we get

$$\|s_{n_k} f_\lambda - s_{m_j} f_\lambda\|_L \leq \|s_{n_k} f - s_{n_{k-1}} f\|_L \cdot \|P_{n_k}^\perp + s_{n_k} \Lambda - s_{m_j} \Lambda + T_{n_{k-1}}\|_L. \quad (10)$$

In view of the arbitrariness of the polynomials  $P_{n_k}^\perp$  and  $T_{n_{k-1}}$ , we may let

$$P_{n_k}^\perp(x) = v_{n_k, n_k} \Lambda(x) - s_{n_k} \Lambda(x)$$

and  $T_{n_{k-1}}$  be the polynomial of the best approximation of  $s_{m_j} \Lambda$ . Hence

$$\|P_{n_k}^\perp + s_{n_k} \Lambda - s_{m_j} \Lambda + T_{n_{k-1}}\|_L = \|v_{n_k, n_k} \Lambda - s_{n_k} \Lambda + s_{n_k} \Lambda - s_{m_j} \Lambda + T_{n_{k-1}}\|_L \leq$$

$$\leq \|v_{n_k, n_k} \Lambda\|_L + \|s_{m_j} \Lambda - T_{n_{k-1}}\|_L.$$

By virtue of (5), the first term on the right is  $O(1)$ , thus

$$\|s_{n_k} f_\lambda - s_{m_j} f_\lambda\|_L \leq \|s_{n_k} f - s_{n_{k-1}} f\|_L \cdot \{O(1) + E_{n_{k-1}}(s_{m_j} \Lambda)_L\}. \quad (11)$$

On the other hand, if we let  $T_{n_{k-1}} \equiv 0$  and take the infimum over all  $P_{n_k}^\perp \in F_{n_k}^\perp$ , we obtain

$$\|s_{n_k} f_\lambda - s_{m_j} f_\lambda\|_L \leq \|s_{n_k} f - s_{n_{k-1}} f\|_L \cdot E_{n_k}^\perp(s_{n_k} \Lambda - s_{m_j} \Lambda)_L. \quad (12)$$

Collecting the estimates (11) and (12), we get

$$\|s_{n_k} f_\lambda - s_{m_j} f_\lambda\|_L \leq \|s_{n_k} f - s_{n_{k-1}} f\|_L \cdot \min\{O(1) + E_{n_{k-1}}(s_{m_j} \Lambda)_L,$$

$$E_{n_k}^\perp(s_{n_k} \Lambda - s_{m_j} \Lambda)_L\} \leq \|s_{n_k} f - s_{n_{k-1}} f\|_L \cdot \{R_j(\lambda) + O(1)\}. \quad (13)$$

Therefore, if the condition (6) holds, then  $f_\lambda \in L_F(\mathfrak{R})$  for every  $f \in L_F(\mathfrak{R})$ .

To prove the necessity of (6), we first introduce some auxiliary notations. For  $m \leq n$  let

$$\sigma_{n,m} \Lambda(x) = \sum_{k=0}^m \left(1 - \frac{k}{m+1}\right) \left(\lambda_{n-k} e^{i(n-k)x} + \lambda_{-(n-k)} e^{-i(n-k)x}\right),$$

and

$$\varrho_{n,m}\Lambda(x) = \sum_{k=1}^m \left(1 - \frac{k}{m+1}\right) \left(\lambda_{n+k} e^{i(n+k)x} + \lambda_{-(n+k)} e^{-i(n+k)x}\right).$$

According to the definition,  $\sigma_{n,m} \in F_n$  and  $\varrho_{n,m} \in F_n^\perp$ . They also have the properties

$$\sigma_{n,m}\Lambda(x) + \varrho_{n,m}\Lambda(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v_{n+m, n+m}\Lambda(t) \{2 \cos n(x-t) \cdot K_m(x-t)\} dt; \quad (14)$$

where  $K_m$  is the symmetrical Fejér kernel, and

$$\sigma_{n,m}\Lambda(x) + \varrho_{n-m-1,m}\Lambda(x) = s_n\Lambda(x) - s_{n-m-1}\Lambda(x). \quad (15)$$

As  $\lambda \in S$ , we have by (5) and Young's inequality that

$$\|\sigma_{n,m}\Lambda + \varrho_{n,m}\Lambda\|_L = O(1). \quad (16)$$

Given a sequence  $\mathfrak{N} = \{n_k\}$ , we define three sets of indices ( $[x]$  denotes the entire part of  $x$ ):

$$I_1(\mathfrak{N}) = \{m : [(n_{k-1} + n_k)/2] < m < n_k\},$$

$$I_2(\mathfrak{N}) = \{m : n_{k-1} < m \leq \min\{[(n_{k-1} + n_k)/2], 2n_{k-1}\}\},$$

$$I_3(\mathfrak{N}) = \{m : 2n_{k-1} < m \leq [(n_{k-1} + n_k)/2]\}.$$

Suppose now that (6) does not hold. Then we can find a sequence  $\{j(p)\}_{p=1}^\infty$  such that

$$R_{j(p)}(\lambda) \geq p^3 \quad (17)$$

and

$$n_{k(p)-1} < m_{j(p)} < n_{k(p)} < m_{j(p+1)}. \quad (18)$$

Let

$$r_j = \begin{cases} n_k - m_j - 1, & \text{if } m_j \in I_1(\mathfrak{N}), \\ m_j - n_{k-1} - 1, & \text{if } m_j \in I_2(\mathfrak{N}), \\ [(m_j - 1)/2], & \text{if } m_j \in I_3(\mathfrak{N}). \end{cases} \quad (19)$$

During the rest of the proof we shall omit some of the common indices, assuming  $k = k(p)$ ,  $m = m_j = m_{j(p)}$  and  $r = r_j = r_{j(p)}$ . If  $m \in I_1(\mathfrak{N}) \cup I_2(\mathfrak{N})$ , let

$$f_p(x) = K_r(x) \cdot 2 \cos mx$$

and in the case  $m \in I_3(\mathfrak{N})$  let

$$f_p(x) = K_r(x) \cdot 2 \cos (2r+1)x.$$

The functions  $f_p$  so defined are trigonometric polynomials of order  $n_k$  orthogonal to  $F_{n_{k-1}}$ . Since  $\|f_p\|_L \leq 2$ , by Levi's theorem the series

$$\sum_{p=1}^{\infty} \frac{1}{p^2} f_p(x) \quad (20)$$

represents an integrable function  $f$ . The estimate

$$\|f - s_n f\|_L = \left\| \sum_{k>l} \frac{1}{p^2} f_p \right\| \leq 2 \sum_{k(p)>l} \frac{1}{p^2} = o(1) \quad (21)$$

( $l \rightarrow \infty$ ) establishes that  $f \in L_F(\mathfrak{N})$ .

To prove that  $f_\lambda$  does not belong to  $L_F(\mathfrak{N})$ , we first suppose that  $m \in I_1(\mathfrak{N})$ . As then  $n_k - r - 1 = m$ , it follows from the construction of  $f$  that

$$\|s_{n_k} f_\lambda - s_m f_\lambda\|_L = \frac{1}{p^2} \|(f_p)_\lambda - s_m (f_p)_\lambda\|_L = \frac{1}{p^2} \|Q_{m,r} A\|_L. \quad (22)$$

By virtue of (15) we may write

$$\begin{aligned} \|Q_{m,r} A\|_L &= \|Q_{m,r} A + \sigma_{n_k,r} A - \sigma_{n_k,r} A + Q_{n_k,r} A - Q_{n_k,r} A\|_L \geq \\ &\geq \|s_{n_k} A - s_m A + Q_{n_k,r} A\|_L - \|\sigma_{n_k,r} A + Q_{n_k,r} A\|_L. \end{aligned}$$

Applying (16) and using the fact that  $Q_{n_k,r} A \in F_{n_k}^\perp$ , we obtain

$$\|Q_{m,r} A\|_L \geq E_{n_k}^\perp (s_{n_k} A - s_m A)_L - O(1), \quad (23)$$

and hence, by (17)

$$\|s_{n_k} f_\lambda - s_m f_\lambda\|_L \geq p - o(1).$$

If  $m \in I_2(\mathfrak{N})$ , let us consider the difference  $s_m f_\lambda - s_{n_{k-1}} f_\lambda$ . As in the preceding argument, we have by the construction of  $f$  that

$$\|s_m f_\lambda - s_{n_{k-1}} f_\lambda\|_L = \frac{1}{p^2} \|s_m (f_p)_\lambda\|_L = \frac{1}{p^2} \|\sigma_{m,r} A\|_L. \quad (24)$$

Observing that in this case  $m - r - 1 = n_{k-1}$  and applying (15) and (16), we get in a similar manner

$$\begin{aligned} \|\sigma_{m,r} A\|_L &= \|\sigma_{m,r} A + Q_{m-r-1,r} A + \sigma_{m-r-1,r} A - \sigma_{m-r-1,r} A - \\ &- Q_{m-r-1,r} A\|_L \geq \|s_m A - s_{n_{k-1}} A + \sigma_{n_{k-1},r} A\|_L - O(1). \end{aligned}$$

Since  $n_{k-1} - r > 0$ , the difference  $s_{n_{k-1}} A - \sigma_{n_{k-1},r} A$  is a polynomial of order  $n_{k-1}$ , and therefore

$$\|\sigma_{m,r} A\|_L \geq E_{n_{k-1}} (s_m A)_L - O(1). \quad (25)$$

By virtue of (17) and (24), this gives us

$$\|s_m f_\lambda - s_{n_{k-1}} f_\lambda\|_L \geq p - o(1).$$

The last case to be considered is  $m \in I_3(\mathfrak{N})$ . In view of the construction of  $f$ , we have by (17) and (5)

$$\begin{aligned} \|s_m f_\lambda - s_{n_{k-1}} f_\lambda\|_L &= \frac{1}{p^2} \|s_m (f_p)_\lambda\|_L = \frac{1}{p^2} \|s_m A - v_{r,r} A\|_L \geq \\ &\geq \frac{1}{p^2} \{\|s_m A\|_L - \|v_{r,r} A\|_L\} \geq p - o(1). \end{aligned} \quad (26)$$

This concludes the proof of Theorem 1.

Let  $\lambda \in S$ . Then, similarly to (14), we may write ( $m \geq r$ )

$$v_{m,r} A(x) - v_{m-r,r} A(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v_{2m,2m} A(t) \cdot \{2 \cos m(x-t) \cdot K_r(x-t)\} dt,$$

and by virtue of (5), applying Young's inequality, we shall obtain

$$\|v_{m,r} A - v_{m-r,r} A\|_L = O(1). \quad (27)$$

Let  $r_j$  be defined by (19) and  $n_{k-1} < m_j < n_k$ . Consider the sequence  $\{Q_j(\lambda)\}$  where

$$Q_j(\lambda) = \begin{cases} \|v_{m_j, r_j+1} A - s_{m_j} A\|_L, & \text{if } m_j \in I_1(\mathfrak{N}), \\ \|s_{m_j} A - v_{n_{k-1}, r_j+1} A\|_L, & \text{if } m_j \in I_2(\mathfrak{N}), \\ \|s_{m_j} A\|_L, & \text{if } m_j \in I_3(\mathfrak{N}). \end{cases}$$

We shall restate Theorem 1 as follows.

**Theorem 2.** *Let  $\lambda \in S$ . Then a necessary and sufficient condition for  $\lambda$  to be in  $(L_F(\mathfrak{N}), L_F(\mathfrak{M}))$  is that*

$$Q_j(\lambda) = O(1) \quad (j \rightarrow \infty). \quad (28)$$

**Proof.** Let  $q_j = r_j + 1$ . In view of (27) we have: if  $m_j \in I_1(\mathfrak{N})$ , then

$$Q_j(\lambda) = \|v_{m_j, q_j} A - s_{m_j} A\|_L \geq \|s_{n_k} A - s_{m_j} A + (v_{n_k, q_j} A - s_{n_k} A)\|_L - \\ - \|v_{n_k, q_j} A - v_{m_j, q_j} A\|_L \geq E_{n_k}^\perp (s_{n_k} A - s_{m_j} A)_L - O(1) \geq R_j(\lambda) - O(1),$$

since  $v_{n_k, q_j} A - s_{n_k} A \in F_{n_k}^\perp$  and  $n_k - q_j = m_j$ ;

if  $m_j \in I_2(\mathfrak{N})$ , then similarly  $v_{n_{k-1}, q_j} A \in F_{n_{k-1}}$  and  $n_{k-1} - q_j > 0$ , hence

$$Q_j(\lambda) = \|s_{m_j} A - v_{n_{k-1}, q_j} A\|_L \geq \|s_{m_j} A - v_{n_{k-1}, q_j} A\|_L - \\ - \|v_{n_{k-1}, q_j} A - v_{n_{k-1}, q_j} A\|_L \geq E_{n_{k-1}} (s_{m_j} A)_L - O(1) \geq R_j(\lambda) - O(1);$$

if  $m_j \in I_3(\mathfrak{N})$ , then the inequality  $Q_j(\lambda) \geq E_{n_{k-1}} (s_{m_j} A)_L \geq R_j(\lambda)$  is obvious.

Therefore, by Theorem 1 the condition (28) is sufficient for  $\lambda$  to be in  $(L_F(\mathfrak{N}), L_F(\mathfrak{M}))$ .

To prove that (28) is also necessary, suppose that it does not hold. Then, as before, we can find a sequence  $\{j(p)\}$  such that

$$Q_{j(p)}(\lambda) \geq p^3, \quad (29)$$

and that (18) holds. Let  $f \in L_F(\mathfrak{N})$  be defined by (20). Since (again assuming  $k = k(p)$ ,  $m = m_j \in m_{j(p)}$  and  $r = r_j = r_{j(p)}$ ): if  $m \in I_1(\mathfrak{N})$ , then

$$p^3 \leq Q_j(\lambda) = \|\omega_{m, q} A - s_m A\|_L = \|Q_{m, r} A\|_L, \quad (30)$$

if  $m \in I_2(\mathfrak{N})$ , then

$$p^3 \leq Q_j(\lambda) = \|s_m A - v_{n_{k-1}, q} A\|_L = \|\sigma_{m, r} A\|_L, \quad (31)$$

and if  $m \in I_3(\mathfrak{N})$ , then, by (5),

$$p^3 \leq Q_j(\lambda) = \|s_m A\|_L \leq \|s_m A - v_{r, r} A\|_L + O(1), \quad (32)$$

it follows, collecting (22) and (30), (24) and (31), (26) and (32), that  $f_\lambda$  is not in  $L_F(\mathfrak{M})$ . That proves the necessity of (28).

Therefore in Theorem 1, instead of the best approximations, we may consider approximations by linear means.

As illustrations we shall prove two corollaries.

**Corollary 1.** *Let  $\lambda \in S$ . Let for every  $j$  ( $n_{k-1} < m_j < n_k$ )  $m_j/n_{k-1} \geq C_1 > 1$  and  $n_k/m_j \geq C_2 > 1$ . Then  $\lambda$  is in  $(L_F(\mathfrak{N}), L_F(\mathfrak{M}))$  if and only if it is in  $(L, L_F(\mathfrak{M}))$ . (Here and later on by  $C, C_1, C_2, \dots$  we shall denote absolute positive constants, not necessarily the same on each occurrence.)*

**Proof.** Let  $r_j$  be defined by (19) and let  $q_j = r_j + 1$ . It is well known (see [9], p. 35) that under the assumptions made,  $\|v_{m_j, q_j}\|_L = O(1)$  and  $\|v_{n_{k-1}, q_j}\|_L = O(1)$ . Thus, by (5), the application of Young's inequality gives us: if  $m_j \in I_1(\mathfrak{N})$ , then

$$\|v_{m_j, q_j} A\|_L = \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} v_{n_k, n_k} A(t) \cdot v_{m_j, q_j}(\cdot - t) dt \right\|_L = O(1),$$

and if  $m_j \in I_2(\mathfrak{N})$ , then

$$\|v_{n_{k-1}, q_j} A\|_L = \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} v_{m_j, m_j} A(t) \cdot v_{n_{k-1}, q_j}(\cdot - t) dt \right\|_L = O(1).$$

Hence  $Q_j(\lambda) = \|s_{m_j} A\|_L + O(1)$ . Since by a theorem by G. Goes [2] the type  $(L, L_F(\mathfrak{N}))$  is characterized by the condition

$$\|s_{m_j} A\|_L = O(1)$$

(Goes proved it for the case where  $\mathfrak{N}$  is the sequence of all positive integers, but it remains valid if  $\mathfrak{N}$  is a subsequence of that sequence), this concludes the proof of the corollary.

Let us consider now only real-valued functions (i.e. let  $c_{-n} = \bar{c}_n$  and  $\lambda_{-n} = \bar{\lambda}_n = \lambda_n$ ). Let  $\lambda$  be a quasi-convex sequence, that is

$$\sum_{n=1}^{\infty} n |\Delta^2 \lambda_n| < \infty,$$

where  $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$  and  $\Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1}$ .

Corollary 2. A necessary and sufficient condition for a quasi-convex sequence  $\lambda$  to be in  $(L_F(\mathfrak{N}), L_F(\mathfrak{N}))$  is that  $(n_{h-1} < m_j < n_h)$

$$\lambda_{m_j} \cdot \log(1 + \min\{n_h - m_j, m_j - n_{h-1}\}) = O(1) \quad (j \rightarrow \infty).$$

Proof. It is known that if  $\lambda$  is quasi-convex, then there exist two constants  $A$  and  $B$  such that

$$\lambda_h = Ak + B + \mu_h$$

and  $\mu_h \rightarrow 0$  ( $h \rightarrow \infty$ ) (see, e.g., [4]). We shall show that the case  $\lambda$  unbounded will not arise. Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k} \cos kx.$$

It is the Fourier series of a square-integrable function  $f$  and therefore converges in integral metrics. But if  $A \neq 0$  then the norms of the partial sums of the transformed series

$$s_n f_{\lambda}(x) = \sum_{k=1}^n \frac{1}{k} \lambda_k \cos kx = A \sum_{k=1}^n \cos kx + \sum_{k=1}^n \frac{1}{k} (B + \mu_k) \cos kx$$

tend to infinity

$$\|s_n f_{\lambda}\|_L = \frac{2}{\pi} |A| \log n + O(1) \quad (n \rightarrow \infty).$$

Thus if  $\lambda \in (L_F(\mathfrak{N}), L_F(\mathfrak{N}))$ , we must have  $A = 0$ . In the following we shall suppose that  $\lambda$  is a bounded sequence. Then it is well known that

$$k \Delta \lambda_{k-1} \rightarrow 0 \quad (k \rightarrow \infty).$$

By Corollary 1 we may consider only such a case where either  $m_j - n_{h-1} = o(m_j)$  or  $n_h - m_j = o(m_j)$ , since (see [8], p. 183) if  $\lambda$  is quasi-convex we have

$$\|s_m, \Lambda\|_L = \frac{4}{\pi^2} |\lambda_m| \log m_j + O(1).$$

This proves the corollary for the case  $m_j \in I_3(\mathfrak{M})$ .

Suppose first that  $m_j - n_{k-1} = o(m_j)$ , that is  $m_j \in I_2(\mathfrak{M})$ . Consider the difference

$$s_m, \Lambda(x) - s_{n_{k-1}}, \Lambda(x) = 2 \sum_{\nu=n_{k-1}+1}^{m_j} \lambda_\nu \cos \nu x.$$

Let  $r_j$  be, as before, defined by (19) and for convenience's sake let us again omit the common indices — if it is not necessary to emphasize them — assuming  $m = m_j$ ,  $r = r_j$  and  $n = n_{k-1}$ . Applying Abel's transformation twice, we get

$$\begin{aligned} s_m \Lambda(x) - s_n \Lambda(x) = & 2 \left\{ \lambda_m \sum_{\nu=0}^r \cos(n+1+\nu)x + \right. \\ & + \Delta \lambda_{m-1} \sum_{\nu=0}^{r-1} (r-\nu) \cos(n+1+\nu)x + \\ & \left. + \sum_{l=0}^{r-2} \Delta^2 \lambda_{n+l+1} \sum_{\nu=0}^l (l+1-\nu) \cos(n+1+\nu)x \right\}. \end{aligned}$$

Let

$$\begin{aligned} T(x) = & 2 \left\{ \lambda_m \sum_{\nu=1}^r \cos(n+1-\nu)x + \Delta \lambda_{m-1} \sum_{\nu=1}^{r-1} (r-\nu) \cos(n+1-\nu)x + \right. \\ & \left. + \sum_{l=0}^{r-2} \Delta^2 \lambda_{n+l+1} \sum_{\nu=1}^l (l+1-\nu) \cos(n+1-\nu)x \right\}. \end{aligned}$$

Since  $m \in I_2(\mathfrak{M})$ , we have  $n - r \geq 0$ , hence  $T$  is a polynomial of order  $n_{k-1}$ , and

$$\begin{aligned} s_m \Lambda(x) - s_n \Lambda(x) + T(x) = & 2 \{ 2\lambda_m D_r(x) + \Delta \lambda_{m-1} \cdot r \cdot K_{r-1}(x) + \\ & + \sum_{l=0}^{r-2} \Delta^2 \lambda_{n+l+1} \cdot (l+1) \cdot K_l(x) \} \cdot \cos(n+1)x, \end{aligned} \quad (33)$$

where  $D_r$  is the Dirichlet kernel of order  $r$ . Thus

$$\begin{aligned} E_{n_{k-1}}(s_m, \Lambda)_L & \leq \|s_m \Lambda - s_n \Lambda + T\|_L \leq \\ & \leq 4 |\lambda_m| \cdot \|D_r\|_L + O(1) \leq C |\lambda_m| \cdot \log r_j + O(1). \end{aligned}$$

On the other hand, let  $\varphi_{n,r}(x) = 4D_r(x) \cos(n+1)x$ . Since  $m - r > n$  and  $\|K_r\|_L \leq 1$ , we have

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_{n,r}(t) \cdot K_r(x-t) \cdot 2 \cos m(x-t) dt = \\ & = \frac{1}{2\pi} \int_{-\pi}^{\pi} \{ \varphi_{n,r}(t) - T_n^*(t) \} \cdot K_r(x-t) \cdot 2 \cos m(x-t) dt, \end{aligned} \quad (34)$$

where  $T_n^*$  is the polynomial of the best approximation of  $\varphi_{n,r}$ . Therefore, applying Young's inequality, we obtain

$$\left\| 2 \sum_{\nu=0}^r \left( 1 - \frac{\nu}{r+1} \right) \cos(m-\nu) \cdot \right\|_L \leq 2E_n(\varphi_{n,r})_L.$$

The sum on the left may be rewritten as



$$(1 + K_r(x)) \cos mx + \tilde{K}_r(x) \sin mx.$$

Here

$$\tilde{K}_r(x) = 2 \sum_{\nu=1}^r \left( 1 - \frac{\nu}{r+1} \right) \sin \nu x$$

is the symmetrical conjugate Fejér kernel. Since

$$\| (1 + K_r) \cos m \cdot \|_L = O(1),$$

we have to estimate the second term. It is known (see, e.g., [8], p. 92) that  $\tilde{K}_r(x) > 0$  for  $0 < x < \pi$ ,  $|\tilde{K}_r(x)| \leq r$  and  $\tilde{K}_r(x) = \cot(x/2) - H_r(x)$ , where  $|H_r(x)| \leq C/\{(r+1)x^2\}$ . We may write

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{K}_r(x) \sin mx| dx &= \frac{1}{\pi} \int_0^{\pi} |\sin mx| \cdot \tilde{K}_r(x) dx = \\ &= \frac{1}{\pi} \int_0^{\pi/r} |\sin mx| \tilde{K}_r(x) dx + \frac{2}{\pi} \int_{\pi/r}^{\pi} |\sin mx| \frac{1}{2} \cot \frac{x}{2} dx - \\ &\quad - \frac{1}{\pi} \int_{\pi/r}^{\pi} |\sin mx| H_r(x) dx, \end{aligned}$$

and in the view of the above mentioned properties, observe that the first and the third integrals are  $O(1)$ . Thus, noting that  $(1/x) - (1/2)\cot(x/2) = O(1)$  ( $0 < x < \pi$ ),

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{K}_r(x) \sin mx| dx = \frac{2}{\pi} \int_{\pi/r}^{\pi} \frac{|\sin mx|}{x} dx + O(1) \geq C \log r + O(1).$$

Therefore

$$E_n(\varphi_{n,r})_L \geq C \log r + O(1). \quad (35)$$

For  $E_{n_{k-1}}(s_{m_j}A)_L$  we obtain by (33)

$$E_{n_{k-1}}(s_{m_j}A)_L = E_{n_{k-1}}(4\lambda_{m_j} \varphi_{n_{k-1}, r_j})_L + O(1) \geq C |\lambda_{m_j}| \log r_j + O(1).$$

This proves the assertion for the case  $m_j \in I_2(\mathfrak{R})$ .

If  $m_j \in I_1(\mathfrak{R})$  and  $r_j = o(m_j)$ , the application of Abel's transformation twice gives us (in this sequel we shall write  $n$  without a lower index for  $n_{l_k}$ )

$$\begin{aligned} s_n A(x) - s_m A(x) &= 2 \left\{ \lambda_{m+1} \sum_{\nu=0}^r \cos(n-\nu)x - \right. \\ &\quad \left. - \Delta \lambda_{m+1} \sum_{\nu=0}^{r-1} (r-\nu) \cos(n-\nu)x + \sum_{l=0}^{r-2} \Delta^2 \lambda_{n-l-2} \sum_{\nu=0}^l (l+1-\nu) \cos(n-\nu)x \right\}. \end{aligned}$$

Adding the polynomial

$$\begin{aligned} P(x) &= 2 \left\{ \lambda_{m+1} \sum_{\nu=1}^r \cos(n+\nu)x - \Delta \lambda_{m+1} \sum_{\nu=1}^{r-1} (r-\nu) \cos(n+\nu)x + \right. \\ &\quad \left. + \sum_{l=0}^{r-2} \Delta^2 \lambda_{n-l-2} \sum_{\nu=1}^l (l+1-\nu) \cos(n+\nu)x \right\}, \end{aligned}$$

we obtain a formula analogous to (29)

$$s_n A(x) - s_m A(x) + P(x) = 2 \left\{ 2\lambda_{m+1} D_r(x) - \right.$$

$$-\Delta\lambda_{m+1} \cdot r \cdot K_{r-1}(x) + \sum_{l=0}^{r-2} \Delta^2\lambda_{n-l-2}(l+1)K_l(x) \} \cdot \cos nx. \quad (36)$$

Since  $P \in F_{n_k}^\perp$  and, in view of the assumptions made,  $r \cdot \Delta\lambda_{m+1} = o(1)$  and

$$\sum_{l=0}^{r-2} |\Delta^2\lambda_{n-l-2}|(l+1) \leq \sum_{l=m+1}^{n-2} (l+1) |\Delta^2\lambda_l| = O(1),$$

we get the estimate from above

$$E_{n_k}^\perp (s_{n_k}A - s_{m_j}A)_L \leq C |\lambda_{m_j+1}| \log r_j + O(1) = C |\lambda_{m_j}| \log r_j + O(1).$$

To obtain the estimate from below, consider the function  $\varphi_{n-1,r}(x) = 4D_r(x) \cos nx$ . As  $m+r+1=n$  we have, similarly to (34), that

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_{n-1,r}(t) K_r(x-t) \cdot 2 \cos m(x-t) dt = \\ & = \frac{1}{2\pi} \int_{-\pi}^{\pi} \{\varphi_{n-1,r}(t) - P(t)\} \cdot K_r(x-t) \cdot 2 \cos m(x-t) dt \end{aligned}$$

for arbitrary  $P \in F_{n_k}^\perp$ , and therefore

$$\left\| 2 \sum_{\nu=1}^r \left(1 - \frac{\nu}{r+1}\right) \cos(m+\nu) \right\|_L \leq 2E_{n_k}^\perp (\varphi_{n-1,r})_L.$$

Since

$$\left\| \sum_{\nu=-r}^r \left(1 - \frac{|\nu|}{r+1}\right) \cos(m+\nu) \right\|_L \leq 1$$

we get, by (35), that

$$E_{n_k}^\perp (\varphi_{n-1,r})_L \geq C \cdot \log r + O(1),$$

and hence from (36) that

$$E_{n_k}^\perp (s_{n_k}A - s_{m_j}A)_L \geq C |\lambda_{m_j+1}| \log r_j + O(1) = C |\lambda_{m_j}| \log r_j + O(1).$$

This concludes the proof of Corollary 2.

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**MÕNE FOURIER' RIDADE KLASSI KOONDUVUSTEGUREIST**

Artiklis on leitud tarvilikud ja piisavad tingimused, et Fourier-Stieltjesi kordajate jada  $\lambda = \{\lambda_n\}$  teisendaks iga integraalses meetrikas mõõda üht etteantud indeksiite jada koonduva Fourier' rea samas meetrikas mõõda teist etteantud indeksiite jada koonduvaks Fourier' reaks. Saadud teoreemi on vaadeldud ka juhtudel, kus 1) indeksiite jadad on lakunaarsed Hadamardi järgi ja 2) multiplikaatorjada  $\lambda$  on kvaasikumer.

Ю. ЛИППУС

**О МНОЖИТЕЛЯХ СХОДИМОСТИ НЕКОТОРЫХ КЛАССОВ РЯДОВ ФУРЬЕ**

Находятся необходимые и достаточные условия для того, чтобы последовательность коэффициентов Фурье—Стилтьеса  $\lambda = \{\lambda_n\}$  переводила каждый ряд Фурье, сходящийся по одной заданной последовательности индексов в среднем, в ряд Фурье, сходящийся по другой заданной последовательности индексов в той же метрике. Полученная теорема иллюстрируется на примере лакунарных последовательностей индексов и на примере квазивыпуклых последовательностей множителей.