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ON MULTIPLIERS OF CONVERGENCE OF SOME CLASSES OF FOURIER SERIES

(Presented by A. Humal)

If X and Y are two classes of 2π-periodic integrable functions, we say that a two-way infinite sequence of complex numbers $\lambda=\{\lambda_k\}$ is a multiplier sequence from X into Y, and we write $\lambda\in(X,Y)$ if whenever

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} \tag{1}$$

is the Fourier series of a function f in X, the series

$$\sum_{k=-\infty}^{\infty} \lambda_k c_k e^{ikx} \tag{2}$$

is the Fourier series of a function f_λ in Y. Let C denote the class of 2π-periodic continuous functions and L the class of 2π-periodic integrable functions. Let C_F and L_F denote the subclasses of those functions whose Fourier series converge in the metrics of the corresponding functional space. The class (C, C_F) was described by J. Karamata [1], and the class (L, L_F) by G. Goes [2]. They showed that a necessary and sufficient condition for λ to be in those classes is that

$$\|s_n\Lambda\|_L=O(1), \quad (n\to\infty) \tag{3}$$

where

$$s_n\Lambda(x)=\sum_{k=-n}^{n}\lambda_k e^{ikx}.$$

If we restrict ourselves to a subclass of C or L, the condition (3) may be relaxed. Multipliers from a class of functions with a given modulus of continuity into the class of functions with convergent Fourier series, have been widely discussed (see e.g. [3–6]).

In the present paper we shall study multipliers from a class of functions whose Fourier series have a convergent subsequence of partial sums, into a class of functions that have another convergent subsequence of partial sums. As far as we know, the first result in this direction was obtained by G. S. Yanakov [7], who considered the case when the first subsequence of partial sums is lacunary, and the other is dense.

We shall prove our results only for integral metrics, since the proof of analogous statements for continuous metrics would differ only in details.

Let

$$s_nf(x)=\sum_{k=-n}^{n} c_k e^{ikx}$$

denote the n-th partial sum of the series (1) and let $\mathfrak{N}=\{n_k\}$ be a strictly increasing sequence of positive integers. We shall say that f belongs to the class $L_F(\mathfrak{N})$ if the sequence $\{s_{n_k}f\}$ converges in integral metrics, i.e. $\|f-s_{n_k}f\|_L=o(1)$ $(k\to\infty)$. Given another increasing sequence of positive integers $\mathfrak{M}$, we may consider the class of multipliers $(L_F(\mathfrak{N}),$ $L_F(\mathfrak{M}))$.

Throughout this paper we shall assume that λ is of the class S of Fourier-Stieltjes sequences (which in particular characterizes the class (L,L)).

Let $v_{n,m}f$ denote the de la Vallée Poussin means of the series (1)

$$v_{n,m}f(x)=\frac{1}{m}\left(s_nf(x)+s_{n+1}f(x)+\ldots+s_{n+m-1}f(x)\right)$$

and let $v_{n,m}(x)$ be the corresponding kernel

$$v_{n,m}f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\,v_{n,m}(x-t)\,\mathrm{d}t.$$

Let

$$K_n(x)=\sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)e^{ikx}=2\left(\frac{1}{2}+\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)\cos kx\right)$$

be the symmetrical Fejér kernel. Then we may write $v_{n,n}(x)=$ $=2K_{2n-1}(x)-K_{n-1}(x)$, and as $\|K_n\|_L\leqslant 1$ we shall have $\|v_{n,n}\|_L\leqslant 3$. Writing $s_n\Lambda$ and $v_{n,m}\Lambda$ respectively for the partial sums and the de la Vallée Poussin means of the series

$$\sum_{k=-\infty}^{\infty}\lambda_k\,e^{ikx},\tag{4}$$

the condition $\lambda\in S$ is equivalent to the condition

$$\|v_{n,n}\Lambda\|_L=O(1)\tag{5}$$

(see [8], p. 137).

Let F_n denote the set of all trigonometric polynomials of an order not higher than n, and $F_n^{\perp}$ the set of all trigonometric polynomials orthogonal to F_n. We shall use the notations $E_n(f)_L=\inf_{P_n\in F_n}\|f-P_n\|_L$, and $E_n^{\perp}(f)_L=\inf_{P_n^{\perp}\in F_n^{\perp}}\|f-P_n^{\perp}\|_L$ for the best approximations of the function f with the respective sets.

Given two sequences of positive integers $\mathfrak{N}$ and $\mathfrak{M}$, we may, for every m_j, consider the constants

$$R_j(\lambda)=\min\{E_{n_{k-1}}(s_{m_j}\Lambda)_L,\ E_{n_k}(s_{n_k}\Lambda-s_{m_j}\Lambda)_L\},$$

where $n_{k-1}\leqslant m_j\leqslant n_k$.

Theorem 1. *Let $\lambda\in S$. Then a necessary and sufficient condition for the sequence λ to be in $(L_F(\mathfrak{N}),\ L_F(\mathfrak{M}))$ is that*

$$R_j(\lambda)=O(1)\quad(j\to\infty).\tag{6}$$

Proof. First we shall prove that the condition is sufficient. As $\lambda\in S$, we have $f_\lambda\in L$, and since that implies

$$\|f_\lambda-v_{n,n}f_\lambda\|_L=o(1)\quad(n\to\infty)$$

we obtain the estimate

$$\|f_\lambda-s_{n_k}f_\lambda\|_L\leqslant\|f_\lambda-v_{n_k,n_k}f_\lambda\|_L+\|v_{n_k,n_k}f_\lambda-s_{n_k}f_\lambda\|_L=$$

$$=o(1)+\left\|\frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(t)-s_{n_k}f(t)\}\cdot v_{n_k,n_k}(\cdot-t)\,dt\right\|_L\leqslant \tag{7}$$

$$\leqslant o(1)+\|f-s_{n_k}f\|_L\cdot\|v_{n_k,n_k}\|_L.$$

Thus if $\lambda\in S$ and $f\in L_F(\mathfrak{N})$, it follows that

$$\|f_\lambda-s_{n_k}f_\lambda\|_L=o(1)\quad(k\to\infty). \tag{8}$$

Therefore it is sufficient to consider the difference $s_{n_k}f_\lambda(x)-s_{m_l}f_\lambda(x)$. We have

$$s_{n_k}f_\lambda(x)-s_{m_l}f_\lambda(x)=$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\{s_{n_k}f(t)-s_{n_{k-1}}f(t)\}\{s_{n_k}\Lambda(x-t)-s_{m_l}\Lambda(x-t)\}\,dt.$$

Observing that the first factor of the integrand is a polynomial of order n_k orthogonal to $F_{n_{k-1}}$, we see that for arbitrary $P^{\perp}_{n_k}\in F^{\perp}_{n_k}$ and $T_{n_{k-1}}\in\in F_{n_{k-1}}$

$$s_{n_k}f_\lambda(x)-s_{m_l}f_\lambda(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\{s_{n_k}f(t)-s_{n_{k-1}}f(t)\}\,\{P^{\perp}_{n_k}(x-t)+$$

$$+s_{n_k}\Lambda(x-t)-s_{m_l}\Lambda(x-t)+T_{n_{k-1}}(x-t)\}\,dt. \tag{9}$$

Applying Young's inequality, we get

$$\|s_{n_k}f_\lambda-s_{m_l}f_\lambda\|_L\leqslant\|s_{n_k}f-s_{n_{k-1}}f\|_L\cdot\|P^{\perp}_{n_k}+s_{n_k}\Lambda-s_{m_l}\Lambda+T_{n_{k-1}}\|_L. \tag{10}$$

In view of the arbitrariness of the polynomials $P^{\perp}_{n_k}$ and $T_{n_{k-1}}$, we may let

$$P^{\perp}_{n_k}(x)=v_{n_k,n_k}\Lambda(x)-s_{n_k}\Lambda(x)$$

and $T_{n_{k-1}}$ be the polynomial of the best approximation of $s_{m_l}\Lambda$. Hence

$$\|P^{\perp}_{n_k}+s_{n_k}\Lambda-s_{m_l}\Lambda+T_{n_{k-1}}\|_L=\|v_{n_k,n_k}\Lambda-s_{n_k}\Lambda+s_{n_k}\Lambda-s_{m_l}\Lambda+T_{n_{k-1}}\|_L\leqslant$$

$$\leqslant\|v_{n_k,n_k}\Lambda\|_L+\|s_{m_l}\Lambda-T_{n_{k-1}}\|_L.$$

By virtue of (5), the first term on the right is $O(1)$, thus

$$\|s_{n_k}f_\lambda-s_{m_l}f_\lambda\|_L\leqslant\|s_{n_k}f-s_{n_{k-1}}f\|_L\cdot\{O(1)+E_{n_{k-1}}(s_{m_l}\Lambda)_L\}. \tag{11}$$

On the other hand, if we let $T_{n_{k-1}}\equiv 0$ and take the infimum over all $P^{\perp}_{n_k}\in F^{\perp}_{n_k}$, we obtain

$$\|s_{n_k}f_\lambda-s_{m_l}f_\lambda\|_L\leqslant\|s_{n_k}f-s_{n_{k-1}}f\|_L\cdot E^{\perp}_{n_k}(s_{n_k}\Lambda-s_{m_l}\Lambda)_L. \tag{12}$$

Collecting the estimates (11) and (12), we get

$$\|s_{n_k}f_\lambda-s_{m_l}f_\lambda\|_L\leqslant\|s_{n_k}f-s_{n_{k-1}}f\|_L\cdot\min\{O(1)+E_{n_{k-1}}(s_{m_l}\Lambda)_L,$$

$$E^{\perp}_{n_k}(s_{n_k}\Lambda-s_{m_l}\Lambda)_L\}\leqslant\|s_{n_k}f-s_{n_{k-1}}f\|_L\cdot\{R_j(\lambda)+O(1)\}. \tag{13}$$

Therefore, if the condition (6) holds, then $f_\lambda\in L_F(\mathfrak{M})$ for every $f\in L_F(\mathfrak{N})$.

To prove the necessity of (6), we first introduce some auxiliary notations. For $m\leqslant n$ let

$$\sigma_{n,m}\Lambda(x)=\sum_{k=0}^{m}\left(1-\frac{k}{m+1}\right)\left(\lambda_{n-k}\,e^{i(n-k)x}+\lambda_{-(n-k)}\,e^{-i(n-k)x}\right),$$

and

$$\varrho_{n,m}\Lambda(x)=\sum_{k=1}^{m}\left(1-\frac{k}{m+1}\right)\left(\lambda_{n+k}\,e^{i(n+k)x}+\lambda_{-(n+k)}\,e^{-i(n+k)x}\right).$$

According to the definition, $\sigma_{n,m}\in F_n$ and $\varrho_{n,m}\in F_n^{\perp}$. They also have the properties

$$\sigma_{n,m}\Lambda(x)+\varrho_{n,m}\Lambda(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} v_{n+m,\,n+m}\Lambda(t)\,\{2\cos n(x-t)\cdot K_m(x-t)\}\,dt; \tag{14}$$

where K_m is the symmetrical Fejér kernel, and

$$\sigma_{n,m}\Lambda(x)+\varrho_{n-m-1,\,m}\Lambda(x)=s_n\Lambda(x)-s_{n-m-1}\Lambda(x). \tag{15}$$

As $\lambda\in S$, we have by (5) and Young's inequality that

$$\|\sigma_{n,m}\Lambda+\varrho_{n,m}\Lambda\|_L=O(1). \tag{16}$$

Given a sequence $\mathfrak{N}=\{n_k\}$, we define three sets of indices ($[x]$ denotes the entire part of x):

$$I_1(\mathfrak{N})=\{m:[(n_{k-1}+n_k)/2]<m<n_k\},$$

$$I_2(\mathfrak{N})=\{m:n_{k-1}<m\leqslant\min\{[(n_{k-1}+n_k)/2],\,2n_{k-1}\}\},$$

$$I_3(\mathfrak{N})=\{m:2n_{k-1}<m\leqslant[(n_{k-1}+n_k)/2]\}.$$

Suppose now that (6) does not hold. Then we can find a sequence $\{j(p)\}_{p=1}^{\infty}$ such that

$$R_{j(p)}(\lambda)\geqslant p^3 \tag{17}$$

and

$$n_{k(p)-1}<m_{j(p)}<n_{k(p)}<m_{j(p+1)}. \tag{18}$$

Let

$$r_j=\begin{cases} n_k-m_j-1, & \text{if } m_j\in I_1(\mathfrak{N}),\\ m_j-n_{k-1}-1, & \text{if } m_j\in I_2(\mathfrak{N}),\\ [(m_j-1)/2], & \text{if } m_j\in I_3(\mathfrak{N}).\end{cases} \tag{19}$$

During the rest of the proof we shall omit some of the common indices, assuming $k=k(p)$, $m=m_j=m_{j(p)}$ and $r=r_j=r_{j(p)}$. If $m\in I_1(\mathfrak{N})\cup\cup I_2(\mathfrak{N})$, let

$$f_p(x)=K_r(x)\cdot 2\cos mx$$

and in the case $m\in I_3(\mathfrak{N})$ let

$$f_p(x)=K_r(x)\cdot 2\cos(2r+1)x.$$

The functions f_p so defined are trigonometric polynomials of order n_k orthogonal to $F_{n_{k-1}}$. Since $\|f_p\|_L\leqslant 2$, by Levi's theorem the series

$$\sum_{p=1}^{\infty}\frac{1}{p^2}f_p(x) \tag{20}$$

represents an integrable function f. The estimate

$$\|f-s_{n_l}f\|_L=\Big\|\sum_{k>l}\frac{1}{p^2}f_p\Big\|\leqslant 2\sum_{k(p)>l}\frac{1}{p^2}=o(1) \tag{21}$$

$(l\to\infty)$ establishes that $f\in L_F(\mathfrak{N})$.

To prove that f_λ does not belong to $L_F(\mathfrak{M})$, we first suppose that $m\in I_1(\mathfrak{N})$. As then $n_k-r-1=m$, it follows from the construction of f that

$$\|s_{n_k}f_\lambda - s_m f_\lambda\|_L = \frac{1}{p^2}\,\|(f_p)_\lambda - s_m(f_p)_\lambda\|_L = \frac{1}{p^2}\,\|\varrho_{m,r}\Lambda\|_L. \tag{22}$$

By virtue of (15) we may write

$$\|\varrho_{m,r}\Lambda\|_L = \|\varrho_{m,r}\Lambda + \sigma_{n_k,r}\Lambda - \sigma_{n_k,r}\Lambda + \varrho_{n_k,r}\Lambda - \varrho_{n_k,r}\Lambda\|_L \geqslant$$
$$\geqslant \|s_{n_k}\Lambda - s_m\Lambda + \varrho_{n_k,r}\Lambda\|_L - \|\sigma_{n_k,r}\Lambda + \varrho_{n_k,r}\Lambda\|_L.$$

Applying (16) and using the fact that $\varrho_{n_k,r}\Lambda \in F^\perp_{n_k}$, we obtain

$$\|\varrho_{m,r}\Lambda\|_L \geqslant E^\perp_{n_k}(s_{n_k}\Lambda - s_{m_j}\Lambda)_L - O(1), \tag{23}$$

and hence, by (17)

$$\|s_{n_k}f_\lambda - s_m f_\lambda\|_L \geqslant p - o(1).$$

If $m \in I_2(\mathfrak{R})$, let us consider the difference $s_m f_\lambda - s_{n_{k-1}} f_\lambda$. As in the preceding argument, we have by the construction of f that

$$\|s_m f_\lambda - s_{n_{k-1}} f_\lambda\|_L = \frac{1}{p^2}\,\|s_m(f_p)_\lambda\|_L = \frac{1}{p^2}\,\|\sigma_{m,r}\Lambda\|_L. \tag{24}$$

Observing that in this case $m - r - 1 = n_{k-1}$ and applying (15) and (16), we get in a similar manner

$$\|\sigma_{m,r}\Lambda\|_L = \|\sigma_{m,r}\Lambda + \varrho_{m-r-1,r}\Lambda + \sigma_{m-r-1,r}\Lambda - \sigma_{m-r-1,r}\Lambda -$$
$$- \varrho_{m-r-1,r}\Lambda\|_L \geqslant \|s_m\Lambda - s_{n_{k-1}}\Lambda + \sigma_{n_{k-1},r}\Lambda\|_L - O(1).$$

Since $n_{k-1} - r > 0$, the difference $s_{n_{k-1}}\Lambda - \sigma_{n_{k-1},r}\Lambda$ is a polynomial of order n_{k-1}, and therefore

$$\|\sigma_{m,r}\Lambda\|_L \geqslant E_{n_{k-1}}(s_m\Lambda)_L - O(1). \tag{25}$$

By virtue of (17) and (24), this gives us

$$\|s_m f_\lambda - s_{n_{k-1}} f_\lambda\|_L \geqslant p - o(1).$$

The last case to be considered is $m \in I_3(\mathfrak{R})$. In view of the construction of f, we have by (17) and (5)

$$\|s_m f_\lambda - s_{n_{k-1}} f_\lambda\|_L = \frac{1}{p^2}\,\|s_m(f_p)_\lambda\|_L = \frac{1}{p^2}\,\|s_m\Lambda - v_{r,r}\Lambda\|_L \geqslant$$
$$\geqslant \frac{1}{p^2}\,\{\|s_m\Lambda\|_L - \|v_{r,r}\Lambda\|_L\} \geqslant p - o(1). \tag{26}$$

This concludes the proof of Theorem 1.

Let $\lambda \in S$. Then, similarly to (14), we may write $(m \geqslant r)$

$$v_{m,r}\Lambda(x) - v_{m-r,r}\Lambda(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v_{2m,2m}\Lambda(t)\cdot\{2\cos m(x-t)\cdot K_r(x-t)\}\,dt,$$

and by virtue of (5), applying Young's inequality, we shall obtain

$$\|v_{m,r}\Lambda - v_{m-r,r}\Lambda\|_L = O(1). \tag{27}$$

Let r_j be defined by (19) and $n_{k-1} < m_j < n_k$. Consider the sequence $\{Q_j(\lambda)\}$ where

$$Q_j(\lambda) = \begin{cases} \|v_{m_j,r_j+1}\Lambda - s_{m_j}\Lambda\|_L, & \text{if } m_j \in I_1(\mathfrak{R}),\\ \|s_{m_j}\Lambda - v_{n_{k-1},r_j+1}\Lambda\|_L, & \text{if } m_j \in I_2(\mathfrak{R}),\\ \|s_{m_j}\Lambda\|_L, & \text{if } m_j \in I_3(\mathfrak{R}). \end{cases}$$

We shall restate Theorem 1 as follows.

Theorem 2. *Let $\lambda \in S$. Then a necessary and sufficient condition for λ to be in $(L_F(\mathfrak{N}), L_F(\mathfrak{M}))$ is that*

$$Q_j(\lambda) = O(1) \quad (j \to \infty). \tag{28}$$

Proof. Let $q_j = r_j + 1$. In view of (27) we have:
if $m_j \in I_1(\mathfrak{N})$, then

$$Q_j(\lambda) = \|v_{m_j, q_j}\Lambda - s_{m_j}\Lambda\|_L \geqslant \|s_{n_k}\Lambda - s_{m_j}\Lambda + (v_{n_k, q_j}\Lambda - s_{n_k}\Lambda)\|_L -$$

$$- \|v_{n_k, q_j}\Lambda - v_{m_j, q_j}\Lambda\|_L \geqslant E^{\perp}_{n_k}(s_{n_k}\Lambda - s_{m_j}\Lambda)_L - O(1) \geqslant R_j(\lambda) - O(1),$$

since $v_{n_k, q_j}\Lambda - s_{n_k}\Lambda \in F^{\perp}_{n_k}$ and $n_k - q_j = m_j$;

if $m_j \in I_2(\mathfrak{N})$, then similarly $v_{n_{k-1}-q_j, q_j}\Lambda \in F_{n_{k-1}}$ and $n_{k-1} - q_j > 0$, hence

$$Q_j(\lambda) = \|s_{m_j}\Lambda - v_{n_{k-1}, q_j}\Lambda\|_L \geqslant \|s_{m_j}\Lambda - v_{n_{k-1}-q_j, q_j}\Lambda\|_L -$$

$$- \|v_{n_{k-1}, q_j}\Lambda - v_{n_{k-1}-q_j, q_j}\Lambda\|_L \geqslant E_{n_{k-1}}(s_{m_j}\Lambda)_L - O(1) \geqslant R_j(\lambda) - O(1);$$

if $m_j \in I_3(\mathfrak{N})$, then the inequality $Q_j(\lambda) \geqslant E_{n_{k-1}}(s_{m_j}\Lambda)_L \geqslant R_j(\lambda)$ is obvious.

Therefore, by Theorem 1 the condition (28) is sufficient for λ to be in $(L_F(\mathfrak{N}), L_F(\mathfrak{M}))$.

To prove that (28) is also necessary, suppose that it does not hold. Then, as before, we can find a sequence $\{j(p)\}$ such that

$$Q_{j(p)}(\lambda) \geqslant p^3, \tag{29}$$

and that (18) holds. Let $f \in L_F(\mathfrak{N})$ be defined by (20). Since (again assuming $k = k(p)$, $m = m_j \in m_{j(p)}$ and $r = r_j = r_{j(p)}$):
if $m \in I_1(\mathfrak{N})$, then

$$p^3 \leqslant Q_j(\lambda) = \|w_{m,q}\Lambda - s_m\Lambda\|_L = \|\varrho_{m,r}\Lambda\|_L, \tag{30}$$

if $m \in I_2(\mathfrak{N})$, then

$$p^3 \leqslant Q_j(\lambda) = \|s_m\Lambda - v_{n_{k-1}, q}\Lambda\|_L = \|\sigma_{m,r}\Lambda\|_L, \tag{31}$$

and if $m \in I_3(\mathfrak{N})$, then, by (5),

$$p^3 \leqslant Q_j(\lambda) = \|s_m\Lambda\|_L \leqslant \|s_m\Lambda - v_{r,r}\Lambda\|_L + O(1), \tag{32}$$

it follows, collecting (22) and (30), (24) and (31), (26) and (32), that f_λ is not in $L_F(\mathfrak{M})$. That proves the necessity of (28).

Therefore in Theorem 1, instead of the best approximations, we may consider approximations by linear means.

As illustrations we shall prove two corollaries.

Corollary 1. Let $\lambda \in S$. *Let for every j $(n_{k-1} < m_j < n_k)$ $m_j/n_{k-1} \geqslant C_1 > 1$ and $n_k/m_j \geqslant C_2 > 1$. Then λ is in $(L_F(\mathfrak{N}), L_F(\mathfrak{M}))$ if and only if it is in $(L, L_F(\mathfrak{M}))$. (Here and later on by $C, C_1, C_2, \ldots$ we shall denote absolute positive constants, not necessarily the same on each occurrence.)*

Proof. Let r_j be defined by (19) and let $q_j = r_j + 1$. It is well known (see [9], p. 35) that under the assumptions made, $\|v_{m_j, q_j}\|_L = O(1)$ and $\|v_{n_{k-1}, q_j}\|_L = O(1)$. Thus, by (5), the application of Young's inequality gives us: if $m_j \in I_1(\mathfrak{N})$, then

$$\|v_{m_j,\,q_j}\Lambda\|_L=\left\|\frac{1}{2\pi}\int_{-\pi}^{\pi} v_{n_k,\,n_k}\Lambda(t)\cdot v_{m_j,\,q_j}(\cdot - t)\,\mathrm{d}t\right\|_L=O(1),$$

and if $m_j\in I_2(\mathfrak{N})$, then

$$\|v_{n_{k-1},\,q_j}\Lambda\|_L=\left\|\frac{1}{2\pi}\int_{-\pi}^{\pi} v_{m_j,\,m_j}\Lambda(t)\cdot v_{n_{k-1},\,q_j}(\cdot - t)\,\mathrm{d}t\right\|_L=O(1).$$

Hence $Q_j(\lambda)=\|s_{m_j}\Lambda\|_L+O(1)$. Since by a theorem by G. Goes [2] the type $(L, L_F(\mathfrak{M}))$ is characterized by the condition

$$\|s_{m_j}\Lambda\|_L=O(1)$$

(Goes proved it for the case where $\mathfrak{M}$ is the sequence of all positive integers, but it remains valid if $\mathfrak{M}$ is a subsequence of that sequence), this concludes the proof of the corollary.

Let us consider now only real-valued functions (i.e. let $c_{-n}=\bar{c}_n$ and $\lambda_{-n}=\overline{\lambda_n}=\lambda_n$). Let λ be a quasi-convex sequence, that is

$$\sum_{n=1}^{\infty} n|\Delta^2\lambda_n|<\infty,$$

where $\Delta\lambda_n=\lambda_n-\lambda_{n+1}$ and $\Delta^2\lambda_n=\Delta\lambda_n-\Delta\lambda_{n+1}$.

Corollary 2. *A necessary and sufficient condition for a quasi-convex sequence λ to be in $(L_F(\mathfrak{N}), L_F(\mathfrak{M}))$ is that* $(n_{k-1}<m_j<n_k)$

$$\lambda_{m_j}\cdot\log(1+\min\{n_k-m_j,\ m_j-n_{k-1}\})=O(1)\quad (j\to\infty).$$

Proof. It is known that if λ is quasi-convex, then there exist two constants A and B such that

$$\lambda_k=Ak+B+\mu_k$$

and $\mu_k\to 0$ $(k\to\infty)$ (see, e.g., [4]). We shall show that the case λ unbounded will not arise. Consider the series

$$\sum_{k=1}^{\infty}\frac{1}{k}\cos kx.$$

It is the Fourier series of a square-integrable function f and therefore converges in integral metrics. But if $A\neq 0$ then the norms of the partial sums of the transformed series

$$s_nf_\lambda(x)=\sum_{k=1}^{n}\frac{1}{k}\lambda_k\cos kx=A\sum_{k=1}^{n}\cos kx+\sum_{k=1}^{n}\frac{1}{k}(B+\mu_k)\cos kx$$

tend to infinity

$$\|s_nf_\lambda\|_L=\frac{2}{\pi}|A|\log n+O(1)\quad (n\to\infty).$$

Thus if $\lambda\in(L_F(\mathfrak{N}), L_F(\mathfrak{M}))$, we must have $A=0$. In the following we shall suppose that λ is a bounded sequence. Then it is well known that

$$k\Delta\lambda_{k-1}\to 0\quad (k\to\infty).$$

By Corollary 1 we may consider only such a case where either $m_j-n_{k-1}=o(m_j)$ or $n_k-m_j=o(m_j)$, since (see [8], p. 183) if λ is quasi-convex we have

$$\|s_{m_j}\Lambda\|_L=\frac{4}{\pi^2}|\lambda_{m_j}|\log m_j+O(1).$$

This proves the corollary for the case $m_j\in I_3(\mathfrak{N})$.

Suppose first that $m_j-n_{k-1}=o(m_j)$, that is $m_j\in I_2(\mathfrak{N})$. Consider the difference

$$s_{m_j}\Lambda(x)-s_{n_{k-1}}\Lambda(x)=2\sum_{\nu=n_{k-1}+1}^{m_j}\lambda_\nu\cos\nu x.$$

Let r_j be, as before, defined by (19) and for convenience's sake let us again omit the common indices — if it is not necessary to emphasize them — assuming $m=m_j$, $r=r_j$ and $n=n_{k-1}$. Applying Abel's transformation twice, we get

$$s_m\Lambda(x)-s_n\Lambda(x)=2\Big\{\lambda_m\sum_{\nu=0}^{r}\cos(n+1+\nu)x+$$
$$+\Delta\lambda_{m-1}\sum_{\nu=0}^{r-1}(r-\nu)\cos(n+1+\nu)x+$$
$$+\sum_{l=0}^{r-2}\Delta^2\lambda_{n+l+1}\sum_{\nu=0}^{l}(l+1-\nu)\cos(n+1+\nu)x\Big\}.$$

Let

$$T(x)=2\Big\{\lambda_m\sum_{\nu=1}^{r}\cos(n+1-\nu)x+\Delta\lambda_{m-1}\sum_{\nu=1}^{r-1}(r-\nu)\cos(n+1-\nu)x+$$
$$+\sum_{l=0}^{r-2}\Delta^2\lambda_{n+l+1}\sum_{\nu=1}^{l}(l+1-\nu)\cos(n+1-\nu)x\Big\}.$$

Since $m\in I_2(\mathfrak{N})$, we have $n-r\geqslant 0$, hence T is a polynomial of order n_{k-1}, and

$$s_m\Lambda(x)-s_n\Lambda(x)+T(x)=2\{2\lambda_m D_r(x)+\Delta\lambda_{m-1}\cdot r\cdot K_{r-1}(x)+$$
$$+\sum_{l=0}^{r-2}\Delta^2\lambda_{n+l+1}\cdot(l+1)\cdot K_l(x)\}\cdot\cos(n+1)x, \tag{33}$$

where D_r is the Dirichlet kernel of order r. Thus

$$E_{n_{k-1}}(s_{m_j}\Lambda)_L\leqslant\|s_m\Lambda-s_n\Lambda+T\|_L\leqslant$$
$$\leqslant 4|\lambda_m|\cdot\|D_r\|_L+O(1)\leqslant C|\lambda_{m_j}|\cdot\log r_j+O(1).$$

On the other hand, let $\varphi_{n,r}(x)=4D_r(x)\cos(n+1)x$. Since $m-r>n$ and $\|K_r\|_L\leqslant 1$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_{n,r}(t)\cdot K_r(x-t)\cdot 2\cos m(x-t)\,dt=$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\varphi_{n,r}(t)-T_n^*(t)\}\cdot K_r(x-t)\cdot 2\cos m(x-t)\,dt, \tag{34}$$

where T_n^* is the polynomial of the best approximation of $\varphi_{n,r}$. Therefore, applying Young's inequality, we obtain

$$\Big\|2\sum_{\nu=0}^{r}\Big(1-\frac{\nu}{r+1}\Big)\cos(m-\nu)\cdot\Big\|_L\leqslant 2E_n(\varphi_{n,r})_L.$$

The sum on the left may be rewritten as

$$(1+K_r(x))\cos mx+\tilde{K}_r(x)\sin mx.$$

Here

$$\tilde{K}_r(x)=2\sum_{\nu=1}^{r}\left(1-\frac{\nu}{r+1}\right)\sin\nu x$$

is the symmetrical conjugate Fejér kernel. Since

$$\|(1+K_r)\cos m\cdot\|_L=O(1),$$

we have to estimate the second term. It is known (see, e. g., [8], p. 92) that $\tilde{K}_r(x)>0$ for $0<x<\pi$, $|\tilde{K}_r(x)|\leqslant r$ and $\tilde{K}_r(x)=\cot(x/2)-H_r(x)$, where $|H_r(x)|\leqslant C/\{(r+1)x^2\}$. We may write

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\tilde{K}_r(x)\sin mx|\,dx=\frac{1}{\pi}\int_{0}^{\pi}|\sin mx|\cdot\tilde{K}_r(x)\,dx=$$

$$=\frac{1}{\pi}\int_{0}^{\pi/r}|\sin mx|\tilde{K}_r(x)\,dx+\frac{2}{\pi}\int_{\pi/r}^{\pi}|\sin mx|\,\frac{1}{2}\cot\frac{x}{2}\,dx-$$

$$-\frac{1}{\pi}\int_{\pi/r}^{\pi}|\sin mx|H_r(x)\,dx,$$

and in the view of the above mentioned properties, observe that the first and the third integrals are $O(1)$. Thus, noting that $(1/x)-(1/2)\cot(x/2)=O(1)\quad(0<x<\pi)$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\tilde{K}_r(x)\sin mx|\,dx=\frac{2}{\pi}\int_{\pi/r}^{\pi}\frac{|\sin mx|}{x}\,dx+O(1)\geqslant C\log r+O(1).$$

Therefore

$$E_n(\varphi_{n,r})_L\geqslant C\log r+O(1). \tag{35}$$

For $E_{n_{k-1}}(s_{m_j}\Lambda)_L$ we obtain by (33)

$$E_{n_{k-1}}(s_{m_j}\Lambda)_L=E_{n_{k-1}}(4\lambda_{m_j}\varphi_{n_{k-1},\,r_j})_L+O(1)\geqslant C|\lambda_{m_j}|\log r_j+O(1).$$

This proves the assertion for the case $m_j\in I_2(\mathfrak{N})$.

If $m_j\in I_1(\mathfrak{N})$ and $r_j=o(m_j)$, the application of Abel's transformation twice gives us (in this sequel we shall write n without a lower index for n_k)

$$s_n\Lambda(x)-s_m\Lambda(x)=2\left\{\lambda_{m+1}\sum_{\nu=0}^{r}\cos(n-\nu)x-\right.$$

$$\left.-\Delta\lambda_{m+1}\sum_{\nu=0}^{r-1}(r-\nu)\cos(n-\nu)x+\sum_{l=0}^{r-2}\Delta^2\lambda_{n-l-2}\sum_{\nu=0}^{l}(l+1-\nu)\cos(n-\nu)x\right\}.$$

Adding the polynomial

$$P(x)=2\left\{\lambda_{m+1}\sum_{\nu=1}^{r}\cos(n+\nu)x-\Delta\lambda_{m+1}\sum_{\nu=1}^{r-1}(r-\nu)\cos(n+\nu)x+\right.$$

$$\left.+\sum_{l=0}^{r-2}\Delta^2\lambda_{n-l-2}\sum_{\nu=1}^{l}(l+1-\nu)\cos(n+\nu)x\right\},$$

we obtain a formula analogous to (29)

$$s_n\Lambda(x)-s_m\Lambda(x)+P(x)=2\left\{2\lambda_{m+1}D_r(x)-\right.$$

$$-\Delta\lambda_{m+1}\cdot r\cdot K_{r-1}(x)+\sum_{l=0}^{r-2}\Delta^2\lambda_{n-l-2}(l+1)K_l(x)\Big\}\cdot\cos nx. \qquad (36)$$

Since $P\in F_{n_k}^{\perp}$ and, in view of the assumptions made, $r\cdot\Delta\lambda_{m+1}=o(1)$ and

$$\sum_{l=0}^{r-2}|\Delta^2\lambda_{n-l-2}|(l+1)\leqslant\sum_{l=m+1}^{n-2}(l+1)|\Delta^2\lambda_l|=O(1),$$

we get the estimate from above

$$E_{n_k}^{\perp}(s_{n_k}\Lambda-s_{m_j}\Lambda)_L\leqslant C|\lambda_{m_j+1}|\log r_j+O(1)=C|\lambda_{m_j}|\log r_j+O(1).$$

To obtain the estimate from below, consider the function $\varphi_{n-1,r}(x)=$ $=4D_r(x)\cos nx$. As $m+r+1=n$ we have, similarly to (34), that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_{n-1,r}(t)K_r(x-t)\cdot 2\cos m(x-t)\,dt=$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\{\varphi_{n-1,r}(t)-P(t)\}\cdot K_r(x-t)\cdot 2\cos m(x-t)\,dt$$

for arbitrary $P\in F_{n_k}^{\perp}$, and therefore

$$\left\|2\sum_{\nu=1}^{r}\left(1-\frac{\nu}{r+1}\right)\cos(m+\nu)\cdot\right\|_L\leqslant 2E_{n_k}^{\perp}(\varphi_{n-1,r})_L.$$

Since

$$\left\|\sum_{\nu=-r}^{r}\left(1-\frac{|\nu|}{r+1}\right)\cos(m+\nu)\cdot\right\|_L\leqslant 1$$

we get, by (35), that

$$E_{n_k}^{\perp}(\varphi_{n-1,r})_L\geqslant C\cdot\log r+O(1),$$

and hence from (36) that

$$E_{n_k}^{\perp}(s_{n_k}\Lambda-s_{m_j}\Lambda)_L\geqslant C|\lambda_{m_j+1}|\log r_j+O(1)=C|\lambda_{m_j}|\log r_j+O(1).$$

This concludes the proof of Corollary 2.

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MÕNE FOURIER' RIDADE KLASSI KOONDUVUSTEGUREIST

Artiklis on leitud tarvilikud ja piisavad tingimused, et Fourier-Stieltjesi kordajate jada $\lambda = \{\lambda_k\}$ teisendaks iga integraalses meetrikas mööda üht etteantud indeksite jada koonduva Fourier' rea samas meetrikas mööda teist etteantud indeksite jada koonduvaks Fourier' reaks. Saadud teoreemi on vaadeldud ka juhtudel, kus 1) indeksite jadad on lakunaarsed Hadamardi järgi ja 2) multiplikaatorjada λ on kvaasikumer.

Ю. ЛИППУС

О МНОЖИТЕЛЯХ СХОДИМОСТИ НЕКОТОРЫХ КЛАССОВ РЯДОВ ФУРЬЕ

Находятся необходимые и достаточные условия для того, чтобы последовательность коэффициентов Фурье—Стилтьеса $\lambda = \{\lambda_k\}$ переводила каждый ряд Фурье, сходящийся по одной заданной последовательности индексов в среднем, в ряд Фурье, сходящийся по другой заданной последовательности индексов в той же метрике. Полученная теорема иллюстрируется на примере лакунарных последовательностей индексов и на примере квазивыпуклых последовательностей множителей.