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AN INEQUALITY FOR INTEGRALS WITH SPHERICALLY SYMMETRIC FUNCTIONS AND ITS APPLICATION TO OPTIMIZATION

(*Presented by N. Alumäe*)

1. Introduction. Several authors have studied the variation of the value of an integral over a convex symmetric set, the last being translated in a given direction [1–3]. In the present paper we consider a different problem: the variation of the integral while the integration domain is rotated around some fixed point. More precisely, let $f(x)$ be a non-negative Lebesgue integrable function, $x \in R^n$, and M — a Borel set in R^n. The question is how the value

$$R(a, A) = \int_{AM+a} f(x)\, dx$$

depends on an n-dimensional orthogonal matrix A and a vector $a \in R^n$.

In the next section we treat the two-dimensional case; the results are generalized in Section 3 for the n-dimensional case and applied in Section 4 to a quadratic stochastic programming problem.

2. Two-dimensional case. Let $M \subset R^2$ be a symmetric set relative to both axes OX and OY, i.e. if $(x, y) \in M$, then $(x', y') \in M$ for each $|x'| = |x|$ and $|y'| = |y|$. Let us introduce polar coordinates $(\varrho, \alpha): x = \varrho \cos \alpha$ and $y = \varrho \sin \alpha$. Suppose that in these coordinates the set M can be described in the following way:

$$M = \{(\varrho, \alpha): \varrho \leqslant \varphi(\alpha), \quad \varrho \geqslant 0, \quad 0 < \alpha < 2\pi\}. \tag{1}$$

By the symmetry of M we get $\varphi(\alpha) = \varphi(-\alpha)$ and $\varphi(\alpha + \pi) = \varphi(\alpha)$, i.e. $\varphi(\alpha)$ is an even periodic function with period of π.

Definition 1. *A symmetric set $M \subset R^2$ is called strongly prolate in OX-direction if the function $\varphi(\alpha)$ in (1) is non-increasing on $(0, \pi/2)$. Strong prolation in L-direction is defined in the same sense with regard to orthogonal rotation of axes so that L coincides with OX.*

Example. Ellipse is strongly prolate along its long axis.

Theorem 1. *Let $g(t)$ be a non-negative, convex and differentiable function on $(0, \infty)$ and M be a set strongly prolate in OX-direction. Then for every $r > 0$ the function*

$$H(\theta) = \int_{M+a_\theta} g(x^2 + y^2)\, dx\, dy,$$

where $a_\theta = (r \cos \theta, r \sin \theta)$, does not increase on $[0, \pi/2]$.

Proof. The function $H(\theta)$ can be written as

$$H(\theta) = \int_{(x - r \cos\theta,\, y - r \sin\theta) \in M} g(x^2 + y^2)\, dx\, dy =$$

$$= \int_{(x,y) \in M} g(x^2 + y^2 + 2r(x \cos\theta + y \sin\theta) + r^2)\, dx\, dy.$$

Differentiating this expression with respect to θ and denoting $x \cos\theta +$ $+ y \sin\theta = s$, $-x \sin\theta + y \cos\theta = t$, we have

$$H'(\theta) = \int_{M_\theta} g'(s^2 + t^2 + 2rs + r^2)\, t\, ds\, dt,$$

where

$$M_\theta = \{(s, t) \in R^2 : (s \cos\theta - t \sin\theta, s \sin\theta + t \cos\theta) \in M\},$$

the set being obtained by turning the set M around its centre of symmetry in the negative direction by an angle θ. Let us present the set M_θ as the union of two non-intersecting sets:

$$M_\theta = \{M_\theta \cap M_{-\theta}\} \cup \{M_\theta \setminus (M_\theta \cap M_{-\theta})\}.$$

By Lemma 1* the set $M_\theta \cap M_{-\theta}$ is symmetric relative to the axis OS, and hence,

$$\int_{M_\theta \cap M_{-\theta}} g'(s^2 + t^2 + 2rs + r^2)\, t\, ds\, dt = 0. \tag{2}$$

The symmetry of $M_\theta \cap M_{-\theta}$ also implies that the set $M_\theta \setminus (M_\theta \cap M_{-\theta})$ is symmetric relative to both axes (OS) and (OT) as a difference of symmetric sets. By Lemma 2 the points of the set $M_\theta \setminus (M_\theta \cap M_{-\theta})$ have Cartesian coordinates with different signs, and hence,

$$\int_{M_\theta \setminus (M_\theta \cap M_{-\theta})} g'(s^2 + t^2 + 2rs + r^2)\, t\, ds\, dt =$$

$$= \int_{\{M_\theta \setminus (M_\theta \cap M_{-\theta})\} \cap \{(s,\,t)\,:\,s \geqslant 0,\, t \leqslant 0\}} (g'(s^2 + t^2 + 2rs + r^2) - g'(s^2 + t^2 - 2rs + r^2))\, t\, ds\, dt. \tag{3}$$

Since the function $g(\cdot)$ is convex and in the formula (3) $s \geqslant 0$, we have for $(s, t) \in M_\theta \setminus (M_\theta \cap M_{-\theta})$:

$$g'(s^2 + t^2 + 2rs + r^2) \geqslant g'(s^2 + t^2 - 2rs + r^2). \tag{4}$$

Combining (2) and (3), we obtain by virtue of (4) that $H'(\theta) \leqslant 0$. Q.E.D.

L e m m a 1. *Set $M_\theta \cap M_{-\theta}$ is symmetric relative to both axes, i.e. if* $(s, t) \in M_\theta \cap M_{-\theta}$, *then* $(s, -t) \in M_\theta \cap M_{-\theta}$ *and* $(-s, t) \in M_\theta \cap M_{-\theta}$.

P r o o f. Let $(s, t) \in M_\theta \cap M_{-\theta}$. Then by definition $(s \cos\theta - t \sin\theta, s \sin\theta + t \cos\theta) \in M$ and also $(s \cos\theta + t \sin\theta, -s \sin\theta + t \cos\theta) \in M$. By symmetry of the set M relative to axis OY we get that $(s \cos\theta - t \sin\theta, -s \sin\theta - t \cos\theta) \in M$ and $(s \cos\theta + t \sin\theta, s \sin\theta - t \cos\theta) \in M$ meaning that $(s, -t) \in M_\theta \cap M_{-\theta}$, i.e. the set $M_\theta \cap M_{-\theta}$ is symmetric relative to the axis OS. Proof of the symmetry relative to the axis OT is analogous.

L e m m a 2. *The following equality holds:*

$$\{M_\theta \setminus (M_\theta \cap M_{-\theta})\} = \{M_\theta \setminus (M_\theta \cap M_{-\theta})\} \cap \{(s, t) \in R^2 : st < 0\}. \tag{5}$$

P r o o f. Equality (5) means that no point in the set $M_\theta \setminus (M_\theta \cap M_{-\theta})$ with Cartesian coordinates has identical signs. From (1) we have

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* The statement and proof of Lemmata 1 and 2 are presented at the end of the Section.

$$M_\theta = \{(\varrho, \alpha) : \varrho \leqslant \varphi(\alpha + \theta), \varrho \geqslant 0, 0 < \alpha < 2\pi\},$$
$$M_{-\theta} = \{(\varrho, \alpha) : \varrho \leqslant \varphi(\alpha - \theta), \varrho \geqslant 0, 0 < \alpha < 2\pi\},$$

and hence,

$$M_\theta \setminus (M_\theta \cap M_{-\theta}) = \{(\varrho, \alpha) : \varphi(\alpha - \theta) \leqslant \varrho \leqslant \varphi(\alpha + \theta), \varrho \geqslant 0, 0 < \alpha < 2\pi\}.$$

We shall show that for points with Cartesian coordinates with identical signs, i.e. in polar coordinates for $0 < \alpha < \pi/2$ and $\pi < \alpha < 3/2\pi$ the inequality

$$\varphi(\alpha + \theta) \leqslant \varphi(\alpha - \theta) \tag{6}$$

is valid and thus no such point belongs to $M_\theta \setminus (M_\theta \cap M_{-\theta})$.

We shall prove (6) only for $0 < \alpha < \pi/2$, the rest of the cases being proved similarly. Note that $\varphi(\alpha + \theta) = \varphi(\alpha - \theta)$ for $\alpha = 0$ and $\alpha = \pi/2$. In addition, by virtue of strong prolation of M, the function

$$\varphi(\alpha + \theta) \begin{cases} \text{is non-increasing for } \alpha \in (0, \pi/2 - \theta), \\ \text{is non-decreasing for } \alpha \in (\pi/2 - \theta, \pi/2), \end{cases}$$

and the function $\tag{7}$

$$\varphi(\alpha - \theta) \begin{cases} \text{is non-decreasing for } \alpha \in (0, \theta), \\ \text{is non-increasing for } \alpha \in (\theta, \pi/2). \end{cases}$$

This implies correctness of (6) for $\alpha \in (0, \theta) \cup (\pi/2 - \theta, \pi/2)$. Now consider the interval $[\theta, \pi/2 - \theta]$. First we prove the inequality (6) for $\alpha \in [\theta, \min(2\theta, \pi/2 - \theta)]$. Taking $\alpha^* = 2\theta - \alpha$, we get $2\theta - \min(2\theta, \pi/2 - \theta) < \alpha^* < \theta$, but for such α^* the inequality (6) has already been proved. Hence $\varphi(\alpha - \theta) = \varphi(\alpha^* - \theta) \geqslant \varphi(\alpha^* + \theta) \geqslant \varphi(\alpha + \theta)$, because $\varphi(\alpha + \theta)$ is non-increasing for $\alpha \in (0, \pi/2 - \theta)$. Thus for $2\theta > \pi/2 - \theta$. Lemma 2 is proved.

Consider the last case: $2\theta < \pi/2 - \theta$. Here for $\alpha \in (0, 2\theta)$ holds $\varphi(\alpha - \theta) \geqslant \varphi(\alpha + \theta)$, while for $\alpha \in (2\theta, \pi/2 - \theta)$ denoting $\alpha^* = \alpha - 2\theta$, we get

$$\varphi(\alpha - \theta) = \varphi(\alpha^* + \theta) \geqslant \varphi(\alpha + \theta),$$

since $\varphi(\alpha + \theta)$ is non-increasing in $(0, \pi/2 - \theta)$. This completes the proof of Lemma 2.

3. n-dimensional case. Here we generalize the results of the previous section to the n-dimensional case. Define the strong prolation in R^n by induction.

Definition 2. *A set $M \subset R^n$ $(n > 2)$ is called strongly prolate in L-direction if every $(n-1)$-dimensional section of the set M parallel to the axis L, is strongly prolate in L-direction in R^{n-1}.*

Example. The ellipsoid is strongly prolate along its longest axis. Define

$$R(a, A) = \int_{AM+a} g(x^T x) dx, \tag{8}$$

where $g(t)$, $0 < t < \infty$, is a Lebesgue integrable function, matrix A belongs to the set of $n \times n$ orthogonal matrices Q^n and the vector a belongs to the n-dimensional sphere $S^n(r)$ with the radius r.

Lemma 3. *For every $b \in S^n(r)$, $B \in Q^n$ and $r > 0$ the following equalities hold*

$$\max_{\substack{A \in Q^n \\ a \in S^n(r)}} R(a, A) = \max_{A \in Q^n} R(b, A) = \max_{a \in S^n(r)} R(a, B). \tag{9}$$

P r o o f. Prove the first equality. Let the first maximum in (9) be attained at $a_0 \in S^n(r)$ and $A_0 \in Q^n$. Then, using in (8) the change of variables $x = Vy$, $Vb = a_0$, $V \in Q^n$, we get

$$\max_{\substack{A \in Q^n \\ a \in S^n(r)}} R(a, A) \geqslant \max_{A \in Q^n} R(b, A) =$$

$$= \max_{A \in Q^n} \int_{Vy \in AM+b} g(y^T y) \, dy \geqslant \int_{A_0 M + a_0} g(y^T y) \, dy = R(a_0, A_0),$$

which implies the first equality in (9). The proof of the second one is analogous.

T h e o r e m 2. *Let $g(t)$ be a non-negative, convex and differentiable function on $(0, \infty)$ and let $M \subset R^n$ be strongly prolate in L-direction. Then for an arbitrary $r > 0$ the following relation holds*

$$\max_{\substack{A \in Q^n \\ a \in S^n(r)}} R(a, A) = R(a_0, A_0), \tag{10}$$

where the vector a_0 is parallel to some axis L_0, along which the set $A_0 M$ is strongly prolate.

R e m a r k. By Lemma 3 the solution of the problem (10) is not unique.

P r o o f. The proof is effected by induction, Theorem 1 being taken as a basis. Let Theorem 2 hold for $n = k - 1$, $k \geqslant 3$. Prove the relation (10) for $n = k$. Choose matrix A_0 so that the set $A_0 M$ would be strongly prolate in OX-direction. Let $a^* = (a_1^*, \ldots, a_n^*)$ be a solution of the problem (10). Then for $i \neq k$

$$\max_{a \in S^n(r)} R(a, A_0) = \int_{(x_i - a_i^*) \in P_{\overline{x}_i}(A_0 M)} dx_i \int_{(\overline{x_i - a_i^*}) \in (A_0 M)_{x_i}} g(x_i^2 + \overline{x}_i^T \overline{x}_i) \, d\overline{x}_i, \tag{11}$$

where $\overline{x}_i = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$, $\overline{a}_i^* = (a_1^*, a_2^*, \ldots, a_{i-1}^*, a_{i+1}^*, \ldots, a_k^*)$, $P_{x_i}(A_0 M)$ is the projection of the set $A_0 M$ to the X_i-axis and $(A_0 M)_{x_i}$ is $(k-1)$-dimensional section of $A_0 M$ with the plane $X_i = x_i$. By inductive assumption the integral

$$\int_{(\overline{x_i - a_i^*}) \in (A_0 M)_{x_i}} g(x_i^2 + \overline{x}_i^T \overline{x}_i) \, d\overline{x}_i$$

is maximized by the vector $b^i = (0, \ldots, 0, b_i, 0, \ldots, b_k)$ independently of the value of x_i since strong prolation of the set $(A_0 M)_{x_i}$ along the axis parallel to X_i-axis. Thus, the existence of such a coordinate $i \geqslant 1$, for which $a_i^* \neq b_i$, contradicts (11) and consequently $a_i^* = 0$, $i = 1, 2, \ldots \ldots, k - 1$, $a_k^* = r$. Q.E.D.

4. A p p l i c a t i o n t o q u a d r a t i c s t o c h a s t i c p r o g r a m m i n g. Consider the following quadratic stochastic programming problem

$$\max_{\substack{a \in A \\ D \in D_\gamma}} P[(X - a)^T D(X - a) \leqslant 1], \tag{12}$$

where P is the symbol of probability, $A \subset R^n$, D_γ is a class of positively defined $n \times n$-matrices such that the Lebesgue measure of the ellipsoid $\{X : (X-a)^T D (X-a) \leqslant 1\}$ is bounded, i. e.

$$\mu((X-a)^T D (X-a) \leqslant 1) \leqslant \gamma, \tag{13}$$

X is a random vector with spherically symmetric distribution, i. e. the density function $p(\cdot)$ of X exists and can be presented in the form

$$p(x) = g(x^T x),$$

where $g(t)$ is convex and differentiable on $(0, \infty)$.

Denote by $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ the eigen-values of the matrix D. Then by Theorem 2 for every $a \in A$

$$P[(X-a)^T D (X-a) \leqslant 1] = \int\limits_{(x-a^*)^T C^T DC(x-a^*) \leqslant 1} g(x^T x)\,dx \leqslant$$

$$\leqslant \int\limits_{\lambda_1(x_1-\|a^*\|)^2 + \sum\limits_{i=2}^{n} \lambda_i x_i^2 \leqslant 1} g(x^T x)\,dx,$$

where $a^* = (\|a\|, 0, \ldots, 0)$ and $a^* = Ca$ (C is an orthogonal matrix). Correctness of the last inequality follows from the strong prolation of the ellipsoid $\{X : (X-a^*)^T C^T DC (X-a^*) \leqslant 1\}$ along the axis which is parallel to the eigen-vector of the matrix $C^T DC$ corresponding to the eigen-value λ_1. From Lemma 1 in [4] and conditions for $g(t)$ we get

$$\int\limits_{\lambda_1(x_1-\|a^*\|)^2 + \sum\limits_{i=2}^{n} \lambda_i x_i^2 \leqslant 1} g(x^T x)\,dx \leqslant \int\limits_{\lambda_1(x_1-r)^2 + \sum\limits_{i=2}^{n} \lambda_i x_i^2 \leqslant 1} g(x^T x)\,dx,$$

where $r = \min\limits_{a \in A} \|a\|$ and, consequently,

$$P[(X-a)^T D (X-a) \leqslant 1] \leqslant \max\limits_{\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n} \int\limits_{\lambda_1(x_1-r)^2 + \sum\limits_{i=2}^{n} \lambda_i x_i^2 \leqslant 1} g(x^T x)\,dx.$$

Lemma 4. *For arbitrary $\lambda_1 > 0$ and $l > 0$*

$$\max\limits_{\lambda_2, \ldots, \lambda_n : \prod\limits_{i=2}^{n} \lambda_i = l^{n-1}} \int\limits_{\lambda_1(x_1-r)^2 + \sum\limits_{i=2}^{n} \lambda_i x_i^2 \leqslant 1} g(x^T x)\,dx = \int\limits_{\lambda_1(x_1-r)^2 + l \sum\limits_{i=2}^{n} x_i^2 \leqslant 1} g(x^T x)\,dx.$$

Proof. Consider two sets in R^{n-1}:

$$Q_\Lambda = \{y = (x_2, \ldots, x_n) \in R^{n-1} : y^T \Lambda y \leqslant 1 - \lambda_1(x_1-r)^2\},$$

$$Q_l = \{y = (x_2, \ldots, x_n) \in R^{n-1} : ly^T y \leqslant 1 - \lambda_1(x_1-r)^2\},$$

where Λ is a diagonal matrix with elements $\lambda_2, \ldots, \lambda_n$. Write the trivial expansion

$$\int\limits_{\lambda_1(x_1-r)^2 + \sum\limits_{i=2}^{n} \lambda_i x_i^2 \leqslant 1} g(x^T x)\,dx =$$

$$= \int\limits_{\lambda_1(x_1-r)^2 \leqslant 1} dx_1 \left(\int\limits_{Q_\Lambda \cap Q_l} g(x_1^2 + y^T y)\,dy + \int\limits_{Q_\Lambda \backslash (Q_\Lambda \cap Q_l)} g(x_1^2 + y^T y)\,dy \right). \tag{14}$$

By the mean value theorem there exists a vector $y_0 \in Q_\Lambda \setminus (Q_\Lambda \cap Q_l)$, so that

$$\int_{Q_\Lambda \setminus (Q_\Lambda \cap Q_l)} g(x_1^2 + y^T y)\, dy = g(x_1^2 + y_0^T y_0)\, \mu(Q_\Lambda \setminus (Q_\Lambda \cap Q_l)),$$

and a vector $y_1 \in Q_l \setminus (Q_\Lambda \cap Q_l)$, so that

$$\int_{Q_l \setminus (Q_\Lambda \cap Q_l)} g(x_1^2 + y^T y)\, dy = g(x_1^2 + y_1^T y_1)\, \mu(Q_l \setminus (Q_\Lambda \cap Q_l)).$$

Note that the restriction $\prod_{i=2}^{n} \lambda_i = l^{n-1}$ means that for an arbitrary Λ the Lebesgue measure $\mu(Q_\Lambda)$ of the ellipsoid Q_Λ is constant, and, consequently,

$$\mu(Q_\Lambda \setminus (Q_\Lambda \cap Q_l)) = \mu(Q_l \setminus (Q_\Lambda \cap Q_l)). \tag{15}$$

In addition, from the definition of the vectors y_0, y_1 and the sets Q_Λ, Q_l, we get

$$y_1^T y_1 \leqslant l^{-1}(1 - \lambda_1 (x_1 - r)^2) < y_0^T y_0. \tag{16}$$

Using (14)—(16), and non-decreasing property of the function $g(t)$, we obtain the inequality

$$\int_{Q_\Lambda \setminus (Q_\Lambda \cap Q_l)} g(x_1^2 + y^T y)\, dy \leqslant \int_{Q_l \setminus (Q_\Lambda \cap Q_l)} g(x_1^2 + y^T y)\, dy,$$

which with (14) proves Lemma 4.

From Lemma 4 we get

$$\max_{0 < \lambda_1 \leqslant \ldots \leqslant \lambda_n} \int_{\lambda_1(x_1-r)^2 + \sum_{i=2}^{n} \lambda_i x_i^2 \leqslant 1} g(x^T x)\, dx =$$

$$= \max_{0 < \lambda_1, l > \lambda_1} \max_{\prod_{i=2}^{n} \lambda_i = l^{n-1}} \int_{\lambda_1(x_1-r)^2 + \sum_{i=2}^{n} \lambda_i x_i^2 \leqslant 1} g(x^T x)\, dx = \max_{0 < \lambda_1 < l} \int_{\lambda_1(x_1-r)^2 + l \sum_{i=2}^{n} x_i^2 \leqslant 1} g(x^T x)\, dx. \tag{17}$$

The restriction (13) is equivalent to the inequality

$$\prod_{i=1}^{n} \lambda_i \geqslant \gamma^{-1} \Gamma^n (1/2) / \Gamma(n/2 + 1) = q^{n-1},$$

and, consequently, $\lambda_1 l^{n-1} \geqslant q^{n-1}$. Denoting $\lambda_1 = \lambda^{n-1}$ and using (17), we obtain the following form of the problem (12):

$$\max_{0 < \lambda < q^{1/n}} \int_{\lambda^{n-1}(x_1-r)^2 + q/\lambda \sum_{i=2}^{n} x_i^2 \leqslant 1} g(x^T x)\, dx,$$

i.e. the problem (12) is reduced to finding the maximum of a one-dimensional function.

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VÕRRATUS SFÄÄRILISELT SÜMMEETRILISEST FUNKTSIOONIST SÕLTUVA INTEGRAALI KOHTA JA SELLE KASUTAMINE OPTIMEERIMISEKS

On vaadeldud integraali väärtuse sõltuvust tugevalt väljavenitatud integreerimispiirkonna pööramisest oma tsentri ümber, kusjuures integreeritav funktsioon on sfääriliselt sümmeetriline. On näidatud, et integraali väärtus on maksimaalne, kui integreerimispiirkonna pikim telg ühtib piirkonna keskpunkti ja koordinaatide alguspunkti läbiva sirgega. Tulemus on kasutatud stohhastilise ruutplaneerimisülesande lahendi leidmisel.

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НЕРАВЕНСТВО ДЛЯ ИНТЕГРАЛОВ ОТ СФЕРИЧЕСКИ СИММЕТРИЧНЫХ ФУНКЦИЙ И ЕГО ПРИМЕНЕНИЕ В ОПТИМИЗАЦИИ

Решается экстремальная задача, за критерий которой принимается значение интеграла от сферически симметричной функции по сильно вытянутому множеству с заданным центром, меняющееся при вращении множества вокруг центра. Доказано, что критерий достигает своего максимума, когда ось сильной вытянутости множества совпадает с прямой, проведенной из центра множества в начало координат. Полученный результат используется для нахождения решения задачи квадратичного стохастического программирования.