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EXTREMAL PROBLEMS FOR CUBATURE FORMULAS

First problems of constructing the optimal quadrature formulas were proposed and solved by A. Sard [1] in the case of fixed nodes and by S. Nikolsky [2] in the case of arbitrary nodes. By now the theory of constructing the optimal quadratures has greatly developed, as shows a survey of fundamental results in this field [3]. The field connected with constructing asymptotically optimal cubature formulas, elaborated by S. Sobolev and his pupils, is of great importance in the theory of cubature formulas [4]. The theory of optimal cubature formulas in general is not developed to such a degree. Main results of this theory are obtained for the sets of functions with derivatives of some orders belonging to L_2 [5,3]. The peculiarity of Hilbert metric was essentially used in obtaining these results.

In this paper we introduce sets of functions of two variables which represent a natural generalization of corresponding sets of functions of one variable, and consider the problem of constructing optimal cubature formulas for these sets with a rectangular mesh of nodes.

Let us introduce notions:

r, s, M, P, Q, m, n, $1 \leqslant q \leqslant \infty$, be given;

$W^r L_q$ is the set of functions $f(x)$ on $[0,1]$ with absolutely continuous derivatives of order $r-1$, and rth derivatives satisfying $\|f^{(r)}(\cdot)\|_{L_q(0,1)} \leqslant 1$;

$W^{r,s} L_q$ is the set of functions $f(x, y)$ on square $0 \leqslant x$, $y \leqslant 1$ with piecewise continuous derivatives

$$f^{(j,l)}(x, y) = \frac{\partial^{j+l}}{\partial x^j \partial y^l} f(x, y) \qquad (j=0, \ldots, r;\ l=0, \ldots, s)$$

and satisfying conditions

$$\Big\|\int_0^1 f^{(r,0)}(\cdot, y)\, dy\Big\|_{L_q(0,1)} \leqslant P, \quad \Big\|\int_0^1 f^{(0,s)}(x, \cdot)\, dx\Big\|_{L_q(0,1)} \leqslant Q,$$

$$\|f^{(r,s)}(\cdot,\cdot)\|_{L_q(0,1;0,1)} \leqslant M.$$

The formula with remainder $R(f)$ is called the optimal one for the set H of functions f, if its parameters are chosen from the condition of minimum of the quantity

$$R[H] = \sup_{f \in H} |R(f)|,$$

which is called the exact bound for the remainder of the formula [3].

We examine the problem of constructing the optimal formula for the set $W^{r,s} L_q$

$$\int_0^1\int_0^1 f(x,y)\,dx\,dy=\sum_{i=1}^{m}\sum_{k=1}^{n}\sum_{j\in J_i}\sum_{l\in L_k}A_{ik}^{jl}f^{(j,l)}(x_i,y_k)+R(f) \tag{1}$$

where $J_i\subseteq\{0,\ldots,r-1\}$ $(i=1,\ldots,m)$, $L_k\subseteq\{0,\ldots,s-1\}$ $(k=1,\ldots,n)$, both with fixed and arbitrary nodes.

We denote by B_{ij}^* the coefficients and by δ_1^* the exact bound for the remainder of the optimal formula for W^rL_q

$$\int_0^1 f(x)\,dx=\sum_{i=1}^{m}\sum_{j\in J_i}B_{ij}f^{(j)}(x_i)+R_1(f) \tag{2}$$

where the nodes $0<x_1<x_2<\ldots<x_m<1$ are fixed. Denote by C_{kl}^* the coefficients and by δ_2^* the exact bound for the remainder of the optimal formula for W^sL_q

$$\int_0^1 f(y)\,dy=\sum_{k=1}^{n}\sum_{l\in L_k}C_{kl}f^{(l)}(y_k)+R_2(f), \tag{3}$$

with fixed nodes $0<y_1<y_2<\ldots<y_n<1$.

It is known [3,6,7] that for constructing for the sets W^rL_q and W^sL_q the optimal formulas (2) and (3) respectively, it is sufficient to construct the monosplines $K_1^*(x)$ and $K_2^*(y)$ of least deviation from zero in $L_p(0,1)$ $(p^{-1}+q^{-1}=1)$ from all monosplines of the form

$$K_1(x)=\frac{x^r}{r!}-\sum_{i=1}^{m}\sum_{j\in J_i}\frac{B_{ij}}{(r-j-1)!}(x-x_i)_+^{r-j-1}, \tag{4}$$

$$K_2(y)=\frac{y^s}{s!}-\sum_{k=1}^{n}\sum_{l\in L_k}\frac{C_{kl}}{(s-l-1)!}(y-y_k)_+^{s-l-1} \tag{5}$$

respectively, satisfying conditions

$$K_1^{(j)}(1)=0 \quad (j=0,\ldots,r-1), \tag{6}$$

$$K_2^{(l)}(1)=0 \quad (l=0,\ldots,s-1) \tag{7}$$

where $u_+^\alpha=u^\alpha$, if $u\geqslant 0$ and $u_+^\alpha=0$, if $u<0$. We note that

$$\|K_1^*(\cdot)\|_{L_p(0,1)}=\delta_1^*,\qquad \|K_2^*(\cdot)\|_{L_p(0,1)}=\delta_2^*.$$

We mark the equalities

$$B_{ij}=(-1)^j[K_1^{(r-j-1)}(x_i-0)-K_1^{(r-j-1)}(x_i+0)] \quad (i=1,\ldots,m;\ j\in J_i), \tag{8}$$

$$C_{kl}=(-1)^l[K_2^{(s-l-1)}(y_k-0)-K_2^{(s-l-1)}(y_k+0)] \quad (k=1,\ldots,n;\ l\in L_k) \tag{9}$$

for the monosplines (4), (5).

L e m m a. *The monospline*

$$K^*(x,y)=K_1^*(x)K_2^*(y)$$

is of least deviation from zero in the L_p (0,1; 0,1) metric from the set of all monosplines

$$K(x,y)=\frac{x^ry^s}{r!s!}-\sum_{i=1}^{m}\sum_{j\in J_i}\frac{b_{ij}(-1)^jy^s(x-x_i)_+^{r-j-1}}{s!(r-j-1)!}-$$

$$-\sum_{k=1}^{n}\sum_{l\in L_k}\frac{c_{kl}(-1)^l x^r (y-y_k)_+^{s-l-1}}{r!(s-l-1)!}+$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{n}\sum_{j\in J_i}\sum_{l\in L_k}\frac{A_{ik}^{jl}(-1)^{j+l}}{(r-j-1)!(s-l-1)!}(x-x_i)_+^{r-j-1}(y-y_k)_+^{s-l-1} \quad (10)$$

satisfying conditions

$$\begin{gathered} K^{(j,0)}(1,y)\equiv 0 \quad (j=0,\ldots,r-1),\\ K^{(0,l)}(x,1)\equiv 0 \quad (l=0,\ldots,s-1). \end{gathered} \quad (11)$$

Proof will be obtained by introducing bases in the spaces of monosplines $K_1(x)$, $K_2(y)$, $K(x,y)$ satisfying (6), (7), (11), respectively, and expanding $K_1(x)$, $K_2(y)$ and $K(x,y)$ into the functions of these bases and using the proper reasonings [8].

Theorem 1. *The optimal formula* (1) *for the set* $W^{r,s}L_q$ *with fixed nodes* $0<x_1<\ldots<x_m<1$, $0<y_1<\ldots<y_n<1$ *has coefficients*

$$A_{ik}^{jl}=B_{ij}^{*}C_{kl}^{*} \quad (12)$$

$$(i=1,\ldots,m;\ j\in J_i;\ k=1,\ldots,n;\ l\in L_k)$$

and exact bound for the remainder

$$R[W^{r,s}L_q]=P\delta_1^*+Q\delta_2^*+M\delta_1^*\delta_2^*. \quad (13)$$

Proof. 1. As is clear from paper [9], the formulas (1) with finite value of quantity $R[W^{r,s}L_q]$ are exact for functions

$$f_1(x,y)=\sum_{\alpha=0}^{r-1}a_\alpha(y)(1-x)^\alpha(\alpha+1),\qquad f_2(x,y)=\sum_{\beta=0}^{s-1}b_\beta(x)(1-y)^\beta(\beta+1)$$

satisfying conditions

$$\begin{gathered} \int_0^1 f_1^{(0,s)}(x,y)\,dx=\sum_{\alpha=0}^{r-1}a_\alpha^{(s)}(y)=0,\\ \int_0^1 f_2^{(r,0)}(x,y)\,dy=\sum_{\beta=0}^{s-1}b_\beta^{(r)}(x)=0 \end{gathered} \quad (14)$$

where $a_\alpha(y)$, $b_\beta(x)$ are arbitrary functions with absolutely continuous derivatives of orders $s-1$, $r-1$, respectively.

Using this fact, we condition

$$\int_0^1\sum_{\alpha=0}^{r-1}a_\alpha(y)\,dy=$$

$$=\sum_{i=1}^{m}\sum_{k=1}^{n}\sum_{j\in J_i}\sum_{l\in L_k}\sum_{\alpha=0}^{r-1}A_{ik}^{jl}\frac{(-1)^j(\alpha+1)!}{(\alpha-j)!}(1-x_i)^{\alpha-j}a_\alpha^{(l)}(y_k)^* \quad (15)$$

on the coefficients of formulas (1) under consideration.

* We suppose $\dfrac{u^{m-j}}{(m-j)!}=0$, if $m<j$.

We denote $a_0(y)+\ldots+a_{r-1}(y)=\varphi(y)$, where $\varphi(y)$ is a polynomial of degree $s-1$, as the functions $a_0(y), \ldots, a_{r-1}(y)$ are connected only by condition (14). Substituting (15) for the value $a_0(y)=\varphi(y)-(a_1(y)+\ldots+a_{r-1}(y))$, we obtain

$$\int_0^1 \varphi(y)\,dy=\sum_{\alpha=1}^{r-1}\sum_{i=1}^{m}\sum_{k=1}^{n}\sum_{j\in J_i}\sum_{l\in L_k} A_{ik}^{jl}\frac{(-1)^j(\alpha+1)!(1-x_i)^{\alpha-j}}{(\alpha-j)!}a_\alpha^{(l)}(y_k)+$$

$$+\sum_{\substack{i=1\\0\in J_i}}^{m}\sum_{k=1}^{n}\sum_{l\in L_k}A_{ik}^{0l}\varphi^{(l)}(y_k)-\sum_{\alpha=1}^{r-1}\sum_{\substack{i=1\\0\in J_i}}^{m}\sum_{k=1}^{n}\sum_{l\in L_k}A_{ik}^{0l}a_\alpha^{(l)}(y_k).$$

We conclude from this formula and from the arbitrariness of numbers $a_\alpha^{(l)}(y_k)$ $(k=1, \ldots, n;\ l\in L_k;\ \alpha=1, \ldots, r-1)$ that the coefficients of formulas (1) with finite value of quantity $R[W^{r,s}L_q]$ satisfy conditions

$$\sum_{\substack{i=1\\0\in J_i}}^{m}A_{ik}^{0l}=\sum_{i=1}^{m}\sum_{j\in J_i}\frac{A_{ik}^{jl}(-1)^j(\alpha+1)!(1-x_i)^{\alpha-j}}{(\alpha-j)!} \tag{16}$$

$$(k=1, \ldots, n;\ l\in L_k;\ \alpha=1, \ldots, r-1).$$

In the same way we conclude that the coefficients of formulas (1) with finite value of quantity $R[W^{r,s}L_q]$ satisfy conditions

$$\sum_{\substack{k=1\\0\in L_k}}^{n}A_{ik}^{j0}=\sum_{k=1}^{n}\sum_{l\in L_k}\frac{A_{ik}^{jl}(-1)^l(\beta+1)!(1-y_k)^{\beta-l}}{(\beta-l)!} \tag{17}$$

$$(i=1, \ldots, m;\ j\in J_i;\ \beta=1, \ldots, s-1).$$

2. Let us consider an arbitrary formula (1) with coefficients satisfying (16), (17) and a function $K(x, y)$ (see (10)) with coefficients

$$c_{kl}=\sum_{\substack{i=1\\0\in J_i}}^{m}A_{ik}^{0l} \quad (k=1, \ldots, n;\ l\in L_k), \tag{18}$$

$$b_{ij}=\sum_{\substack{k=1\\0\in L_k}}^{n}A_{ik}^{j0} \quad (i=1, \ldots, m;\ j\in J_i). \tag{19}$$

Let $f(x,y)\in W^{r,s}L_q$, $x_0=y_0=1-x_{m+1}=1-y_{n+1}=0$.
Integrating by parts, we obtain

$$\int_0^1\int_0^1 f(x,y)\,dx\,dy=\sum_{i=0}^{m}\sum_{k=0}^{n}\int_{x_i}^{x_{i+1}}\int_{y_k}^{y_{k+1}} f(x,y)K^{(r,s)}(x,y)\,dx\,dy=$$

$$=\sum_{j=0}^{r-1}\sum_{l=0}^{s-1}(-1)^{j+l}\Bigg[\sum_{i=1}^{m}\sum_{k=1}^{n}f^{(j,l)}(x_i, y_k)\,(K^{(r-j-1,\,s-l-1)}(x_i-0, y_k-0)-$$

$$-K^{(r-j-1,\,s-l-1)}(x_i-0, y_k+0)-K^{(r-j-1,\,s-l-1)}(x_i+0, y_k-0)+$$

$$+K^{(r-j-1,\,s-l-1)}(x_i+0, y_k+0))+\sum_{k=1}^{n}f^{(j,l)}(1, y_k)\,(K^{(r-j-1,\,s-l-1)}(1, y_k-0)-$$

$$-K^{(r-j-1,\,s-l-1)}(1,y_k+0))-\sum_{k=1}^{n}f^{(j,l)}(0,y_k)\,(K^{(r-j-1,\,s-l-1)}(0,\,y_k-0)-$$

$$-K^{(r-j-1,\,s-l-1)}(0,y_k+0))+\sum_{i=1}^{m}f^{(j,l)}(x_i,1)\,(K^{(r-j-1,\,s-l-1)}(x_i-0,\,1)-$$

$$-K^{(r-j-1,\,s-l-1)}(x_i+0,1))-\sum_{i=1}^{m}f^{(j,l)}(x_i,0)\,(K^{(r-j-1,\,s-l-1)}(x_i-0,\,0)- \tag{20}$$

$$-K^{(r-j-1,\,s-l-1)}(x_i+0,0))+f^{(j,l)}(1,1)\,K^{(r-j-1,\,s-l-1)}(1,1)-$$

$$-f^{(j,l)}(1,0)\,K^{(r-j-1,\,s-l-1)}(1,0)-f^{(j,l)}(0,1)\,K^{(r-j-1,\,s-l-1)}(0,1)+$$

$$+f^{(j,l)}(0,0)\,K^{(r-j-1,\,s-l-1)}(0,0)\Big]+(-1)^r\int_0^1\int_0^1 f^{(r,0)}(x,y)\,K^{(0,s)}(x,y)\,dx\,dy+$$

$$+(-1)^s\int_0^1\int_0^1 f^{(0,s)}(x,y)\,K^{(r,0)}(x,y)\,dx\,dy-$$

$$-(-1)^{r+s}\int_0^1\int_0^1 f^{(r,s)}(x,y)\,K(x,y)\,dx\,dy.$$

Let us examine some properties of monospline $K(x,\,y)$. We suppose that $(i,\,k,\,j,\,l)\in V$, if $i\in\{1,\,\ldots,\,m\}$, $j\in J_i$, $k\in\{1,\,\ldots,\,n\}$, $l\in L_k$. Denote by

$$\Lambda_{ik}^{jl}=K^{(r-j-1,\,s-l-1)}(x_i-0,\;y_k-0)-K^{(r-j-1,\,s-l-1)}(x_i-0,\;y_k+0)-$$
$$-K^{(r-j-1,\,s-l-1)}(x_i+0,\;y_k-0)+K^{(r-j-1,\,s-l-1)}(x_i+0,\;y_k+0). \tag{21}$$

Then it follows from (10) that

$$\Lambda_{ik}^{jl}=\begin{cases}(-1)^{j+l}A_{ik}^{jl}, & \text{if } (i,k,j,l)\in V\\ 0, & \text{if } (i,k,j,l)\,\overline{\in}\, V.\end{cases} \tag{22}$$

We also obtain from (10) the identities

$$K^{(r-j-1,0)}(0,y)\equiv K^{(0,s-l-1)}(x,0)\equiv 0\quad (j=0,\,\ldots,\,r-1;\ l=0,\,\ldots,\,s-1). \tag{23}$$

From conditions (16), (17), (10), (18), (19) it follows that

$$K^{(r-j-1,0)}(1,y)\equiv K^{(0,s-l-1)}(x,1)\equiv 0\quad (j=0,\,\ldots,\,r-1;\ l=0,\,\ldots,\,s-1). \tag{24}$$

Using (21)—(24) we assertain that (20) is the cubature formula (1) with remainder

$$R(f)=(-1)^r\int_0^1\int_0^1 f^{(r,0)}(x,y)\,K^{(0,s)}(x,y)\,dx\,dy+$$
$$+(-1)^s\int_0^1\int_0^1 f^{(0,s)}(x,y)\,K^{(r,0)}(x,y)\,dx\,dy-$$
$$-(-1)^{r+s}\int_0^1\int_0^1 f^{(r,s)}(x,y)\,K(x,y)\,dx\,dy. \tag{25}$$

Let us note that the monosplines

$$K^{(0,s)}(x,y)=\frac{x^r}{r!}-\sum_{i=1}^{m}\sum_{j\in J_i}\frac{(-1)^i b_{ij}}{(r-j-1)!}(x-x_i)_+^{r-j-1}=K_1(x), \tag{26}$$

$$K^{(r,0)}(x,y)=\frac{y^s}{s!}-\sum_{k=1}^{n}\sum_{l\in L_k}\frac{(-1)^l c_{kl}}{(s-l-1)!}(y-y_k)_+^{s-l-1}=K_2(y) \qquad (27)$$

are the functions of one variable.

3. For arbitrary function $f(x, y)\in W^{r,s}L_q$ we obtain by Hölder's inequality an estimate

$$|R(f)|\leqslant P\,\|K_1\|_{L_p(0,1)}+Q\,\|K_2\|_{L_p(0,1)}+M\,\|K\|_{L_p(0,1;0,1)} \qquad (28)$$

where K_1, K_2 and K are determined by (26), (27), (10), (18), (19).

Let $q>1$, $M_1(x)$, $M_2(y)$, $M(x, y)$ be the functions satisfying conditions

$$M_1^{(r)}(x)=\|K_1\|_{L_p(0,1)}^{1-p}(-1)^r|K_1(x)|^{p-1}\operatorname{sign}K_1(x),$$

$$M_2^{(s)}(y)=\|K_2\|_{L_p(0,1)}^{1-p}(-1)^s|K_2(y)|^{p-1}\operatorname{sign}K_2(y),$$

$$M^{(r,s)}(x,y)=\|K\|_{L_p(0,1;0,1)}^{1-p}(-1)^{r+s+1}|K(x,y)|^{p-1}\operatorname{sign}K(x,y).$$

Then the function

$$f_0(x,y)=PM_1(x)+QM_2(y)+MM(x,y)-M\int_0^1 M(x,y)\,dx-M\int_0^1 M(x,y)\,dy$$

belongs to the set $W^{r,s}L_q$, and the inequality (28) turns into equality for this function, as it follows from (25). Thus, if $q>1$ then

$$R[W^{r,s}L_q]=P\,\|K_1\|_{L_p(0,1)}+Q\,\|K_2\|_{L_p(0,1)}+M\,\|K\|_{L_p(0,1;0,1)}. \qquad (29)$$

This equality is also correct in the case $q=1$, and may be obtained from (29) by limit passage $p\to\infty$.

4. Thus, for constructing the optimal formula (1) for the set $W^{r,s}L_q$ it is sufficient to minimize the right part of (29). It follows from lemma and identities (24) that inequality

$$P\,\|K_1\|_{L_p(0,1)}+Q\,\|K_2\|_{L_p(0,1)}+M\,\|K\|_{L_p(0,1;0,1)}\geqslant$$

$$\geqslant P\,\|K_1^*\|_{L_p(0,1)}+Q\,\|K_2^*\|_{L_p(0,1)}+M\,\|K_1^*\|_{L_p(0,1)}\|K_2^*\|_{L_p(0,1)}$$

is accomplished for every monospline (10), satisfying (16)—(19).

Therefore the theorem will be proved if we show that the coefficients of monospline $K^*(x,y)=K_1^*(x)K_2^*(y)$ are calculated by formula (22) and satisfy conditions (12), (16)—(19).

In fact, by using (21), (22), (8), (9) we obtain the equalities

$$A_{ik}^{*jl}=(-1)^{j+l}[K_1^{*(r-j-1)}(x_i-0)-K_1^{*(r-j-1)}(x_i+0)]\,[K_2^{*(s-l-1)}(y_k-0)-$$

$$-K_2^{*(s-l-1)}(y_k+0)]=B_{ij}^*C_{kl}^*$$

$$(i=1,\ldots,m;\quad k=1,\ldots,n;\quad j\in J_i;\quad l\in L_k).$$

By this equality and (6), (7) we obtain that

$$\sum_{i=1}^{m}\sum_{j\in J_i}\frac{A_{ik}^{*jl}(-1)^j(\alpha+1)!(1-x_i)^{\alpha-j}}{(\alpha-j)!}=$$

$$=C_{kl}^{*}\sum_{i=1}^{m}\sum_{j\in J_i}\frac{\overset{*}{B}_{ij}(-1)^{j}(\alpha+1)!(1-x_i)^{\alpha-j}}{(\alpha-j)!}=C_{kl}^{*}\sum_{\substack{i=1\\0\in J_i}}^{m}B_{i0}^{*}=\sum_{\substack{i=1\\0\in J_i}}^{m}A_{ik}^{*0l}$$

$$(\alpha=1,\ \ldots,\ r-1),$$

$$\sum_{k=1}^{n}\sum_{l\in L_k}\frac{\overset{*jl}{A_{ik}}(-1)^{l}(\beta+1)!(1-y_k)^{\beta-l}}{(\beta-l)!}=$$

$$=B_{ij}^{*}\sum_{k=1}^{n}\sum_{l\in L_k}\frac{\overset{*}{C}_{kl}(-1)^{l}(\beta+1)!(1-y_k)^{\beta-l}}{(\beta-l)!}=$$

$$=B_{ij}^{*}\sum_{\substack{k=1\\0\in L_k}}^{n}C_{k0}^{*}=\sum_{\substack{k=1\\0\in L_k}}^{n}A_{ik}^{*j0}\quad(\beta=1,\ \ldots,\ s-1).$$

Satisfaction of conditions (18), (19) for the monospline $K^{*}(x,y)=$ $=K_{1}^{*}(x)K_{2}^{*}(y)$ is evident.

Thus the quality (29) has its minimal value if $K(x,y)=K_{1}^{*}(x)K_{2}^{*}(y)$. The theorem is proved.

Note 1. *The proved statements remain valid if the equalities* $x_1=0$, $y_1=0$, $x_m=1$, $y_n=1$ (*or part of them*) *are fulfilled.*

Let now the nodes of the formulas (1)—(3) be arbitrary. Let us denote by $\bar{B}_{ij}$, $\bar{x}_i$ $(i=1,\ \ldots,\ m;\ j\in J_i)$, $\bar{\delta}_1$ the coefficients, nodes and exact bound for the remainder of the optimal formula (2) for the set $W^{r}L_q$, and by $\bar{C}_{kl}$, $\bar{y}_k$ $(k=1,\ \ldots,\ n;\ l\in L_k)$. $\bar{\delta}_2$ the coefficients, nodes and exact bound for the remainder of the optimal formula (3) for $W^{s}L_q$.

Theorem 2. *The optimal formula* (1) *for* $W^{r,s}L_q$ *has the coefficients and nodes*

$$A_{ik}^{jl}=\bar{B}_{ij}\bar{C}_{kl},\quad x_i=\bar{x}_i,\quad y_k=\bar{y}_k$$

$$(i=1,\ \ldots,\ m;\ k=1,\ \ldots,\ n;\ j\in J_i;\ l\in L_k)$$

and exact bound for the remainder

$$R[W^{r,s}L_q]=P\bar{\delta}_1+Q\bar{\delta}_2+M\bar{\delta}_1\bar{\delta}_2.$$

Proof at once follows from the estimate (13).

Note 2. *Similar result remains valid if some of nodes of formulas* (1)—(3) *are fixed, e. g., if the extreme nodes (or part of them) of these formulas coincide with integration limits.*

REFERENCES

1. Sard, A., Amer. J. Math., LXXI, 80 (1949).
2. Никольский С. М., Успехи матем. наук, **V**, 2(36), 165 (1950).
3. Никольский С. М., Квадратурные формулы, М., 1974.
4. Соболев С. Л., Введение в теорию кубатурных формул, М., 1974.
5. Левин М., Изв. АН ЭССР, Физ. Матем., **25**, 331 (1976).
6. Левин М. И., Гиршович Ю. М., Арро В. К., Докл. АН СССР, **226**, 51 (1976).
7. Levin, M., Girshovich, Y., ENSV TA Toimet., Füüs. Matem., **24**, 264 (1975).
8. Левин М., Изв. АН ЭССР, Физ. Матем., **18**, 464 (1969).
9. Левин М., Изв. АН ЭССР, Физ. Матем., **20**, 90 (1971).

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KUBATUURVALEMITE EKSTREEMUMÜLESANDED

Olgu $W^{r,s}L_q$ $(1 \leqslant q \leqslant \infty)$ kõigi selliste funktsioonide $f(x, y)$ hulk, mis omavad ruudus $0 \leqslant x, y \leqslant 1$ tükati pidevaid tuletisi

$$f^{(j,l)}(x, y) = \frac{\partial^{j+l}}{\partial x^j \partial y^l} f(x, y) \qquad (j=0, \ldots, r; \ l=0, \ldots, s)$$

ja mis rahuldavad tingimusi

$$\| \int_0^1 f^{(r,0)}(\cdot, y) dy \|_{L_q(0,1)} \leqslant P, \qquad \| \int_0^1 f^{(0,s)}(x, \cdot) dx \|_{L_q(0,1)} \leqslant Q,$$

$$\| f^{(r,s)}(\cdot, \cdot) \|_{L_q(0,1;\, 0,1)} \leqslant M.$$

Olgu $\overline{B}_{ij}$, $\overline{x}_i$ $(i=1, \ldots, m; \ j \in J_i)$, $\overline{\delta}_1$ hulgal W^rL_q parima kvadratuurvalemi (2) vastavalt kordajad, sõlmed ja jääkliikme täpne hinnang ning $\overline{C}_{kl}$, $\overline{y}_k$ $(k=1, \ldots, n; \ l \in L_k)$, $\overline{\delta}_2$ hulgal W^sL_q parima kvadratuurvalemi (3) vastavalt kordajad, sõlmed ja jääkliikme täpne hinnang.

Teoreem. *Hulgal $W^{r,s}L_q$ parima kubatuurvalemi* (1) *kordajad ja sõlmed on*

$$A_{ik}^{jl} = \overline{B}_{ij}\overline{C}_{kl}, \quad x_i = \overline{x}_i, \quad y_k = \overline{y}_k$$

$$(i=1, \ldots, m; \ k=1, \ldots, n; \ j \in J_i; \ l \in L_k)$$

ja jääkliikme täpne hinnang on

$$R[W^{r,s}L_q] = P\overline{\delta}_1 + Q\overline{\delta}_2 + M\overline{\delta}_1\overline{\delta}_2.$$

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ЭКСТРЕМАЛЬНЫЕ ЗАДАЧИ ДЛЯ КУБАТУРНЫХ ФОРМУЛ

Обозначим через $W^{r,s}L_q$ $(1 \leqslant q \leqslant \infty)$ множество всех функций $f(x, y)$, имеющих на квадрате $0 \leqslant x, y \leqslant 1$ кусочно-непрерывные производные

$$f^{(j,l)}(x, y) = \frac{\partial^{j+l} f(x, y)}{\partial x^j \partial y^l} \qquad (j=0, \ldots, r; \ l=0, \ldots, s)$$

и удовлетворяющих условиям

$$\| \int_0^1 f^{(r,0)}(\cdot, y) dy \|_{L_q(0,1)} \leqslant P, \qquad \| \int_0^1 f^{(0,s)}(x, \cdot) dx \|_{L_q(0,1)} \leqslant Q,$$

$$\| f^{(r,s)}(\cdot, \cdot) \|_{L_q(0,1;\, 0,1)} \leqslant M.$$

Через $\overline{B}_{ij}$, $\overline{x}_i$ $(i=1, \ldots, m; \ j \in J_i)$ и $\overline{\delta}_1$ обозначим веса, узлы и точную оценку остатка наилучшей на множестве W^rL_q квадратурной формулы (2), а через $\overline{C}_{kl}$, $\overline{y}_k$ $(k=1, \ldots, n; \ l \in L_k)$ и $\overline{\delta}_2$ — веса, узлы и точную оценку остатка наилучшей на множестве W^sL_q формулы (3). Тогда справедлива

Теорема. *Наилучшая на множестве $W^{r,s}L_q$ кубатурная формула* (1) *имеет веса, узлы*

$$A_{ik}^{jl} = \overline{B}_{ij}\overline{C}_{kl}, \quad x_i = \overline{x}_i, \quad y_k = \overline{y}_k$$

$$(i=1, \ldots, m; \ k=1, \ldots, n; \ j \in J_i; \ l \in L_k)$$

и точную оценку остатка

$$R[W^{r,s}L_q] = P\overline{\delta}_1 + Q\overline{\delta}_2 + M\overline{\delta}_1\overline{\delta}_2.$$