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T. VIIK

SOURCE FUNCTIONS FOR AN ISOTROPICALLY SCATTERING HOMOGENEOUS SLAB BOUNDED BY A PERFECT SPECULAR REFLECTOR

The source function for an isotropically scattering homogeneous slab bounded by a perfect specular reflector is found using the discrete ordinate solution of the Ambartsumian-Sobolev equation.

This paper deals with the problem of finding source functions for an absorbing and isotropically scattering homogeneous slab bounded by a perfect specular reflector and illuminated by uniform parallel rays.

Let us denote the albedo for single scattering by λ and the optical thickness of the slab by x. Let the angle of incidence of parallel rays be arc cos u. We wish to determine the source function which represents the total rate of production of scattered energy per unit volume per unit solid angle.

Let us denote the source function by S^*,

$$S^*=S^*(t,x,u),\quad 0\leqslant t\leqslant x,\ 0\leqslant u\leqslant 1,$$

where t is the optical altitude above the reflecting bottom.

It is well known that in this case the source function satisfies the Fredholm integral equation [1]

$$S^*(t,x,u)=\frac{1}{4}\lambda\left[e^{-\frac{x-t}{u}}+e^{-\frac{x+t}{u}}\right]+ \\ +\frac{1}{2}\lambda\int_0^x[E_1(|t-y|)+E_1(t+y)]S^*(y,x,u)\,dy, \tag{1}$$

where

$$E_1(z)=\int_0^1 e^{-z/w}\,dw/w.$$

Casti, Kagiwada and Kalaba [2] have shown, that by formally extending of the source function to the interval $-x\leqslant t\leqslant 0$ one obtains an integral equation

$$S^*(t,x,u)=\frac{1}{4}\lambda\left[e^{-\frac{x-t}{u}}+e^{-\frac{x+t}{u}}\right]+ \\ +\frac{1}{2}\lambda\int_{-x}^{x}E_1(|t-y|)S^*(y,x,u)\,dy, \tag{2}$$

where the symmetry relation

$$S^*(-t, x, u) = S^*(t, x, u) \tag{3}$$

has been used. Comparing equation (2) with the integral equation for the standard case, i. e., when there is no reflecting bottom at $t = 0$,

$$S(t, x, u) = \frac{1}{4}\lambda e^{-\frac{x-t}{u}} + \frac{1}{2}\lambda \int_0^x E_1(|t-y|)\, S(y, x, u)\, dy, \tag{4}$$

one can see that the following relationship holds [2]

$$S^*(t, x, u) = S(x+t, 2x, u) + S(x-t, 2x, u). \tag{5}$$

This important relationship gives us the possibility of expressing the source function for the problem of a perfect specular reflector by means of the solution of the standard problem. We have found a semi-analytic solution to the standard problem [3] (if only $\lambda \neq 1$) in the form

$$S(t, x, u_i) = \frac{1}{2}\lambda u_i \sum_{n=1}^{N} A_n \left[\frac{U_{in}\, e^{-\beta_n (x-t)}}{1-\beta_n u_i} + \frac{V_{in}\, e^{-\beta_n t}}{1+\beta_n u_i}\right], \tag{6}$$

where

$$\begin{aligned} U_{in} &= X(u_i, x) - Y(u_i, x)\, C_n, \\ V_{in} &= X(u_i, x)\, C_n - Y(u_i, x), \end{aligned} \tag{7}$$

and

$$C_n = -\frac{G_n}{1-D_n}. \tag{8}$$

Here X and Y are the well-known Ambartsumian-Chandrasekhar functions.

The functions G_n and D_n are defined by the following formulae

$$\begin{aligned} D_n &= \frac{1}{2}\lambda \sum_{m=1}^{N} \frac{a_m X(u_m, x)}{1+\beta_n u_m}, \\ G_n &= \frac{1}{2}\lambda \sum_{m=1}^{N} \frac{a_m Y(u_m, x)}{1+\beta_n u_m}, \end{aligned} \tag{9}$$

and β_n are the roots of the characteristic equation

$$1 - \lambda \sum_{m=1}^{N} \frac{a_m}{1-\beta^2 u_m^2} = 0. \tag{10}$$

The a_i and u_i are the weights and the points of the gaussian quadrature formula of order N, normalized to the interval (0,1), and A_n are the coefficients to be determined from the boundary conditions

$$S(0, x, u_i) = \frac{1}{4}\lambda Y(u_i, x),$$

or (11)

$$S(x, x, u_i) = \frac{1}{4}\lambda X(u_i, x).$$

Using equation (6) in equation (5), we get immediately

$$S^*(t,x,u_i)=\frac{1}{2}\lambda u_i\sum_{n=1}^{N}A_n[e^{-\beta_n(x+t)}+e^{-\beta_n(x-t)}]\left(\frac{U_{in}}{1-\beta_n u_i}+\frac{V_{in}}{1+\beta_n u_i}\right). \quad (12)$$

One must be careful in using formula (12) since the quantities A_n, U_{in} and V_{in} are to be determined for the slab of optical thickness $2x$.

For comparison we have calculated the reflection function for the case $x=0.1$ and $\lambda=0.9$ using equation [4]

$$\frac{\partial}{\partial x}R(v,u,x)=$$

$$=-(1/u+1/v)R(v,u,x)+\lambda\left[1+e^{-2x/u}+\frac{1}{2}\int_0^1 R(w,u,x)\,dw/w\right]\times \quad (13)$$

$$\times\left[1+e^{-2x/v}+\frac{1}{2}\int_0^1 R(v,w,x)\,dw/w\right],$$

$R(v,u,0)=0.$

In this equation $\arccos v$ is the angle of reflection. It should be mentioned that here the normalization of the reflection function differs slightly from the one used in [4].

On the other hand, we have obtained the reflection function by using the source function [4]

$$R(v,u,x)=4\int_0^x S(y,x,u)\left[e^{-\frac{x-y}{v}}+e^{-\frac{x+y}{v}}\right]dy. \quad (14)$$

In formula (14) the source function has been found according to the techniques described above. For numerical integration the Simpson rule has been used. The results are shown in the Table. In general, at least five significant figures of agreement have been found.

Comparison of two calculations for the diffuse reflection function ($x=0.1, \lambda=0.9$)
$N=7$

$\arccos u_i$	$R(v_1,u_i,x)$ Eq. (14)	$R(v_1,u_i,x)$ Eq. (13)	$R(v_4,u_i,x)$ Eq. (14)	$R(v_4,u_i,x)$ Eq. (13)	$R(v_7,u_i,x)$ Eq. (14)	$R(v_7,u_i,x)$ Eq. (13)
88°.54	0.013232	0.013232	0.048669	0.048669	0.053209	0.053209
82°.57	0.031826	0.031826	0.196845	0.196846	0.215872	0.215872
72°.72	0.043324	0.043324	0.281847	0.281847	0.309162	0.309162
60°.00	0.048669	0.048669	0.319242	0.319243	0.350195	0.350195
45°.34	0.051279	0.051279	0.337119	0.337119	0.369808	0.369809
29°.45	0.052601	0.052601	0.346089	0.346090	0.379650	0.379650
12°.95	0.053209	0.053209	0.350195	0.350195	0.384154	0.384154

Summarizing we can say that our results may prove useful in the interpretation of radiation measurements, since the air — sea interface may be considered a specular reflector in the ultraviolet region [2].

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Allikfunktsiooni leidmiseks on kasutatud Ambartsumjan-Sobolevi võrrandi diskreetsete ordinaatide meetodil saadud lahendit.

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ФУНКЦИЯ ИСТОЧНИКА ДЛЯ ИЗОТРОПНО РАССЕИВАЮЩЕГО ОДНОРОДНОГО СЛОЯ, ГРАНИЧАЩЕГО С ИДЕАЛЬНОЙ ЗЕРКАЛЬНОЙ ПОВЕРХНОСТЬЮ

Для нахождения функции источника использовано решение уравнения Амбарцумяна-Соболева для альбедной задачи.