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AN OPERATIONAL APPROACH TO THE ENERGY OF GRAVITATIONAL WAVES

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Р. ТАММЕЛО. ОПЕРАЦИОННЫЙ ПОДХОД К ЭНЕРГИИ ГРАВИТАЦИОННЫХ ВОЛН

The aim of the present paper is to compare the angular distribution of the energy flux ensuing from the Landau-Lifschitz pseudotensor, on the one hand, and the angular distribution of the energy of gravitational waves, directly available to detectors, on the other. It is found that in the case of a weak gravitational field and monochromatic waves the angular distributions coincide for the time-averaged pseudotensor, and for a special two-dimensional detector. As the energy absorbed by a detector has a clear-cut physical meaning, it follows that the time-averaged Landau-Lifschitz pseudotensor gives the correct angular dependence of gravitational energy flux.

Consider gravitational radiation in the wave zone. The metric can be taken in the following transverse form:

$$ds^2 = du^2 + 2du\,dr - r^2[(1+2\gamma)\,d\vartheta^2 + 4\delta \sin\vartheta\, d\vartheta\, d\varphi + (1-2\gamma)\sin^2\vartheta\, d\varphi^2],$$

where γ, δ are functions of $u = t - r$, r, ϑ, φ. As we shall see, the only components of the curvature tensor $R_{\alpha\beta\gamma\delta}$ necessary for calculating the energy absorbed by a detector are $R_{\alpha 0\gamma 0}$. In the first approximation

$$R_{\alpha 0\gamma 0} = \frac{\ddot{c}}{r} m_\alpha m_\gamma + \frac{\ddot{\bar{c}}}{r} \bar{m}_\alpha \bar{m}_\gamma + \ldots,$$

where

$$\frac{c}{r} = \gamma - i\delta + O\left(\frac{1}{r^2}\right), \qquad m_\alpha = -\frac{r}{\sqrt{2}}(\delta^2_\alpha + i\sin\vartheta\ \delta^\beta_\alpha).$$

A dot above a symbol denotes differentiation with respect to u, and a bar above a function denotes the complex conjugate of the function. Here c is the news function of Bondi and Sachs. In the quadrupole approximation [1].

$$c=\frac{4\pi\sqrt{2}}{5\sqrt{3}}\sum_{m=-2}^{2}Y^{2}_{-2m}\frac{\partial^2}{\partial u^2}\int(T^{0}_{0}-T^{S}_{S})r^{2}\bar{Y}^{2}_{0m}\,dV,$$

where $T_{\mu\nu}$ is the energy tensor of the matter in the approximation of flat space-time, and Y^{l}_{nm} are the spin spherical harmonics of Goldberg *et al.* [2].

To find the energy available to detectors, we first discuss an idealized one-dimensional Weber detector which consists of two masses m bound by a spring, and is characterized by elasticity k and dissipation D. We refer the motion of the masses to the local Cartesian coordinates $x^{\underline{i}}$ whose origin coincides with the detector's centre of mass *. The detector is in free fall. The orientation of the detector is arbitrary. The small changes ξ from the equilibrium length l of the detector, due to gravitational waves, are governed by the equation

$$\frac{d^2}{dt^2}\xi+\frac{D}{m}\frac{d}{dt}\xi+\frac{k}{m}\xi=F_{gr}.$$

The wave-produced gravitational force along the arbitrarily oriented detector is approximately [3]

$$F_{gr}=-\frac{1}{l}R_{\underline{i0k0}}X^{\underline{i}}X^{\underline{k}},$$

where $X^{\underline{i}}$ is the connecting vector between the two masses. In the case of incident sinusoidal waves with frequency ω the power absorbed by the detector is

$$E(\vartheta,\varphi)=\frac{\omega^2m^2D}{(k-\omega^2m)^2+\omega^2D^2}\langle F^{2}_{gr}\rangle,$$

where $\langle f\rangle$ stands for time-averaged f.

We examine spherical waves at large distances from the source. To specify the local Cartesians, let $x^{\underline{1}}$ be oriented in the radial direction (in the direction of propagation of gravitational waves) and let $x^{\underline{2}}$, $x^{\underline{3}}$ be tangent to the coordinate lines of ϑ, φ, respectively. After that the driving force along the detector can readily be found:

$$F_{gr}=\frac{\mathrm{Re}\,\ddot{c}}{lr}(X^{\underline{2}2}-X^{\underline{2}2})+\frac{\mathrm{Im}\,\ddot{c}}{lr}2X^{\underline{2}}X^{\underline{3}}+O\left(\frac{1}{r^2}\right).$$

It is convenient to introduce the angles Θ, Φ which determine the direction of the detector relative to the local axes

$$X^{\underline{1}}=l\cos\Theta,\quad X^{\underline{2}}=l\sin\Theta\cos\Phi,\quad X^{\underline{3}}=l\sin\Theta\sin\Phi.$$

Now the power absorbed by the one-dimensional detector can be written in the explicit form:

$$E(\vartheta,\varphi,\Theta,\Phi)=\frac{\omega^2m^2Dl^2\sin^4\Theta}{(k-\omega^2m)^2+\omega^2D^2}\frac{1}{r^2}[\langle(\mathrm{Re}\,\ddot{c})^2\rangle\cos^2 2\Phi+$$
$$+\langle(\mathrm{Im}\,\ddot{c})^2\rangle\sin^2 2\Phi-\langle\mathrm{Re}\,\ddot{c}\,\mathrm{Im}\,\ddot{c}\rangle\sin 4\Phi].$$

* The range and summation conventions are assumed for Latin indices over 1, 2, 3 and for Greek indices over 0, 1, 2, 3. Bars under indices denote the physical components of the corresponding tensor.

A detector whose power is independent of the angle Φ can be easily designed by simply combining two identical one-dimensional detectors with orientations (Θ, Φ) and $(\Theta, \Phi+45°)$. This special two-dimensional detector will be used for our comparison. The power absorbed is

$$E^{*}(\vartheta, \varphi, \Theta)=E(\vartheta, \varphi, \Theta, \Phi)+E(\vartheta, \varphi, \Theta, \Phi+45°)=\frac{\omega^2 m^2 D l^2 \sin^4\Theta}{(k-\omega^2 m)^2+\omega^2 D^2}\frac{\langle \dddot{c}\dddot{c}\rangle}{r^2}.$$

Next turn to the Landau-Lifschitz gravitational pseudotensor $t_{\alpha\beta}$. It gives for the density and radial flux of energy the following expression [4]:

$$\underline{t_{01}}=\underline{t_{00}}=\frac{1}{16\pi}\left[\dot{\underline{h}}_{23}^2+\frac{1}{4}(\dot{\underline{h}}_{22}-\dot{\underline{h}}_{33})^2\right]+O\left(\frac{1}{r^3}\right)=\frac{1}{4\pi}\frac{\dot{\ddot{c}}\dot{\ddot{c}}}{r^2}+O\left(\frac{1}{r^3}\right),$$

where $\underline{h}_{\alpha\beta}$ are the physical components of the metric perturbation. In order to compare the angular distributions of $\underline{t_{01}}$ and E^{*}, one has to average $\underline{t_{01}}$ over time because E^{*} is a time-averaged quantity in its essence. Taking into consideration that for monochromatic radiation $\langle \dddot{c}\dddot{c}\rangle=\omega^2\langle \dot{\ddot{c}}\dot{\ddot{c}}\rangle$, we arrive at the result that the quotient

$$\frac{E^{*}}{\langle \underline{t_{01}}\rangle}=\frac{4\pi\omega^4 m^2 D l^2\sin^4\Theta}{(k-\omega^2 m)^2+\omega^2 D^2}=\text{const.}$$

The expression points out that the angular distributions of E^{*} and $\langle \underline{t_{01}}\rangle$ are identical.

Lastly, consider an example of a spinning rod. The energy flux ensuing from the pseudotensor (without averaging over time) is

$$\underline{t_{01}}=\frac{\pi\omega^6 I^2}{200 r^2}[\cos^4\vartheta+6\cos^2\vartheta+1-(\cos^2\vartheta-1)^2\cos 2(\omega u-2\varphi)],$$

where I is the moment of inertia of the rod. The power extracted from the gravitational field of the spinning rod by a one-dimensional detector is

$$E=\frac{\pi^2\omega^{10} m^2 D l^2 I^2\sin^4\Theta}{100 r^2[(k-\omega^2 m)^2+\omega^2 D^2]}[\cos^4\vartheta+6\cos^2\vartheta+1+(\cos^2\vartheta-1)^2\cos 4\Phi].$$

So one can see that, after averaging $\underline{t_{01}}$ over u, and forming the special two-dimensional detector as described before, the angular distributions of $\underline{t_{01}}$ and the power absorbed coincide.

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