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RADIATIVE TRANSFER IN A SPHERICAL ATMOSPHERE OF FREE ELECTRONS

1. In constructing the model stellar atmospheres and interpreting the observed spectral line profiles, one of the most difficult points is the problem of scattering. The methods for the solution of the equation of radiative transfer and temperature correction which are powerful in the absorbing layers of an atmosphere, fail in the layers where scattering becomes the main interaction between the radiation and matter. Scattering of the radiation in continuum is important not only in the early-type stars where free electrons are responsible for the process, but even in the late-type stars where the Rayleigh scattering from atomic and molecular hydrogen becomes significant. That is the reason why the problems of scattering have remained topical up to the present time.

In any discussion considering the radiative transfer problem involving scattering, much attention has been given to the source function. For the case where the radiation is scattered from the Maxwellian gas of free electrons, the source function was first derived by Dirac [1]. He expanded the scattering coefficient in powers of $\alpha \equiv kT/mc^2$ where T is the temperature and mc^2 — the rest energy of the electron, and found the explicit expression for the source function correct to order $\sqrt{\alpha}$. Somewhat better results were obtained by Edmonds [2] who computed the source function to order α. The results of Dirac and Edmonds were used by Münch [3] and Edmonds [4,5] for computing the radiation field in a stellar atmosphere.

Chandrasekhar [6] has considered the computing of the radiation field in a stellar atmosphere with a somewhat different method. In solving the problem, he retained all first order terms in $\gamma = h\nu/mc^2$ (regarding $\alpha = 0$), but a certain feature of his solution is rather curious. He found that the emergent intensity did not vanish for $\nu > \nu_0$ in the case of an incident monochromatic flux of frequency ν_0. This is, of course, an incorrect behaviour, physically, because no photon could increase its frequency by passing through the scattering atmosphere of zero temperature. Chandrasekhar placed the blame on the use of only a finite number of terms in the expansion of the source function in powers of γ.

However, this cannot be the reason of the incorrect shift as it became apparent in the paper by Pomraning [7]. His treatment of the problem was based on the simplified equation of transfer suggested by Frazer [8].

The source function in the case of the Compton scattering of the radiation from a Maxwellian gas of free electrons can be conveniently expanded in powers of $\sqrt{\alpha}$. The coefficients by these powers are terms containing γ. Frazer expands these terms to order γ, which should be quite accurate for photon energies of astrophysical interest, and finds a simple equation of transfer. This equation contains a second-order differential scattering operator instead of the integral one.

Pomraning considered a purely scattering plane-parallel atmosphere with spatially independent properties and zero-temperature and found no shift of the emergent intensity to the violet.

In the previous papers [10, 11] we presented the expressions for the emergent intensity in a spherical stellar atmosphere of free electrons. These expressions were obtained following the techniques elaborated by Chandrasekhar [6] and Münch [3] for the case of $\alpha=0$ and $\alpha\neq 0$, respectively. As the treatment of scattering in our papers was based on the oversimplified source functions, it is of interest to use the Pomraning equation in the case of a spherical stellar atmosphere in order to get more reliable expression for the emergent radiation.

The case considered here will be that of a pure electron scattering spherical atmosphere of finite thickness τ_1 with the inner radius R_1 and the outer radius R_2.

In the case of spherical symmetry, the equation derived by Pomraning and Freeman [9] turns into

$$\left(\frac{\partial}{\partial R}+\frac{2}{R}\right)\frac{1}{\sigma n_e}\frac{\partial J}{\partial R}+3\sigma n_e\left[\alpha\frac{\partial^2}{\partial\nu^2}+(\gamma-2\alpha)\nu\frac{\partial}{\partial\nu}+\gamma\right]J=0, \quad (1.1)$$

where J is the mean intensity, σ — the Thomson cross section and n_e — the electron density.

It should be emphasized that the equation (1.1) is valid in the case of dilute radiation.

2. First we shall consider the case $\alpha=0$, i. e. where the thermal motions of the electrons are neglected.

We shall use the Marshak boundary conditions because experience has shown that the Marshak conditions are somewhat more accurate in the lower approximation [12]. If $\Gamma(\nu)$ denotes the incoming distribution of intensity on a boundary surface $R=R_1$, then the Marshak boundary conditions are that the assumed intensity distribution gives the correct incoming flux. In our case, the Marshak boundary conditions take the form

$$\Gamma(\nu)=\frac{1}{4}J(R_1,\nu)-\frac{1}{6\sigma n_e}\frac{\partial J(R_1,\nu)}{\partial R}, \quad (2.1)$$

$$0=\frac{1}{4}J(R_2,\nu)+\frac{1}{6\sigma n_e}\frac{\partial J(R_1,\nu)}{\partial R}. \quad (2.2)$$

In our further discussion, we shall assume that the opacity varies inversely as the n-th power of the geometric radius of the layer. Therefore we set

$$\sigma n_e\sim R^{-n}, \quad (2.3)$$

where $n>0$.

Now it is convenient to define a dimensionless wavelength variable η, measured in units of the Compton shift, i. e.

$$\eta=\frac{1}{\gamma}=\frac{mc^2}{h\nu}. \quad (2.4)$$

On substituting equation (2.4) into equation (1.1) and keeping in view that $\alpha=0$ we readily find

$$\left(\frac{\partial}{\partial R}+\frac{2}{R}\right)\frac{1}{\sigma n_e}\frac{\partial u}{\partial R}-3\sigma n_e\frac{\partial u}{\partial\eta}=0, \quad (2.5)$$

where

$$u(R,\eta)=\eta^{-1}J(R,\nu). \tag{2.6}$$

The physical significance of the function u is that it is essentially a photon number distribution function per unit wavelength, as it can be seen from equation (2.6).

Defining the radial optical depth by

$$d\tau=-\sigma n_e\,dR, \tag{2.7}$$

we find for the case $n=1$ that

$$\tau=\frac{1}{a}\ln\frac{R_2}{R}, \tag{2.8}$$

and for the case $n\neq 1$ that

$$\tau=\frac{b}{n-1}\left(R^{1-n}-R_2^{1-n}\right), \tag{2.9}$$

where a and b are certain constants.

Use of equation (2.7) in boundary conditions (2.1) and (2.2) yields

$$12f(\eta)=3u(\tau_1,\eta)+2\frac{\partial u(\tau_1,\eta)}{\partial\tau}, \tag{2.10}$$

$$0=3u(0,\eta)-2\frac{\partial u(0,\eta)}{\partial\tau}, \tag{2.11}$$

where

$$f(\eta)=\eta^{-1}\Gamma(\nu).$$

Now it is convenient to separate the treatment of our problem into two parts. At first we shall consider the case $n=1$.

3. Substituting equation (2.8) into equation (2.5) we find

$$\frac{\partial^2 u}{\partial\tau^2}-2a\frac{\partial u}{\partial\tau}-3\frac{\partial u}{\partial\eta}=0. \tag{3.1}$$

The parabolic equation (3.1) can easily be solved by Laplace transform techniques with respect to the variable η.

We define

$$\bar{u}(\tau,s)=\int_0^\infty e^{-s\eta}u(\tau,\eta)\,d\eta \tag{3.2}$$

and

$$\bar{f}(s)=\int_0^\infty e^{-s\eta}f(\eta)\,d\eta. \tag{3.3}$$

In the transform space, equations (3.1), (2.10) and (2.11) become

$$\bar{u}''-2a\bar{u}'-3s\bar{u}=0, \tag{3.4}$$

$$12\bar{f}(s)=3\bar{u}(\tau_1,s)+2\bar{u}'(\tau_1,s), \tag{3.5}$$

$$0=3\bar{u}(0,s)-2\bar{u}'(0,s). \tag{3.6}$$

In deriving equation (3.4), we have assumed that

$$u(\tau,0)\equiv 0.$$

Physically it means that no photons are present with an infinite frequency. The solution of equations (3.4) through (3.6) is given by

$$\bar{u}(\tau, s) = e^{-a(\tau_1-\tau)}\bar{f}(s)\frac{(3-2a)\,\mathrm{sh}\,\delta\tau + 2\delta\,\mathrm{ch}\,\delta\tau}{(^3/_4+s)\,\mathrm{sh}\,\delta\tau_1 + \delta\,\mathrm{ch}\,\delta\tau_1}, \tag{3.7}$$

where

$$\delta^2 = a^2 + 3s.$$

Our solution differs only slightly from that of Pomraning [7] found in the case of plane-parallel atmosphere, so we can follow his techniques.

The net flux can be found by Fick's law of diffusion which has been used in deriving the radiative transfer equation (1.1). We have

$$H = -\frac{1}{3\sigma n_e}\frac{\partial J}{\partial R} = \frac{1}{3}\frac{\partial J}{\partial \tau}. \tag{3.8}$$

Now it is better to use the absolute value of the net flux w per unit wavelength at $\tau = 0$. (Function w is also a photon number, not energy, distribution function). Then we have

$$\bar{w} = \frac{1}{3}\frac{\partial u}{\partial \tau} = \bar{g}(s)\bar{f}(s), \tag{3.9}$$

where $\bar{w}$ is the Laplace transform of the function w and

$$\bar{g}(s) = \frac{\delta e^{-a\tau_1}}{(^3/_4+s)\,\mathrm{sh}\,\delta\tau_1 + \delta\,\mathrm{ch}\,\delta\tau_1}. \tag{3.10}$$

By the convolution theorem for Laplace transforms we can write

$$w(\eta) = \int_0^\eta g(\eta-\eta';\tau_1)f(\eta')d\eta', \tag{3.11}$$

where $g(\eta;\tau_1)$ is the inverse transform of $\bar{g}(s)$.

Equation (3.11) may be recast in the form

$$H(\nu) = \frac{1}{\gamma}\int_{1/\nu}^{\infty} g\left(\frac{\nu'-\nu}{\gamma\nu'};\tau_1\right)\Gamma(\nu')\frac{d\nu'}{\nu'}. \tag{3.12}$$

Now let the incident distribution at $\tau = \tau_1$ be a sharp pulse with the maximum at ν_0; then, equation (3.12) yields

$$H(\nu) = \frac{1}{\gamma\nu_0}g\left(\frac{\nu_0-\nu}{\gamma\nu_0};\tau_1\right)H(\nu_0-\nu), \tag{3.13}$$

where $H(\nu_0-\nu)$ is the unit step function. (It should not be confused with the flux $H(\nu)$. In this paper it is the only place where the unit step function is used). It arises because ν_0 may lie outside the integration range. The unit step function causes the vanishing of emergent flux for $\nu > \nu_0$ in the case of an incident monochromatic flux of frequency ν_0, which is the physically correct behaviour. Our next step should be the inverting of $\bar{g}(s)$ in order to complete the solution, but we leave it for the appendix.

4. Now we consider the case $n \neq 1$. Upon substituting equation (2.9) into equation (2.5), we find

$$\frac{\partial^2 u}{\partial \tau^2} - \frac{2}{\omega\tau+\varphi}\frac{\partial u}{\partial \tau} - 3\frac{\partial u}{\partial \eta} = 0, \tag{4.1}$$

where $\omega = n - 1$ and $\varphi = bR_2^{1-n}$. If we define

$$\frac{\omega\tau + \varphi}{\omega} = \xi,$$

then in the transform space equation (4.1) is written

$$\xi\bar{u}'' - 2\omega^{-1}\bar{u}' - 3s\xi\bar{u} = 0. \tag{4.2}$$

The boundary conditions have the form of equations (3.5) and (3.6). Equation (4.2) is readily solved, and, taking into account the boundary conditions, the solution is

$$\bar{u}(\xi, s) = \xi^{\varkappa}\xi_1^{-\varkappa}\bar{f}(s)\,G^{-1}(s)\,[3(K^0_{\varkappa}I_{\varkappa} - I^0_{\varkappa}K_{\varkappa}) + 2\sqrt{3s}\,(I_{\varkappa}K^0_{\varkappa-1} + K_{\varkappa}I^0_{\varkappa-1})], \tag{4.3}$$

where I and K are the modified Bessel functions,

$$G(s) = 3(I^1_{\varkappa}K^0_{\varkappa} - I^0_{\varkappa}K^1_{\varkappa}) + 2\sqrt{3s}\,(I^1_{\varkappa}K^0_{\varkappa-1} + I^0_{\varkappa-1}K^1_{\varkappa} + I^1_{\varkappa-1}K^0_{\varkappa} + I^0_{\varkappa}K^1_{\varkappa-1}) + 4s(I^1_{\varkappa-1}K^0_{\varkappa-1} - I^0_{\varkappa-1}K^1_{\varkappa-1}), \tag{4.4}$$

$$\xi_0 = \frac{\varphi}{\omega}\sqrt{3s}, \quad \xi_1 = \frac{\omega\tau_1 + \varphi}{\omega}\sqrt{3s}, \quad \varkappa = \frac{1}{\omega} + \frac{1}{2} = \frac{n+1}{2(n-1)},$$

and the superscripts 0 and 1 of modified Bessel functions mean that the value of the function must be taken at ξ_0 and ξ_1, respectively.

The general solution is quite awkward to deal with, so let us consider a special case when $n = 2$. Then the modified Bessel functions can be expressed in hyperbolic functions, as follows:

$$I_{3/2}(z) = \sqrt{\frac{2}{\pi z}}\left(\operatorname{ch} z - \frac{1}{z}\operatorname{sh} z\right), \qquad I_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\operatorname{sh} z,$$

$$K_{3/2}(z) = \sqrt{\frac{\pi}{2z}}\left(\operatorname{ch} z + \frac{1}{z}\operatorname{ch} z - \operatorname{sh} z - \frac{1}{z}\operatorname{sh} z\right), \tag{4.5}$$

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}\,(\operatorname{ch} z - \operatorname{sh} z).$$

According to equation (4.3) we find

$$\bar{u}(\xi, s) = \bar{f}(s)\,M^{-1}(s)\left[\sqrt{3s}\left(\tau + \frac{2}{3}\xi\xi_0\right)\operatorname{ch}\tau\sqrt{3s} + \left(\xi\xi_0 - \frac{2}{3}\sqrt{3s}\,\xi_0 - 1\right)\operatorname{sh}\tau\sqrt{3s}\right], \tag{4.6}$$

where

$$M(s) = \tau_1\sqrt{3s}\,[(\varrho - 1)^2 + 4s\tau_1\varrho]\operatorname{ch}\tau_1\sqrt{3s} + [4s^2\tau_1^2\varrho + 3s\tau_1^2\varrho + 2s\tau_1(\varrho - 1)^2 - (\varrho - 1)^2]\operatorname{sh}\tau_1\sqrt{3s},$$

and

$$\varrho = R_2/R_1.$$

As we are interested in the outcoming flux, then, using equation (3.9), we have

$$\overline{w}(0, s) = 4\overline{f}(s)\, s\tau_1^2 \sqrt{3s}\, M^{-1}(s). \tag{4.7}$$

When $\varrho \to 1$, we have the formula found for the case of a plane-parallel atmosphere with inessential differences in some coefficients.

Now it is of interest to find out how the curvature of the layers of the atmosphere ϱ, the optical depth τ_1 and the index of the density gradient n tell on the outcoming flux, incoming flux at $\tau = \tau_1$ being monochromatic.

Passing through the scattering atmosphere, the preliminary sharp monochromatic flux is "smeared" over a long range of wavelengths. Figures 1 and 2 show that this "smearing" will become more severe when any of the three characteristics ϱ, τ_1 or n increase.

For even moderately thin atmospheres, the maximum of the outcoming flux is shifted to considerably longer wavelengths, to say nothing about thick atmospheres.

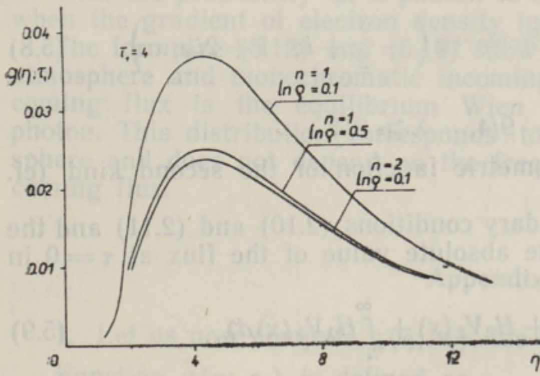

Fig. 1.

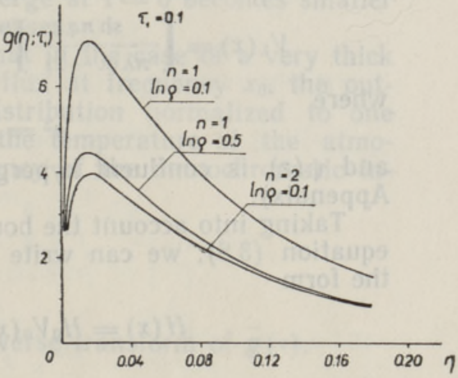

Fig. 2.

Figures 1 and 2 also show an essential feature of our solution: there is no increase of frequency when radiation passes through the atmosphere. That is why the present solution which was found following Pomraning [7] is preferable.

5. Here we consider the case $\alpha \neq 0$. We suppose that the temperature is constant throughout the atmosphere, and so we can define the new variable

$$x = \gamma/\alpha = h\nu/kT. \tag{5.1}$$

Let $n = 1$, then equation (1.1) takes the form

$$\left(\frac{\partial^2}{\partial\tau^2} - 2a\frac{\partial}{\partial\tau}\right) J + 3\alpha \left[x^2 \frac{\partial^2}{\partial x^2} + (x-2)\, x \frac{\partial}{\partial x} + x \right] J = 0, \tag{5.2}$$

where the variables τ and x can be separated, writing

$$J_\lambda(\tau, x) = U_\lambda(\tau)\, V_\lambda(x). \tag{5.3}$$

Substituting equation (5.3) into (5.2) we find

$$\frac{\partial^2 U_\lambda}{\partial\tau^2} - 2a\frac{\partial U_\lambda}{\partial\tau} - 3\alpha\lambda U_\lambda = 0 \tag{5.4}$$

and

$$x^2 \frac{\partial^2 V_\lambda}{\partial x^2} + (x-2)x \frac{\partial V_\lambda}{\partial x} + (x+\lambda) V_\lambda = 0, \tag{5.5}$$

where λ is the separation constant.

Equation of the type (5.4) has already been solved (see eq. (3.4)). Equation (5.5) can be approached by means of an eigenfunction expansion. Pomraning [14] has recently examined this problem very thoroughly and found that the spectrum of equation (5.5) is quite complicated. It consists of two discrete points $\lambda=0$ and $\lambda=2$, and a continuum, $\lambda \geqslant 9/4$. The normalized eigenfunctions, corresponding to the discrete eigenvalues, are elementary functions, namely

$$V_0(x) = \frac{1}{\sqrt{2}} x^3 e^{-x}, \tag{5.6}$$

$$V_2(x) = \frac{1}{\sqrt{2}} x^2 (x-2) e^{-x}, \tag{5.7}$$

and the normalized continuum eigenfunctions are

$$V_\lambda(x) = \left[\frac{\operatorname{sh} \pi q}{\pi\lambda(\lambda-2)}\right]^{1/2} x^{3/2+iq} e^{-x} \psi\left(-\frac{3}{2}+iq,\ 1+2iq; x\right), \tag{5.8}$$

where

$$q^2 = \lambda - 9/4, \quad \lambda \geqslant 9/4$$

and $\psi(z)$ is confluent hypergeometric function of the second kind (cf. Appendix).

Taking into account the boundary conditions (2.10) and (2.11) and the equation (3.8), we can write the absolute value of the flux at $\tau=0$ in the form

$$H(x) = H_0 V_0(x) + H_2 V_2(x) + \int_{9/4}^{\infty} H_\lambda V_\lambda(x)\, d\lambda, \tag{5.9}$$

where

$$H_\mu = \frac{\varrho^{-1}\sqrt{a^2+3a\mu}\, f_\mu}{(3/4+a\mu)\operatorname{sh}\tau_1\sqrt{a^2+3a\mu} + \sqrt{a^2+3a\mu}\operatorname{ch}\tau_1\sqrt{a^2+3a\mu}}, \tag{5.10}$$

and the f_μ are the expansion coefficients of the incoming flux distribution, i. e.

$$f_\mu = \int_0^\infty x^{-4} e^x f(x)\, V_\mu(x)\, dx. \tag{5.11}$$

In the case of a very thick atmosphere, equation (5.9) reduces to

$$H(x) \xrightarrow[\tau_1\to\infty]{} \frac{1}{2} x^3 e^{-x} \left(\frac{\varrho^{-1}}{1+3/4\,\tau_1}\right) \int_0^\infty f(z) \frac{dz}{z}. \tag{5.12}$$

The expression in brackets represents the probability that a photon will emerge at $\tau=0$.

Comparing this expression with that found in the plane-parallel case [14], we see that the corresponding probability is smaller by the factor ϱ which characterizes the geometric dilution of the radiation.

The expression of the outcoming flux at $\tau=0$ can be derived for arbitrary n (see § 4) but because of the awkwardness of the formulae we confine ourselves to the case $n=2$.

For the case $n=2$ the formula for the coefficient takes the form

$$H_\lambda = \alpha\lambda\tau_1^2 M^{-1} f_\lambda \sqrt{3\alpha\lambda}, \quad \lambda \neq 0, \tag{5.13}$$

$$H_0 = \frac{f_\lambda}{\varrho + {}^3/_4\,\varrho\tau_1 + {}^1/_2(\varrho-1)^2}, \quad \lambda = 0 \tag{5.14}$$

where

$$M = \tau_1\sqrt{3\alpha\lambda}\,[(\varrho-1)^2 + 4\alpha\lambda\tau_1]\operatorname{ch}\tau_1\sqrt{3\alpha\lambda} + \\ + [4\alpha^2\lambda^2\tau_1^2 + 3\alpha\lambda\tau_1^2 + 2\alpha\lambda\tau_1(\varrho-1)^2 - (\varrho-1)^2]\operatorname{sh}\tau_1\sqrt{3\alpha\lambda}. \tag{5.15}$$

Again, for a very thick atmosphere, i. e., $\tau_1 \to \infty$, the absolute value of the flux at $\tau = 0$ is

$$H(x) \xrightarrow[\tau_1\to\infty]{} \frac{1}{2}x^3 e^{-x}\left[\frac{1}{\varrho + {}^3/_4\tau_1\varrho + {}^1/_2(\varrho-1)^2}\right]\int_0^\infty f(z)\frac{dz}{z}. \tag{5.16}$$

Thus the probability for a photon to emerge at $\tau = 0$ becomes smaller when the gradient of electron density increases.

The formulae (5.12) and (5.16) show that in the case of a very thick atmosphere and monochromatic incoming flux at frequency x_0, the outcoming flux is the equilibrium Wien distribution normalized to one photon. This distribution corresponds to the temperature of the atmosphere and does not depend on the frequency of the monochromatic incoming flux.

Appendix

1. Let us now compute $g(\eta;\tau_1)$, the inverse transform of $\bar g(s)$.

Function $\bar g(\eta;\tau_1)$ is defined as

$$g(\eta;\tau_1) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{s\eta}\bar g(s)\,ds, \tag{A.1}$$

where

$$\bar g(s) = \frac{\varrho^{-1}\sqrt{a^2+3s}}{({}^3/_4+s)\operatorname{sh}\tau_1\sqrt{a^2+3s} + \sqrt{a^2+3s}\operatorname{ch}\tau_1\sqrt{a^2+3s}}, \tag{A.2}$$

and the path of integration is parallel to the imaginary axis, ε being nonnegative and chosen so that all singularities of function $\bar g(s)$ lie to the left of the integration path. We have to find all singularities of the function $\bar g(s)$. Obviously all singularities are simple poles and they can be found, looking for the zeros of the denominator.

Writing

$$({}^3/_4 - {}^1/_3 a^2 - {}^1/_3 p_n^2)\sin p_n\tau_1 + p_n\cos p_n\tau_1 = 0, \tag{A.3}$$

where

$$-p_n^2 = a^2 + 3s_n, \tag{A.4}$$

we see that if p_n is a root, so is $-p_n$. In spite of the fact that $s_n = 0$ is a root of equation (A.3), it is not a singular point of $\bar g(s)$.

Writing now the root p_n in a complex form

$$p_n = x_n + iy_n, \tag{A.5}$$

where x_n and y_n are real, we show that under a certain condition the imaginary part of p_n is zero, i. e., all the roots of equation (A.3) are real.

Using equation (A.5), we find

$$\begin{aligned} &3y_n (x_n^2 + y_n^2 + 9/4 - a^2)(1 + \mathrm{th}^2\, y_n\tau_1) + \\ &+ \mathrm{th}\, y_n\tau_1 [(x_n^2 - y_n^2 - 9/4 + a^2)^2 + 9y_n^2 + 4x_n^2 y_n^2 + 9x_n^2] = 0. \end{aligned} \tag{A.6}$$

It is clear that if $a \leqslant 3/2$, then the only root of equation (A.6) is $y_n = 0$. It shows that all the singularities of $\bar{g}(s)$ lie on the negative real axis and hence the path of integration in equation (A.1) can be taken as the imaginary axis. We separate the integral (A.1) into two parts, according to [9]

$$g(\eta; \tau_1) = \frac{1}{2\pi i}\left[\int_{-i\infty}^{0} e^{s\eta}\bar{g}(s)\,ds + \int_{0}^{i\infty} e^{s\eta}\bar{g}(s)\,ds\right]. \tag{A.7}$$

Putting $s = -i\zeta^2$ in the first integral, and $s = i\zeta^2$ in the second, we find

$$\begin{aligned} g(\eta; \tau_1) &= \frac{1}{\pi}\int_0^\infty \left[\bar{g}(i\zeta^2)\, e^{i\zeta^2\eta} + \bar{g}(-i\zeta^2)\, e^{-i\zeta^2\eta}\right]\zeta\, d\zeta = \\ &= \frac{2}{\pi}\int_0^\infty \mathrm{Re}\left[\bar{g}(i\zeta^2)\, e^{i\zeta^2\eta}\right]\zeta\, d\zeta. \end{aligned} \tag{A.8}$$

Using equation (A.2), we obtain

$$g(\eta; \tau_1) = \frac{2}{\pi\varrho}\int_0^\infty \frac{\sqrt{a^4 + 9\zeta^4}\,[v_1(\zeta)\cos\zeta^2\eta + v_2(\zeta)\sin\zeta^2\eta]}{v_1^2(\zeta) + v_2^2(\zeta)}\,\zeta\, d\zeta, \tag{A.9}$$

where

$$\begin{aligned} v_1(\zeta) = \sqrt{a^4 + 9\zeta^4}\,\mathrm{ch}\,\tau_1 z_1 \cos\tau_1 z_2 + (3/4\, z_1 + \zeta^2 z_2)\,\mathrm{sh}\,\tau_1 z_1 \cos\tau_1 z_2 + \\ + (3/4\, z_2 - \zeta^2 z_1)\,\mathrm{ch}\,\tau_1 z_1 \sin\tau_1 z_2, \end{aligned} \tag{A.10}$$

$$\begin{aligned} v_2(\zeta) = \sqrt{a^4 + 9\zeta^4}\,\mathrm{sh}\,\tau_1 z_1 \sin\tau_1 z_2 - (3/4\, z_1 + \zeta^2 z_2)\,\mathrm{ch}\,\tau_1 z_1 \sin\tau_1 z_2 + \\ + (3/4\, z_2 - \zeta^2 z_1)\,\mathrm{sh}\,\tau_1 z_1 \cos\tau_1 z_2), \end{aligned} \tag{A.11}$$

$$z_1 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{a^4 + 9\zeta^4} + a^2} \tag{A.12}$$

and

$$z_2 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{a^4 + 9\zeta^4} - a^2}. \tag{A.13}$$

2. Now we consider the two limiting cases of small and large τ_1, for $n=1$ and $n=2$. The optical depth τ_1 being small, we can expand the hyperbolic functions in the denominator of $\bar{g}(s)$, carrying terms through τ_1^4 in these expansions, in order to get a quadratic in s. So we can, analytically, find the poles, and finally we have

$$g(\eta;\tau_1)=\varrho^{-1}Q^{-1/2}(e^{-s_1\eta}-e^{-s_2\eta}),\quad \eta>0,\ \tau_1\ll 1,\ s_2>s_1, \tag{A.14}$$

where

$$Q=\tau_1^2\left[1+\tau_1+\frac{1}{3}\ln^2\varrho+\tau_1^4\left(\frac{3}{8}-\frac{\ln^2\varrho}{6\tau_1^2}\right)^2\right], \tag{A.15}$$

and s_1 and s_2 are the roots of the quadratic

$$As^2-Bs+C=0, \tag{A.16}$$

with

$$A={}^1/_2\tau_1^3(1+{}^3/_4\tau_1),$$
$$B=\tau_1[1+{}^3/_2\tau_1+{}^3/_8\tau_1^2+{}^1/_6\ln^2\varrho(1+{}^3/_2\tau_1)]$$

and

$$C=1+{}^3/_4\tau_1+{}^1/_2\ln^2\varrho[1+{}^1/_4\tau_1+{}^1/_{12}\ln^2\varrho].$$

It is easy to show that both roots of equation (A.16) are real and positive for all values of τ_1 and ϱ.

For large values of τ_1, equation (A.1) takes the form

$$g(\eta;\tau_1)=\frac{1}{2\pi i}\frac{8}{\varrho\sqrt{3}}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\sqrt{s}\,e^{s\eta}e^{-\tau_1\sqrt{3s}}\,ds.$$

Original function can be readily found according to [13]

$$g(\eta;\tau_1)=\frac{4}{3}\sqrt{\frac{3}{\pi}}\varrho^{-1}\eta^{-5/2}\left(\frac{3}{2}\tau_1^2-\eta\right)\exp\left(-\frac{3\tau_1^2}{4\eta}\right). \tag{A.17}$$

In the case of $n=2$ we go through the same simplifications as in the case of $n=1$. Namely, τ_1 being small, we expand the hyperbolic functions in the denominator of $\bar{g}(s)$, getting a quadratic in s, and the inverse Laplace transform has the form

$$g(\eta;\tau_1)=P^{-1/2}(e^{-s'_1\eta}-e^{-s'_2\eta}),\quad \eta>0,\ \tau_1\ll 1,\ s'_2>s'_1, \tag{A.18}$$

where

$$P=16+\tau_1(48+8\beta)+\tau_1^2(48+12\beta+\beta^2)-$$
$$-\tau_1^3(14+13\beta)-\tau_1^4({}^{87}/_4+8\beta), \tag{A.19}$$

and s_1' and s_2' are the roots of the quadratic

$$A's^2-B's+C'=0 \tag{A.20}$$

with

$$A=2\tau_1^3,$$
$$B=\tau_1(4+\beta\tau_1+6\tau_1+{}^3/_2\tau_1^2),$$
$$C=\tau_1(3+\beta)+2\beta+4,$$

and

$$\beta=\frac{(\varrho-1)^2}{\varrho}.$$

The roots of the quadratic (A.20) are positive and real for those values of τ_1 and ϱ which are of interest.

The optical depth τ_1 being large, we find from equation (4.7) that original for the $\bar{g}(s)$ is as follows:

$$g(\eta;\tau_1)=$$

$$=4\sqrt{\frac{3}{\pi}}\,\eta^{-5/2}\left(\frac{3}{2}\tau_1^2-\eta\right)\left[3\varrho+\frac{2}{\tau_1}(\varrho-1)^2\right]^{-1}\exp\left(-\frac{3\tau_1^2}{4\eta}\right). \qquad (A.21)$$

For $\tau_1 \gg 1$, equations (A.17) and (A.21) coincide.

3. In computing the expansion coefficients of the incoming flux distribution in the case of continuum eigenfunctions, we can use the integral representation of the confluent hypergeometric functions of the second kind:

$$\psi(a,c;x)=\frac{1}{\sqrt{\pi}}\,2^{a-\frac{c}{2}+1}\,e^{\frac{x}{2}}x^{\frac{1}{2}-\frac{c}{2}}\times \qquad (A.22)$$

$$\times\int_0^\infty e^{-1/2\,\mathrm{sh}^2 t}\,D_{c-2a}(\sqrt{2x}\,\mathrm{ch}\,t)\,\mathrm{ch}\,[(c-1)t]\,dt,$$

where D_ν is the parabolic cylinder function of the order ν [15]. In our case,

$$\psi\left(-\frac{3}{2}+iq,\,1+2iq;\,x\right)= \qquad (A.23)$$

$$=\frac{1}{2\sqrt{\pi}}\,x^{-iq}e^{1/2 x}\int_0^\infty e^{-1/2\,\mathrm{sh}^2 t}\,D_4(\sqrt{2x}\,\mathrm{ch}\,t)\cos 2qt\,dt.$$

Taking into account that [16]

$$D_n(z)=2^{-1/2 n}\,e^{-1/4 z^2}H_n\left(\frac{1}{\sqrt{2}}z\right),$$

where n is positive integer or zero and $H_n(z)$ is the Hermite polynomial of the order n, the formula (5.8) for the continuum eigenfunction is

$$V_\lambda(x)=\frac{1}{2\pi}\left[\frac{\mathrm{sh}\,\pi q}{\lambda(\lambda-2)}\right]^{1/2}x^3\int_0^\infty \exp\left[-\frac{1+x}{2}\,\mathrm{sh}^2 t\right]\times$$

$$\times(4x^2\,\mathrm{ch}^4 t-12x\,\mathrm{ch}^2 t+3)\cos 2qt\,dt.$$

REFERENCES

1. Dirac P. A. M., Monthly Notices Roy. Astron. Soc., **85**, 825 (1925).
2. Edmonds F. N. Jr., Astrophys. J., **117**, 298 (1953).
3. Münch G., Astrophys. J., **108**, 116 (1948).
4. Edmonds F. N. Jr., Astrophys. J., **119**, 58 (1954).
5. Edmonds F. N. Jr., Astrophys. J., **119**, 425 (1954).
6. Chandrasekhar S., Proc. Roy. Soc. A, **192**, 508 (1948).
7. Pomraning G. C., JQSRT, **8**, 1087 (1968).
8. Frazer A. R., AWRE Report No. 0-82/65 (1966).
9. Pomraning G. C., Freeman G. E., JQSRT, **8**, 909 (1963).

10. Вийк Т., В сб.: Некоторые проблемы теории звездных атмосфер, Тарту, 1968, с. 71.
11. Вийк Т., В сб.: Некоторые проблемы теории звездных атмосфер, Тарту, 1968, с. 91.
12. Дэвисон Б., Теория переноса нейтронов, М., 1960.
13. Handbook of Mathematical Functions, Ed. by M. Abramowitz and I. Stegun, Dover Publ., Inc., N. Y., 1965.
14. Pomraning G. C., Astrophys. J., 152, 808 (1968).
15. Бейтмен Г., Эрдейи А., Высшие трансцендентные функции, т. 1, М., 1965, с. 262.
16. Бейтмен Г., Эрдейи А., Высшие трансцендентные функции, т. 2, М., 1965, с. 123.

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T. VIIK

KIIRGUSE ÜLEKANNE VABADEST ELEKTRONIDEST KOOSNEVAS SFÄÄRILISES TÄHEATMOSFÄÄRIS

Sfäärilises atmosfääris esineva kiirguse ülekande võrrandi lahendamiseks tuleb ette anda vabade elektronide tiheduse jaotus funktsioonina vastava kihi raadiusest. Kõigepealt vaadeldakse juhtu, kus elektronide soojusliikumise võib arvestamata jätta. Kiirguse ülekande võrrand lahendatakse Laplace'i meetodil ja leitakse atmosfäärist väljuva kiirgusvoo avaldis. Uurimise teises osas arvestatakse ka elektronide soojusliikumist.

Näidatakse, et kui atmosfäär on küllalt paks ja optilisel sügavusel τ_1 siseneb atmosfääri monokromaatne kiirgusvoog, siis allub atmosfäärist väljuv kiirgusvoog Wieni jaotusseadusele. Seejuures selgub väljuva kiirgusvoo avaldisest, et nii vabade elektronide tiheduse gradiendi kui ka atmosfäärikihtide kõveruse suurendamine vähendab footoni väljumise tõenäosust.

Т. ВИЙК

ПЕРЕНОС ИЗЛУЧЕНИЯ В СОСТОЯЩЕЙ ИЗ СВОБОДНЫХ ЭЛЕКТРОНОВ СФЕРИЧЕСКОЙ АТМОСФЕРЕ ЗВЕЗДЫ

Для решения уравнения переноса излучения в сферической атмосфере надо задать распределение плотности свободных электронов, которое является функцией расстояния от центра звезды. В первой части рассматривается случай, когда тепловым движением электронов можно пренебречь. Уравнение переноса излучения решается методом Лапласа и получается выражение выходящего потока на границе атмосферы.

Во второй части работы учитывается и тепловое движение электронов. Оказывается, что если оптическая толщина атмосферы достаточно большая и в атмосферу на глубине τ_1 входит монохроматический поток, то частотное распределение выходящего из атмосферы потока подчиняется закону Вина. При этом из выражения выходящего потока следует, что как увеличение градиента плотности свободных электронов, так и увеличение кривизны атмосферных слоев звезды уменьшает вероятность выхода светового кванта.