EESTI NSV TEADUSTE AKADEEMIA TOIMETISED. XVIII KÖIDE
FÜÜSIKA * MATEMAATIKA. 1969, NR. 2

ИЗВЕСТИЯ АКАДЕМИИ НАУК ЭСТОНСКОЙ ССР. ТОМ XVIII
ФИЗИКА * МАТЕМАТИКА. 1969, № 2

https://doi.org/10.3176/phys.math.1969.2.05

V. UNT

A COMBINED BONDI AND FAST APPROXIMATION METHOD IN GENERAL RELATIVITY. I. THE AXI-SYMMETRIC CASE

In this paper the fast approximation expansion of the Einstein field equations is used and Bondi's method is applied to integrate the n-th order equations. In part A equations (21) to (23) for the Fourier coefficients (of the transverse metric and integration functions) are given as well as the expressions for the metric. In this part the outlines of the derivation will also be found.

Part B deals with the integration of the equations (21) to (23). The following results are obtained. The mass loss of a source is described in the metric by an angle-independent quantity. Expressions are given determining the recoil of a source and the deflection of rays in the field of the mass carried by waves. A new time coordinate is introduced in such a way that at least in the second approximation the conservation laws of Newman and Penrose no longer exist. As an example, the solutions of the equations (21) to (23) for the quadrupole radiation in the second approximation are given.

Introduction

In recent years, many new exact solutions of the Einstein field equations have been suggested. But they are too far removed from what can be called realistic physical situations. In order to examine the gravitational fields of physically realistic sources which are not spherically symmetric, one has to use approximations. This paper deals with axi-symmetric outgoing gravitational waves in an asymptotically flat space far removed from the sources of radiation. To tackle the problem, Bondi's method and the fast approximation method have been adapted. In the latter case, the metric is expanded in powers of the gravitational constant or some equivalent parameter, and meaningful operations are found for each stage of approximation, but difficulties arise in connection with the convergence of the solutions. In this paper, the fast approximation expansion is retained, and the common linear field theory technique of decomposition into spherical harmonics is introduced in a natural way. Thus the difficulties arising from the convergence of solutions can be kept under control. In the main, we have used Bondi's coordinate system, and our treatment resembles rather that of Bondi et al. [1] than the usual fast approximation method. On the whole, one might say that we have applied Bondi's method to the approximate field equations (n-th order equations).

In the case of exact field equations, cumbersome expressions have to be handled, and it is often difficult to find the physical meaninig of various terms. Approximate solutions can play a useful part in providing insight into Bondi's method in its original form. Let us list some of the problems that have remained unsolved up to now. How to choose a news function so as to get a convergent solution? Why should the mass loss of a source via radiation be described in the metric in terms of the mass aspect which depends on polar angles? An answer to these questions in the case of approximate field equations will be proposed, as well as a new interpretation of the conservation laws of Newman and Penrose. We shall single out some terms from the approximate solutions and give their physical interpretations.

From the mathematical point of view, the key-step of the paper is the derivation of ordinary differential equations (21) to (23) to contain only the Fourier coefficients of the transverse metric, of the integration functions and of the known functions. Thus the whole problem is reduced to that of finding convergent solutions of these equations. The main physical inferences of the paper stem from the conditions of convergence.

The same assumptions are made as in the paper of Bondi et al. [1], and its notation will be followed.

Part A

THE DERIVATION OF EQUATIONS FOR FOURIER COEFFICIENTS

1. The idea of derivation

The Einstein equations corresponding to the metric

$$ds^2 = (Vr^{-1} e^{2\beta} - U^2 r^2 e^{2\gamma}) d u^2 + 2e^{2\beta} du\, dr +$$

$$+ 2U r^2 e^{2\gamma} du\, d\vartheta - r^2 (e^{2\gamma} d \vartheta^2 + e^{-2\gamma} \sin^2 \vartheta\, d \varphi^2) \quad (1)$$

have been divided into four "main equations" and two "supplementary conditions". The main equations have the following structure *:

$$\beta_{,1} = \tfrac{r}{2}(\gamma_{,1})^2, \quad (2)$$

$$[r^4 e^{2(\gamma-\beta)} U_{,1}]_{,1} = \{\beta, \gamma, 1, 2\}, \quad (3)$$

$$V_{,1} = \{U, \beta, \gamma, 1, 2\} \quad (4)$$

$$(r\gamma)_{,01} = \{U, V, \beta, \gamma, 1, 2\}, \quad (5)$$

where the suffix after the comma denotes ordinary differentiation with respect to the corresponding coordinate $x^\alpha = (u, r, \vartheta, \varphi)$, $\alpha = 0, 1, 2, 3$, while the symbol $\{\beta, \gamma, 1, 2\}$ etc. denotes an expression depending only on β, γ, and their derivatives with respect to (and combinations with) r and ϑ. Integrating the equations (3) and (4), we have

$$r^4 e^{2(\gamma-\beta)} U_{,1} = -6N(u, \vartheta) + \ldots, \quad V = -2M(u, \vartheta) + \ldots.$$

* This form of the equations has been given by Bondi [2]. The full form of the equations is found in the paper by Bondi et al. [1].

The integration functions N and M satisfy the supplementary conditions ($\dot{f}=f_{,0},\ f'=f_{,2}$)

$$\dot{M}=-\dot{c}^2+\tfrac{1}{2}(\dot{c}''+3\dot{c}'\cot\vartheta-2\dot{c}),\tag{6}$$

$$-3\dot{N}=M'+3c\dot{c}'+4c\dot{c}\cot\vartheta+\dot{c}c',\tag{7}$$

where $\dot{c}$ is the news function.

In order to find β, U and V we make the formal assumption that γ is a given function and has the following structure

$$\gamma=\sum_{s\geqslant 1}a_s(u,\vartheta)r^{-s}.\tag{8}$$

By integrating the equations (2) to (4) we get

$$\beta(u,r,\vartheta)=\sum_s\overset{s}{\beta}(u,\vartheta)r^{-s},\ U(u,r,\vartheta)=\sum_s\overset{s}{U}(u,\vartheta)r^{-s},$$

$$V(u,r,\vartheta)=\sum_s\overset{s}{V}(u,\vartheta)r^{-s},$$

where

$$\overset{s}{\beta}=\{a_1,\ldots,\ a_k,2\},$$

$$\overset{s}{U}=\{\overset{r}{\beta},\ldots,\overset{r+l}{\beta},\ a_1,\ldots,a_k,\ N,2\}=\{a_1,\ldots,\ a_k,N,2\},$$

$$\overset{s}{V}=\{a_1,\ldots,a_k,M,N,2\}.$$

The meaning of the curly brackets was explained above. After inserting the last expressions into (5) we get the following recurrent equations:

$$\dot{a}_{s+1}=\{a_1,\ldots,a_s,M,N,2\}.\tag{9}$$

The whole problem is now reduced to the integration of the equations (6), (7) and (9).

But the equations (9), if worked out in detail, are so complicated that in order to integrate them, approximation methods have to be used. For this reason we shall derive the equations (9) not in the exact form, but by making use of an approximation method from the very beginning. Thus we make the fast approximation expansion for a_s,γ,β,U and V.

$$a_s(u,\vartheta)=\sum_n\underset{n}{a_s}(u,\vartheta)\ \text{etc.},$$

where the index n indicates the order of magnitude of the term. We find $\underset{n}{\gamma},\underset{n}{\beta},\underset{n}{U}$ and $\underset{n}{V}$ from the equations $\underset{n}{R_{\mu\nu}}=0$, where $R_{\mu\nu}$ is the Ricci tensor. In order to simplify the notation, the index n indicating the order of magnitude of the term will be omitted and henceforth we assume that the n-th order part of any summand is taken. Only occasionally will the index be written.

2. The equations for the Fourier coefficients of the transverse metric and of the integration functions

The following forms of the equations (2) to (5) are used:

$$0=R_{11}=-4\beta_{,1}r^{-1}+<11>, \tag{10}$$

$$0=R_{12}=-\frac{1}{2r^2}\,[r^4U_{,1}]_{,1}+(\beta_{,12}-\gamma_{,12}-2\beta_{,2}r^{-1}-2\gamma_{,1}\cot\vartheta)+<12>, \tag{11}$$

$$\left.\begin{aligned}0=R_{22}+\sin^{-2}\vartheta\,R_{33}=2V_{,1}-\frac{1}{r^2}[r^4(U_{,2}+U\cot\vartheta)]_{,1}+\\+2\beta_{,22}+2\beta_{,2}\cot\vartheta-4\beta-2(\gamma_{,22}+\\+3\gamma_{,2}\cot\vartheta-2\gamma)+<22+33>,\end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned}0=R_{22}-\sin^{-2}\vartheta\,R_{33}=-4r\,(r\gamma)_{,01}+2r\,(r\gamma)_{,11}+2\,(\beta_{,22}-\\-\beta_{,2}\cot\vartheta)-(r^2U_{,2}-r^2U\cot\vartheta)_{,1}+<22-33>.\end{aligned}\right\} \tag{13}$$

Here the symbol $<\mu\nu>$ denotes nonlinear terms in the expression for $R_{\mu\nu}$ in Bondi's coordinates. At any stage of approximation $\underset{n}{<\mu\nu>}$ can be calculated via $\underset{m}{\gamma}$, $\underset{m}{\beta}$, $\underset{m}{U}$, $\underset{m}{V}$, $m<n$, and may be considered as a known function.

$$<22\pm 33>=<22>\pm\sin^{-2}\vartheta<33>.$$

By carrying out the calculations following the scheme given in the previous section, we get the following equations for the n-th order terms in (9):

$$\dot{a}_{s+1}=-\frac{(s-1)}{2\,(s-2)\,(s+1)}\left\{a_s''+a_s'\cot\vartheta+\left[s\,(s-1)-\frac{4}{\sin^2\vartheta}\right]a_s\right\}-$$
$$-\frac{1}{4}\,(N'-N\cot\vartheta)\,\delta_{s2}+\frac{1}{2s}\overset{s}{Q}, \tag{14}$$

where $\overset{s}{Q}$ are nonlinear in a's, and at any stage of approximation they may be considered as known functions. The eigenfunctions of the differential operator in curly brackets are the associated Legendre polynomials $P_k^{(2)}$. Hence the natural decomposition for a_s is

$$a_s(u,\vartheta)=\sum_k \alpha_{sk}(u)\,P_k^{(2)}\,(\cos\vartheta). \tag{15}$$

We shall also make the following expansions in spherical harmonics:

$$M\,(u,\vartheta)=\sum_k \mu_k\,(u)\,P_k\,(\cos\vartheta), \tag{16}$$

$$N\,(u,\vartheta)=\sum_k \nu_k\,(u)\,P_k^{(1)}\,(\cos\vartheta), \tag{17}$$

$$\overset{s}{Q}\,(u,\vartheta)=\sum_k q_{sk}\,(u)\,P_k^{(2)}\,(\cos\vartheta),\ s\geqslant 2, \tag{18}$$

$$\dot{c}^2 = -\sum_k q_{0k}(u)\, P_k(\cos\vartheta), \tag{19}$$

$$3c\dot{c}' + 4c\dot{c}\cot\vartheta + \dot{c}c' = \sum_k q_{1k}(u)\, P_k^{(1)}(\cos\vartheta). \tag{20}$$

By inserting (15) to (20) into the equations (6), (7) and (14) we have in the n-th approximation *

$$\dot{\mu}_k(u) = \frac{(k+2)!}{2(k-2)!}\dot{\alpha}_{1k}(u) + q_{0k}(u),\ k \geqslant 0, \tag{21}$$

$$-3\dot{\nu}_k(u) = \mu_k(u) + q_{1k}(u),\ k \geqslant 1, \tag{22}$$

$$\dot{\alpha}_{s+1,k} = \frac{(s-1)(k-s+1)(k+s)}{2(s-2)(s+1)}\alpha_{sk} - \frac{1}{4}\nu_k\delta_{s2} + \frac{1}{2s}q_{sk},\ s \geqslant 2, k \geqslant 2, \tag{23}$$

where $q_{sk}(u)$ are known functions. In linear approximation $q_{sk} = 0$. An integral of the equations (21) to (23) is determined apart from the integration constants by $q_{sk}(u)$ and the Fourier coefficients $\dot{\alpha}_{1k}(u)$ of the news function. To get a convergent solution of the Einstein field equations, it is sufficient to find a convergent solution of the equations (21) to (23).

A more detailed expression for q_{sk}, $s \geqslant 2$ is needed. Let us now decompose $<\mu\nu>$ ($<s, \mu\nu, k>$ are functions of u only)

$$<\mu\nu> = \sum_s <s, \mu\nu> r^{-s} = \sum_{s,k} <s, \mu\nu, k> P_k^{(m)}(\cos\vartheta) r^{-s},$$

where the index m of the associated Legendre polynomials has the following values

$$m = 0, \text{ if } \mu\nu = 00,\ 11,\ 22+33,$$
$$m = 1, \text{ if } \mu\nu = 02,\ 12,$$
$$m = 2, \text{ if } \mu\nu = 22\text{—}33.$$

Now we get for the non-vanishing q_{sk}, $s \geqslant 2$ the following expression

$$\left.\begin{aligned} q_{sk}(u) = -\frac{1}{(s+1)(s-2)}[\tfrac{1}{2}<s+2, 11, k> + (s-1)<s+1, 12, k>] - \\ -\tfrac{1}{2}<s, 22\text{—}33, k>, \text{ if } s \geqslant 2,\ k \geqslant 2. \end{aligned}\right\} \tag{24}$$

The expressions for $<\mu\nu>$ may be taken from the exact form of the equations (2) to (5). If the linear field equations are satisfied, then in the second approximation we have

$$<11> = 2(\gamma_{,1})^2, \tag{25}$$

$$<12> = -r^{-2}[r^4\gamma U_{,1}]_{,1} + 2\gamma_{,1}\gamma', \tag{26}$$

$$<22+33> = \tfrac{1}{2}r^4(U_{,1})^2 + 4\gamma'^2 + 4\gamma(\gamma'' + 3\gamma'\cot\vartheta - \gamma), \tag{27}$$

$$\left.\begin{aligned} <22-33> = \tfrac{1}{2}r^4(U_{,1})^2 + 2(r\gamma_{,1}V)_{,1} - 2r^2\gamma_{,1}U' - \\ -2r^2\gamma' U_{,1} - 2rU(2r\gamma_{,1}' + r\gamma_{,1}\cot\vartheta + 2\gamma'). \end{aligned}\right\} \tag{28}$$

---

* A more detailed derivation of the equations (21) to (23) as well as their application in the case of electromagnetic radiation in the linear approximation will be found in our previous papers [3, 4].

For all components of the metric tensor, their linear approximation values are to be taken here.

3. The metric

In this section the expressions for γ, β, U and V are given in terms of α_{sk}, μ_k, ν_k and $\langle s, \mu\nu, k \rangle$. In order to calculate $\langle s, \mu\nu, k \rangle$ we must know γ, β, U and V in approximations of a lower order than n; $\underset{n}{\alpha_{sk}}$, $\underset{n}{\mu_k}$ and $\underset{n}{\nu_k}$ must be determined from the equations (21) to (23).

$$\underset{n}{\gamma} = \sum_{s,k} \underset{n}{\alpha_{sk}}(u)\, P_k^{(2)}(\cos\vartheta)\, r^{-s}, \tag{29}$$

$$\underset{n}{\beta} = -\tfrac{1}{4} \sum_{s,k} \frac{1}{s} \underset{n}{\langle s+2, 11, k \rangle} P_k(\cos\vartheta)\, r^{-s}, \tag{30}$$

$$\left.\begin{aligned} \underset{n}{U} = \sum_{s,k} \frac{2}{s(s-3)} \Big[\underset{n}{\langle s, 12, k \rangle} + \frac{(s+1)}{4(s-1)} \underset{n}{\langle s+1, 11, k \rangle} - \\ -(s-1)(k-1)(k+2)\, \underset{n}{\alpha_{s-1,k}} \Big] P_k^{(1)} r^{-s} + 2 \sum_k \underset{n}{\nu_k} P_k^{(1)} r^{-3}, \end{aligned}\right\} \tag{31}$$

$$\left.\begin{aligned} \underset{n}{V} = \sum_{s>0,k} \Big\{ \frac{1}{2s} \Big[\underset{n}{\langle s+1, 22+33, k \rangle} - \frac{2k(k+1)(s-2)}{(s-1)(s+2)} \underset{n}{\langle s+2, 12, k \rangle} + \\ + \frac{1}{s+1} \underset{n}{\langle s+3, 11, k \rangle} + \frac{2k(k+1)}{2(s-1)(s+1)(s+2)} \underset{n}{\langle s+3, 11, k \rangle} \Big] - \\ - \frac{2(k+2)!\, \underset{n}{\alpha_{s+1,k}}}{(k-2)!\,(s-1)(s+2)} \Big\} P_k r^{-s} - \sum_k [2\underset{n}{\mu_k} - k(k+1) \underset{n}{\nu_k} r^{-1}] P_k. \end{aligned}\right\} \tag{32}$$

Part B

THE INTEGRATION OF THE EQUATIONS (21) TO (23) THE NATURE OF THE SOLUTIONS

1. The linear approximation

Before embarking on a study of higher approximations, it is worth while paying some attention to the equations (21) to (23) in linear approximation. Here $q_{sk} = 0$, and the integrals are determined by $\underset{1}{\alpha_{1k}}$ apart from the integration constants. Let us first find the meaning of these constants.

The Schwarzschild solution is

$$\left.\begin{aligned} &\mu_k = 0 \text{ unless } k = 0; \ \nu_k = \alpha_{sk} = 0; \\ &\mu_0 = m, \ m = \text{const.} \end{aligned}\right\} \tag{33}$$

Hence μ_0 is the mass of the source. It is easy to verify that (33) is an exact solution, since q_{sk} vanish at any higher stage of approximation. To (33) corresponds the metric

$$ds^2 = \left(1 - \frac{2m}{\tilde{r}}\right) d\tilde{u}^2 + 2d\tilde{u}d\tilde{r} - \tilde{r}^2 (d\tilde{\vartheta}^2 + \sin^2\tilde{\vartheta}\, d\varphi^2). \qquad (34)$$

Next let us consider the solution:

$$\left.\begin{aligned} &\mu_k = 0 \text{ unless } k = 0, 1; \quad \nu_k = 0 \text{ unless } k = 1; \ \alpha_{sk} = 0; \\ &\mu_0 = m, \ \mu_1 = 3P, \quad \nu_1 = -Pu - D, \quad P = \text{const}, \ D = \text{const}. \end{aligned}\right\} \qquad (35)$$

The corresponding line element is

$$ds^2 = [1 - \frac{2m}{r} - \left(\frac{6P}{r} + \frac{2Pu}{r^2} + \frac{2D}{r^2}\right)\cos\vartheta]\, du^2 +$$
$$+ \frac{4}{r}(Pu + D)\sin\vartheta\, d\vartheta\, du - r^2 (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2). \qquad (36)$$

If one introduces the new constants v and L (v being velocity, L distance)

$$P = mv, \qquad (37)$$

$$D = mL, \qquad (38)$$

it is possible to get (36) from (34) by a non-homogeneous Lorentz transformation (the squares and higher powers of v and $\frac{L}{r}$ are neglected).

$$\tilde{r} = r - v(u + r)\cos\vartheta - L\cos\vartheta + \ldots,$$
$$\tilde{\vartheta} = \vartheta + \frac{v}{r}(u + r)\sin\vartheta + \frac{L}{r}\sin\vartheta + \ldots,$$
$$\tilde{u} = u + vu\cos\vartheta + L\cos\vartheta + \ldots.$$

Hence P is the momentum of the source, D and $-\nu_1$ are its dipole moments at the time $u = 0$ and u. The retarded time occurs instead of the usual time because we are looking at the gravitational field far from the source. In higher approximations we shall also call μ_0 and μ_1 the mass and the momentum of the source.

To the static multipole field corresponds the solution:

$$\left.\begin{aligned} &\mu_k = 0 \text{ unless } k = 0; \ \nu_k = 0; \ \alpha_{sk} = 0 \text{ unless } s = k + 1; \\ &\mu_0 = m, \ \alpha_{k+1,k} = \frac{1}{k(k+1)} D_k, \ D_k \equiv D_0^{(k)} = \text{const}. \end{aligned}\right\}$$

The definition of $D_0^{(k)}$ is given by L. D. Landau and E. M. Lifshitz [5] as

$$D_0^{(k)} = \int \varrho r^k P_k(\cos\vartheta)\, d\omega,$$

where $\varrho = T_0^0 - T_i^i$, T_μ^ν is the energy-momentum tensor in Galilean coordinates, and $d\omega$ is the volume element.

It is difficult to give simple physical interpretations to the remaining integration constants, and we put them equal to zero, since otherwise they would give rise to unbounded Fourier coefficients. With respect to the solutions that contain (do not contain) the unbounded coefficients ν_k, $k \geqslant 2$ and α_{sk} the word "divergent" ("convergent") will be used. In the following sections an attempt will be made to apply Bondi's method effectually, while using this highly limiting notion of convergency at any stage of approximation.* The resulting difficulties and ways of overcoming them are examined in the following sections.

The radiative solutions of equations (21) to (23) are determined by $\dot{\alpha}_{1k}$. The latter are equal to the $(k+1)$-th retarded time derivatives of the 2^k-pole moments of the source multiplied by constant factors.

Let us consider the quadrupole radiation in linear approximation. Then

$$\alpha_{12} = \frac{1}{12}\ddot{Q}(u), \quad \alpha_{1k} = 0, \text{ if } k \geqslant 3.$$

Here Q is the quadrupole moment:

$$Q = Q_{zz} = -2Q_{xx} = -2Q_{yy} = 2D_2.$$

By integrating the equations (21) to (23) and inserting α_{s2}, μ_2 and ν_2 into (29) to (32) we have

$$\left.\begin{aligned} \gamma &= \frac{1}{12}\left(\frac{\ddot{Q}(u)}{r} + \frac{Q(u)}{r^3}\right) P_2^{(2)}(\cos\vartheta), \\ \beta &= 0, \\ U &= \left(\frac{\ddot{Q}}{3r^2} - \frac{2\dot{Q}}{3r^3} - \frac{Q}{2r^4}\right) P_2^{(1)}, \\ V &= -2m - \left(2\ddot{Q} + \frac{2\dot{Q}}{r} + \frac{Q}{r^2}\right) P_2. \end{aligned}\right\} \tag{39}$$

2. On the convergence of the integrals of the equations (21) to (23) in higher approximations

In this section the divergencies that may be produced by q_{rs} are examined. From the equations (21) to (23) it follows that the monotonic changes of μ_k, ν_k and α_{sk}, $k \geqslant 2$, $s > 1$, $s \neq k+1$ (if they occur) would give rise to unbounded terms. Hence such changes are to be excluded. Let us suppose that one of the equations (23)

$$\dot{\underset{n}{\alpha}}_{s+1,k} = \frac{(s-1)(k-s+1)(k+s)}{2(s-2)(s+1)} \underset{n}{\alpha}_{sk} + \frac{1}{2s} \underset{n}{q}_{sk}(\underset{m}{\alpha}_{rl}), \; m < n,$$

gives

$$\underset{n}{\alpha}_{s+1,k} = F(u) + \ldots,$$

where F is a monotonically changing function. Now if $s < k+1$, $\alpha_{s+1,k}$ depends on $\underset{n}{\alpha}_{1k}$. Taking $\underset{n}{\alpha}_{1k} = A\frac{d^s F}{du^s}$ and choosing an appropriate value for the constant A, we can "cut off" the monotonic change of $\underset{n}{\alpha}_{s+1,k}$. No difficulties connected with the divergence of the solution have so far appeared.

The monotonic changes of $\underset{n}{\mu}_0$ and $\underset{n}{\mu}_1$ are not affected by the choice of $\underset{n}{\alpha}_{1k}$, since $\alpha_{10} = \alpha_{11} = 0$. The necessary and sufficient condition for the latter is the regularity of the solution on the polar axis $\sin\vartheta = 0$.

In the case $s \geqslant k+1$ no "cut-off" is possible because of the factor

* It may be that further investigations will show that some less limiting notion of convergency must be introduced. Such would clearly be the case if it really turned out that gravitational waves do not adhere to the Huygens principle, and the terms const $(u+2r)^{-m}$ that describe the incoming radiation cannot be avoided.

$k-s+1$ in the equation (23). Real difficulties may appear at this stage of approximation only. Now one could try to make coordinate transformations that produce in addition to q_{sk} the coefficients $\tilde{q}_{sk}$ to cancel the divergencies. In section 7 we shall see that some divergencies are really removable by a slight modification of the metric (1). As we shall see later, direct calculations in the case of quadrupole radiation show that no difficulties arise in connection with the terms which are quadratic in either quadrupole moment or its derivatives.

3. The loss of mass

Let us now contemplate the case of an initially and eventually quiescent transmitter, i. e. $\dot{\alpha}_{1k}(u)=0$ if $u \leqslant u_0$ and $u \geqslant u_1$. Consider the equation (21) in the second approximation:

$$\underset{2}{\dot{\mu}_0} = \underset{2}{q_{00}}, \tag{40}$$

$$\underset{2}{q_{00}} = -\tfrac{1}{2}\int \underset{1}{\dot{c}^2} \sin\vartheta \, d\vartheta. \tag{41}$$

The integral of (40) is

$$\underset{2}{\mu_0}(u) = -\Delta m, \ \Delta m = -\int_{u_0}^{u} \underset{2}{q_{00}}(\tau)\, d\tau.$$

The monotonic changes of $\underset{2}{\mu_1}$, if they occur, may be removed by a coordinate transformation.

Consider the case $k \geqslant 2$. Let us denote the monotonically changing part of the function with a barred letter, and choose $\underset{2}{c}=\underset{2}{\bar{c}}+\ldots$, where the Fourier coefficients of $\underset{2}{\bar{c}}$ satisfy the equations

$$\frac{1}{2}\,\frac{(k+2)!}{(k-2)!}\,\underset{2}{\dot{\bar{\alpha}}_{1k}} + \underset{2}{q_{0k}} = 0, \ k \geqslant 2, \tag{42}$$

$$\underset{2}{q_{0k}} = -\frac{2k+1}{2}\int_0^{\pi} (\underset{1}{\dot{c}})^2\, P_k(\cos\vartheta)\,\sin\vartheta\, d\vartheta.$$

Then we have

$$\underset{2}{\bar{\mu}_k} = 0, \text{ if } k \geqslant 2,$$

and

$$M = m - \Delta m\,(u) + \ldots .$$

The mass carried away by gravitational waves is expressed in the metric by $\Delta m\,(u)$ and does not depend on polar angles.

We have not taken into consideration the periodically changing part of $\underset{2}{c}$; its contribution to the second order solution is similar to that of $\underset{1}{c}$ to the first order solution.

4. The gravitational recoil

Consider the equation

$$\dot{\mu}_1 = q_{01}, \tag{43}$$

or retaining the interpretation given in (35) and (37)

$$\dot{P}(u) = \tfrac{1}{3} q_{01}(u), \tag{44}$$

where P is the momentum of the source. From the last expression it follows that $\frac{1}{3} q_{01}$ is a force acting upon the source along the polar axis. It may be different from zero only in the case where there is an emission of gravitational waves, and we interpret it as the force of the gravitational recoil. It has the following value:

$$\tfrac{1}{3} q_{01} = -\tfrac{1}{2} \int_0^\pi \dot{c}^2(u, \vartheta) \cos\vartheta \sin\vartheta \, d\vartheta.$$

5. The deflection of gravitational rays

The conditions (42) eliminate the monotonic changes of $\underset{2}{\mu}_k$, $k > 2$, and produce, instead, the monotonic changes of $\underset{2}{\alpha}_{1k}$. We shall attempt to show that these changes can be explained as a deflection of the gravitational rays; they may be introduced or removed by a coordinate transformation. Let us define

$$A_k(u) = \int_{u_0}^{u} q_{0k}(\tau) \, d\tau.$$

Then

$$\underset{2}{\alpha}_{1k}(u) = -\frac{2(k-2)!}{(k+2)!} A_k(u) + \ldots .$$

We can write out the terms containing A_k in the metric as follows:

$$\gamma = -\frac{2}{r} \sum_k \frac{(k-2)!}{(k+2)!} A_k(u) \, P_k^{(2)}(\cos\vartheta), \tag{45}$$

$$U = -\frac{2}{r^2} \sum_k A_k(u) \frac{1}{k(k+1)} P_k^1(\cos\vartheta), \tag{46}$$

$$\beta = V = 0.$$

The significance of the monotonically changing terms in (45) and (46) may be understood by considering what happens to the flat space metric

$$ds^2 = d\tilde{u}^2 + 2\, d\tilde{u} d\tilde{r} - \tilde{r}^2 (d\tilde{\vartheta}^2 + \sin^2\tilde{\vartheta} \, d\varphi^2), \tag{47}$$

if the following co-ordinate transformation is made

$$\tilde{u} = u - H(u, \vartheta), \tag{48}$$

$$\tilde{r} = r - \frac{1}{2\sin\vartheta} (\sin\vartheta \, H')', \tag{49}$$

$$\tilde{\vartheta} = \vartheta + H' r^{-1}, \tag{50}$$

$$H = -\sum_k \frac{4(k-2)!}{(k+2)!} \quad A_k(u) P_k(\cos\vartheta), \ k \geqslant 2. \tag{51}$$

We get from (47) a new line element with (Δ^* being the Laplace operator on the surface of a sphere)

$$\gamma = \frac{1}{2r} (H'' - H' \cot\vartheta), \tag{52}$$

$$U = -\frac{1}{r^2} [H' + \tfrac{1}{2} (\Delta^* H)'], \tag{53}$$

$$\beta = 0 + \ldots, \ V = 0 + \ldots .$$

The terms containing $\dot{H}$ have been omitted, they do not change monotonically. After inserting the values of H from (51) we find that the terms containing A_k in (52) and (53) are equal to the corresponding terms in (45) and (46). When the radiation ceases, $\dot{H} = 0$, and (48) to (50) is a supertranslation.

This phenomenon may be interpreted in the following manner. When there is no radiation, we have on each "test ray"

$$\tilde{\vartheta} = \text{const}, \tag{54}$$

when radiation is present — $\vartheta = \text{const}$, or

$$\tilde{\vartheta} - \frac{H'}{r} = \text{const}. \tag{55}$$

The last expression gives us the path of the deflected rays as compared to (54).

6. The conservation laws of Newman and Penrose

In this section, some difficulties connected with the divergence of α_{sk}, $s \geqslant k+2$ will be examined in the second approximation. These divergences are caused by q_{sk} which contain the mass m as a factor. By taking only such terms into account in q_{sk} we have

$$q_{sk} = -\tfrac{1}{2} < s, 22-33, k > + \ldots, \tag{56}$$

$$<22-33> = -4m (r \underset{1}{\gamma}_{,11} + \underset{1}{\gamma}_{,1}) + \ldots . \tag{57}$$

Now consider the quadrupole radiation. Inserting $\underset{1}{\gamma}$ from (39) into (57) and (56), we have

$$q_{32} = 0, \quad q_{42} = \frac{3}{2} m Q,$$

and the equation (23) for $s = 4$, $k = 2$ becomes

$$\dot{\alpha}_{42} = 0. \tag{58}$$

This is the conservation law * of Newman and Penrose [6].
By integrating, we get

$$\alpha_{42} = A, \tag{59}$$

where A is a conserved quantity. In order to interpret it, let us consider the equation (23) for $s=5$, $k=2$

$$\dot{\alpha}_{52} = -\frac{9}{10} A + \frac{3}{16} m Q.$$

In the static case $\dot{\alpha}_{52} = 0$, and

$$A = \frac{5}{24} m Q.$$

The solution is divergent unless the values of $Q(u)$ before and after the radiation are equal, i. e.

$$Q(u_0) = Q(u_1).$$

Now let us make the same calculations in the case of the 2^k-pole moment. Let

$$\underset{1}{\alpha}_{k+1,\,k} = \frac{1}{k(k+1)} D_k(u),$$

where D_k is the 2^k-pole moment. From (23) follows

$$\underset{1}{\alpha}_{kk} = \frac{(k-2)}{k^2(k-1)} \dot{D}_k(u).$$

Inserting the corresponding γ into (57) and (56), we have

$$q_{k+1,\,k} = \frac{2(k-2)}{(k-1)} m \dot{D}_k(u), \tag{60}$$

$$q_{k+2,\,k} = \frac{2(k+1)}{k} m D_k. \tag{61}$$

Equations (23) now become

$$\dot{\alpha}_{k+2,\,k} = 0 \cdot \alpha_{k+1,\,k} + \frac{(k-2)}{(k+1)(k-1)} m \dot{D}_k, \tag{62}$$

$$\dot{\alpha}_{k+3,\,k} = -\frac{(k+1)^2}{k(k+3)} \alpha_{k+2,\,k} + \frac{(k+1)}{k(k+2)} m D_k. \tag{63}$$

In the initially static case $\dot{D}_k = 0$,

$$\alpha_{k+2,\,k} = A,$$

$$\dot{\alpha}_{k+3,\,k} = \frac{(k+1) m D_k}{k(k+2)} - \frac{(k+1)^2 A}{k(k+3)},$$

$$A = \frac{(k+3)}{(k+1)(k+2)} m D_k(u_0).$$

---

* The conservation law has been given in this form by Van der Burg [7]. In an axisymmetric case there is only one conservation law.

During radiation

$$\alpha_{k+2,\,k} = A + \frac{(k-2)}{(k-1)(k+1)}\, m D_k(u),$$

$$\dot{\alpha}_{k+3,\,k} = \left[\frac{(k+1)}{k(k+2)} - \frac{(k-2)(k+1)}{k(k-1)(k+3)}\right] m D_k(u) - \frac{(k+1)^2}{k(k+3)} A,$$

$$\dot{\alpha}_{k+4,\,k} = \text{const} \cdot \alpha_{k+3,\,k}, \; .$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

To get the last expressions after radiation has ceased, we simply replace $D_k(u)$ by $D_k(u_1) = \text{const}$. Now we have a divergent solution unless $D_k(u_1)$ takes a specific value. If $k=2$, then $D_k = \frac{1}{2}Q$, and we have the case of quadrupole radiation. We shall not try to give any interpretation to the results obtained. Further on it will be demonstrated that the mathematical difficulties encountered here can be removed by the specific choice of the coordinate system.

7. The new time coordinate

Now we shall repeat the calculations of the previous section, using a new time coordinate τ instead of u. In order to avoid long and tedious calculations, let us ignore for the time being the tensor character of the $g_{\mu\nu}$ and of equations (2) to (5), and interpret the coordinate transformation as the introduction of the new independent variable τ. Let

$$\tau = u + 2m \ln r. \tag{64}$$

In the first approximation equations (21) to (23) and their solutions remain unchanged. In higher approximations, the equations also retain their form, except for the additional terms $\tilde{q}_{sk}$. Replacing $\frac{\partial}{\partial r}$ by $\frac{\partial}{\partial r} + \frac{2m}{r}\frac{\partial}{\partial \tau}$ we get the new second order terms

$$< \tilde{11} > = 0,$$

$$< \tilde{12} > = -3m\dot{U} - 2mr\dot{U}_{,1} - \frac{2m}{r}(\dot{\gamma}' + 2\dot{\gamma}\cot\vartheta),$$

$$< \tilde{22} - \tilde{33}) > = -8mr\ddot{\gamma} + 4m\dot{\gamma} + 8mr\dot{\gamma}_{,1} - 2mr(\dot{U}' - \dot{U}\cot\vartheta),$$

$$< \tilde{22} + \tilde{33} > = \frac{4m}{r}\dot{V} - 2mr(\dot{U}' + \dot{U}\cot\vartheta).$$

Only the terms that may cause divergences are of interest here, viz.:

$$\tilde{q}_{k+1,\,k} = \frac{2(2k^2+3k+1)}{(k-1)(k+1)(k+2)}\, m\dot{D}_k, \tag{65}$$

$$q_{k+2,\,k} = 0.$$

The coordinate transformation (64) leaves $\underset{2}{<\mu\nu>}$ and $\underset{2}{q_{sk}}$ unchanged, because they are quadratic in the first order terms, which themselves remain unchanged. Now we have

$$\dot{\alpha}_{k+2,k}(\tau) = \frac{1}{2(k+1)}[\tilde{q}_{k+1,k}(\tau) + q_{k+1,k}(\tau)],$$

$$\dot{\alpha}_{k+3,k}(\tau) = -\frac{(k+1)^2}{k(k+3)}\alpha_{k+2,k}(\tau) + \frac{1}{2(k+2)}q_{k+2,k}(\tau).$$

Insert q_{sk} and $\tilde{q}_{sk}$ from (60), (61) and (65). By integrating, we get

$$\alpha_{k+2,k}(\tau) = \frac{(k+3)}{(k+1)(k+2)}\, mD_k,$$

$$\alpha_{k+3,k}(\tau) = 0.$$

No restrictions exist for the final state. It is thus possible to pass from an initially static state to an eventually static state via an interlude of radiation.

But now another difficulty appears. We have new kinds of non-vanishing $< s, \tilde{\mu\nu}, k >$

$$< 0, \tilde{22} - \tilde{33}, k > = -8m\, \ddot{\alpha}_{1k}, \tag{66}$$

$$< 1, \tilde{22} - \tilde{33}, k > = -4m\, \dot{\alpha}_{1k} - 2m\,(k-1)(k+2)\, \dot{\alpha}_{1k}, \tag{67}$$

$$< 2, \tilde{12}, k > = 3m\,(k-1)(k+2)\, \dot{\alpha}_{1k}, \tag{68}$$

$$< 1, \tilde{22} + \tilde{33}, k > = 0. \tag{69}$$

These terms cannot be put into the formulae which have been deduced above, and we must find their contribution to the metric separately. By inserting (66) to (69) into (11) to (13) and integrating we have

$$\underset{2}{\gamma} = -\frac{2m\,\underset{1}{\dot{c}}\ln r}{r}, \tag{70}$$

$$\underset{2}{U} = \frac{2m\ln r}{r^2}(\underset{1}{\dot{c}'} + 2\underset{1}{\dot{c}}\cot\vartheta), \tag{71}$$

$$\underset{2}{V} = m\ln r\,(\dot{c}'' + 3\dot{c}'\cot\vartheta - 2\dot{c}). \tag{72}$$

Now we can use the formulae deduced above, and add the latter expressions to our second order solution.

8. Quadrupole radiation in the second approximation

Let us now proceed to find γ in the second approximation. After that has been done, the other components of the metric tensor can be evaluated by simple algebraic operations.

First of all, consider the second order terms that contain m as a factor. We shall make use of the new time coordinate introduced by (60). The non-vanishing q_{sk} and $\tilde{q}_{sk}$ now take the form of

$$q_{22} = \frac{1}{6} m \ddot{Q}, \qquad q_{42} = \frac{3}{2} m Q.$$

$$\tilde{q}_{22} = \frac{1}{3} m \ddot{Q}, \qquad \tilde{q}_{32} = \frac{5}{4} m \dot{Q}.$$

By integrating (23), we get

$$\alpha_{32} = \frac{1}{8} m \dot{Q}, \; \alpha_{42} = \frac{5}{24} m Q, \; \alpha_{52} = 0.$$

We must now find the values for the non-vanishing q_{sk} that are quadratic in either Q or its derivatives

$$q_{00} = -\frac{1}{30} \dddot{Q}^2,$$

$$q_{02} = \frac{1}{21} \dddot{Q}^2, \qquad q_{04} = -\frac{1}{70} \dddot{Q}^2,$$

$$q_{12} = -\frac{1}{7} \ddot{Q} \dddot{Q}, \qquad q_{14} = \frac{3}{70} \ddot{Q} \dddot{Q},$$

$$q_{22} = -\frac{2}{21} \ddot{Q}^2, \qquad q_{24} = \frac{1}{35} \ddot{Q}^2,$$

$$q_{32} = 0, \qquad q_{34} = \frac{1}{10} \dot{Q} \ddot{Q},$$

$$q_{42} = -\frac{3}{7} (Q \ddot{Q} + \dot{Q}^2), \qquad q_{44} = \frac{23}{70} Q \ddot{Q} - \frac{6}{35} \dot{Q}^2,$$

$$q_{52} = -\frac{5}{3} Q \dot{Q}, \qquad q_{54} = -\frac{1}{3} Q \dot{Q},$$

$$q_{62} = -\frac{15}{14} Q^2, \qquad q_{64} = -\frac{5}{28} Q^2,$$

$$\tilde{q}_{sk} = 0.$$

Let us choose the following $\underset{2}{\alpha}_{1k}$

$$\alpha_{12} = -\frac{5}{42} \Delta m + \frac{5}{168} \frac{d}{d\tau} \ddot{Q}^2, \; \Delta m(\tau) = \frac{1}{30} \int_{\tau_0}^{\tau} \dddot{Q}^2 (u)\, du,$$

$$\alpha_{14} = \frac{1}{420} \Delta m - \frac{1}{1680} \frac{d}{d\tau} \ddot{Q}^2 + \frac{1}{1200} \frac{d^3}{d\tau^3} \dot{Q}^2.$$

By integrating (21) to (23), we have

$$\mu_0 = -\Delta m(\tau),$$

$$\mu_2 = \frac{5}{7} \ddot{Q} \dddot{Q}, \qquad \mu_4 = -\frac{3}{28} \frac{d \ddot{Q}^2}{d\tau} + \frac{3}{20} \frac{d^3 \dot{Q}^2}{d\tau^3},$$

$$\nu_2 = -\frac{2}{21} \ddot{Q}^2, \qquad \nu_4 = \frac{1}{35} \ddot{Q}^2 - \frac{1}{20} \frac{d^2 \dot{Q}^2}{d\tau^2},$$

$$\alpha_{32} = 0, \qquad \alpha_{34} = \frac{1}{40}\dot{Q}\ddot{Q},$$

$$\alpha_{42} = 0, \qquad \alpha_{44} = \frac{5}{96}\dot{Q}^2,$$

$$\alpha_{52} = -\frac{3}{56}Q\dot{Q}, \qquad \alpha_{54} = \frac{23}{560}Q\dot{Q},$$

$$\alpha_{62} = -\frac{1}{24}Q, \qquad \alpha_{64} = -\frac{1}{60}Q^2,$$

$$\alpha_{72} = 0, \qquad \alpha_{74} = 0.$$

No difficulties arise in connection with the divergence of α_{sk}, $s \geqslant k+2$. If we want to get the expressions for β, U, V as well, we need only to find $\underset{2}{\langle s, \mu\nu, k \rangle}$, after which α_{sk}, μ_k, ν_k and $\langle s, \mu\nu, k \rangle$, inserted into expressions (29) to (32) will give us, together with (70) to (72), $\underset{2}{\gamma}$, $\underset{2}{\beta}$, $\underset{2}{U}$ and $\underset{2}{V}$.

9. Conclusion

We have considered the application of Bondi's method to the approximate Einstein field equations. The most important result is the following: the monotonically decreasing part of Bondi's mass aspect does not depend on polar angles. The most difficult problem which remains is connected with the choice of the coordinate system and the convergence of the solutions for α_{sk}, $s \geqslant k+2$. It may be that slight modifications of the coordinate system (1) are necessary. But these would inevitably destroy the elegant structure of the equations (2) to (5), on account of the appearance of the time derivatives in (2) to (4), and produce logarithmic terms.

The author is deeply indebted to Prof. H. Bondi, Prof. F. A. E. Pirani, Prof. R. Penrose and Dr. T. Morgan for their help in elucidating Bondi's method. The author became interested in the problems discussed here during his stay at King's College, London University, and wishes to make use of this opportunity for gratefully acknowledging the generous hospitality he received there.

REFERENCES

1. Bondi H., Van der Burg M. G. J. and Metzner A. W. K., Proc. Roy. Soc., A **269**, 21 (1962).
2. Bondi H., Lectures on General Relativity, Brandeis Summer Institute in Theor. Phys., New Jersey: Prentice-Hall, 1964.
3. Унт В., Изв. АН ЭССР, Физ. Матем., **17**, **№ 2**, **164** (1968).
4. Унт В., Изв. АН ЭССР, Физ. Матем., **17**, **№ 3**, **290** (1968).
5. Ландау Л. Д., Лифшиц Е. М., Теория поля, М., 1960.
6. Newman E. T., Penrose R., Phys. Rev. Lett., **15**, 231 (1965).
7. Van der Burg M. G. J., Proc. Roy. Soc., A **294**, **112** (1966).

Academy of Sciences of the Estonian SSR,
Institute of Physics and Astronomy

Received
Sept. 6, 1968

V. UNT

KOMBINEERITUD BONDI JA KIIRETE LÄHENDITE MEETOD ÜLDISES RELATIIVSUSTEOORIAS. I. AKSIAALSÜMMEETRILINE JUHT

Artiklis vaadeldakse gravitatsioonilaineid, mille allikas on aksiaalsümmeetriline ja lõplike mõõtmetega. Resümeeritakse mõningaid meie poolt varem saadud tulemusi ja esitatakse uusi. Einsteini võrrandid arendatakse gravitatsioonikonstandi astmete järgi ritta, n-dat järku võrrandid integreeritakse Bondi meetodil.

Artikli esimeses osas esitatakse võrrandid transversaalse meetrika ja integreerimisfunktsioonide Fourier' kordajate jaoks, teises osas integreeritakse neid võrrandeid ja interpreteeritakse tulemusi. Saadud matemaatilised avaldised kirjeldavad massi ärakandmist gravitatsioonilainete poolt, allika gravitatiivset tagasipõrget ja gravitatsioonikiirte kõrvalekaldumist gravitatsioonilainete poolt kantava massi väljas. Kasutatakse sellist ajakoordinaati, mille puhul Newman-Penrose'i jäävuse seadused teises lähenduses lakkavad eksisteerimast. Näitena vaadeldakse kvadrupolkiirgust teises lähenduses.

В. УНТ

МЕТОД БОНДИ В СОЕДИНЕНИИ С МЕТОДОМ БЫСТРЫХ ПРИБЛИЖЕНИЙ В ОБЩЕЙ ТЕОРИИ ОТНОСИТЕЛЬНОСТИ. I. АКСИАЛЬНО-СИММЕТРИЧНЫЙ СЛУЧАЙ

Резюмируются работы автора по гравитационным волнам и приводятся новые результаты. Уравнения Эйнштейна разлагаются в ряд по степеням гравитационной постоянной, а для интегрирования уравнений n-го порядка применяется метод Бонди.

В первой части работы приведены уравнения (21)—(23) для коэффициентов Фурье трансверсальной метрики и произвольных функций интегрирования, указана общая схема их вывода. Даны также выражения для компонент метрического тензора в n-м приближении. Во второй части работы интегрируются уравнения (21)—(23) для коэффициентов Фурье и дается физическая интерпретация различных членов в решении уравнений Эйнштейна. Получены следующие результаты. Унос массы сказывается в линейном элементе в монотонном убывании величины, описывающей массу центрального тела. Даны выражения, определяющие гравитационную отдачу источника и отклонение гравитационных лучей в поле массы, переносимой этими волнами. Введена новая координата времени, так что по крайней мере во втором приближении законы сохранения Ньюмэна—Пенроуза недействительны. В виде примера рассматривается квадрупольное излучение во втором приближении.