Proc. Estonian Acad. Sci. Phys. Math., 2007, **56**, 1, 17–27
https://doi.org/10.3176/phys.math.2007.1.02

Solution of nonlinear Fredholm integral equations via the Haar wavelet method

Ülo Lepik and Enn Tamme

Institute of Applied Mathematics, University of Tartu, Liivi 2, 50409 Tartu, Estonia; ulo.lepik@ut.ee, enn.tamme@ut.ee

Received 12 September 2006, in revised form 3 January 2007

Abstract. A numerical method for solving nonlinear Fredholm integral equations, based on the Haar wavelet approach, is presented. Its efficiency is tested by solving four examples for which the exact solution is known. This allows us to estimate the exactness of the obtained numerical results. High accuracy of the results even in the case of a small number of grid points is observed.

Key words: nonlinear integral equations, Haar wavelets, collocation method.

1. INTRODUCTION

Many problems from physics and other disciplines lead to linear or nonlinear integral equations. Several methods have been proposed for numerical solution of these equations (see, e.g., [1]). One of the favourite techniques is the collocation method; of numerous papers about this approach we would cite here the papers [2–4].

Since 1991 the wavelet method has been applied to solving integral equations. Various wavelet bases have been employed. In addition to the conventional Daubechies wavelets [5], the Hermite-type trigonometric wavelets [6], linear B-splines [7], Walsh functions [8], Cohen [9] and Albert [10] wavelets have been used. These solutions are often quite complicated, therefore simplifications are welcome. One possibility is to make use of the Haar wavelets, which are mathematically the simplest wavelets. For linear integral equations this approach has been realized in [11,12].

In the present paper the Haar wavelet method is applied to solving linear Fredholm, Volterra, and integro-differential equations. Weakly singular integral equations are also discussed. The paper is an extension of [11]. For solving nonlinear integral equations an iterative method, for which the number of collocation points is doubled at each iteration, is proposed. The method is tested with the help of four numerical examples the exact solution of which is known.

In this paper only the Fredholm integral equations of second kind are considered, but as it follows from [11], this approach can be easily applied also to solving other types of integral equations.

2. HAAR WAVELETS

The Haar wavelet family is

$$h_i(t) = \begin{cases} 1 & \text{for} \qquad t \in [t^{(1)}, t^{(2)}), \\ -1 & \text{for} \qquad t \in [t^{(2)}, t^{(3)}), \\ 0 & \text{elsewhere.} \end{cases} \tag{1}$$

Here the notations

$$t^{(1)} = \frac{k}{m}, \quad t^{(2)} = \frac{k+0.5}{m}, \quad t^{(3)} = \frac{k+1}{m} \tag{2}$$

are introduced. The integer $m = 2^j$, $j = 0, 1, ..., J$, indicates the level of the wavelet; $k = 0, 1, ..., m-1$ is the translation parameter. The integer J determines the maximal level of resolution. The index i is calculated from the formula $i = m + k + 1$. Here the minimal value is $i = 2$ (then $m = 1,\ k = 0$); the maximal value is $i = 2M$, where $M = 2^J$. The index $i = 1$ corresponds to the scaling function

$$h_1(t) = \begin{cases} 1 & \text{for} \qquad 0 \le t \le 1, \\ 0 & \text{elsewhere.} \end{cases} \tag{3}$$

Let us divide the interval $t \in [0, 1]$ into $2M$ parts of equal length $\Delta t = 1/(2M)$; the grid points are

$$\tau_l = (l-1)\Delta t, \quad l = 1, 2, ..., 2M+1. \tag{4}$$

Since in the following the integral equations are solved by the collocation method, the collocation points

$$t_l = (l-0.5)\Delta t, \quad l = 1, 2, ..., 2M, \tag{5}$$

are introduced.

Following Chen and Hsiao [12], the Haar coefficient matrix H is introduced. It is a $2M \times 2M$ matrix with the elements $H(i, l) = h_i(t_l)$.

3. SOLVING INTEGRAL EQUATIONS BY THE WAVELET METHOD

Consider the Fredholm integral equation of the second kind

$$u(x) = \int_0^1 K(x, t, u(t))\mathrm{d}t + f(x), \quad 0 < x < 1, \tag{6}$$

where K and f are given functions.

The function u to be determined is expanded into the wavelet series

$$u(x) = \sum_{i=1}^{2M} a_i h_i(t), \tag{7}$$

where a_i are wavelet coefficients.

Putting (6) in the collocation points $x_l = (l - 0.5)\Delta t$, we obtain

$$u(x_l) = \int_0^1 K(x_l, t, u(t))\mathrm{d}t + f(x_l), \quad l = 1, 2, ..., 2M. \tag{8}$$

Substituting (7) into (8), we get an algebraic system of equations for evaluating the coefficients a_i. This system is in general nonlinear and some numerical procedure must be applied to solve it.

The following procedure is used to solve (8).

We assume that (8) is already solved for some value $J - 1$ to which there correspond $M = 2^J$ collocation points. For the next iteration the number of collocation points is doubled (the value J is increased by one). The new values for $u(t)$ and a_i are estimated as

$$\hat{u}^{(\nu)}(t) = \sum_{i=1}^{2M} \hat{a}_i^{(\nu)} h_i(t), \tag{9}$$

where

$$\hat{a}_i^{(\nu)} = \begin{cases} a_i^{(\nu-1)} & \text{for} \quad i = 1, ..., M, \\ 0 & \text{for} \quad i = M+1, ..., 2M. \end{cases} \tag{10}$$

The approximation (10) holds if the coefficients a_i for $i = M + 1, ..., 2M$ are relatively small. In all examples, solved in Section 5, this assumption was justified.

These estimates are corrected with the aid of the Newton method, which leads to the equation

$$\sum_{p=1}^{2M} \left[h_p(x_l) - \int_0^1 K_u(x_l, t, \hat{u}^{(\nu)}(t)) h_p(t)\mathrm{d}t \right] \Delta a_p$$
$$= -\hat{u}^{(\nu)}(x_l) + \int_0^1 K(x_l, t, \hat{u}^{(\nu)}(t))\mathrm{d}t + f(x_l), \quad l = 1, 2, ..., 2M, \tag{11}$$

where $K_u = \partial K / \partial u$ and $x_l = (l - 0.5)\Delta t$.

By solving (11) we find $\Delta a_p^{(\nu)}$, and the corrected values for (9), (10) are

$$a_p^{(\nu)} = \hat{a}_p^{(\nu)} + \Delta a_p^{(\nu)}, \quad u^{(\nu)}(t) = \sum_{p=1}^{2M} a_p^{(\nu)} h_p(t). \tag{12}$$

In all examples considered in Section 5 it was sufficient to apply the procedure (11) only once (the next step is carried out for a redoubled number of collocation points). If it turns out that at some level the corrections $|\Delta a_p^{(\nu)}|$ are too large, then the iteration by the Newton method must be repeated until

$$\max_p |\Delta a_p^{(\nu)}| < \eta, \tag{13}$$

where $\eta > 0$ is a small fixed parameter.

It is suitable to put (11) into the matrix form. For this purpose the symbols

$$\varphi(l) = \int_0^1 K(x_l, t, \hat{u}^{(\nu)}(t))\mathrm{d}t, \tag{14}$$

$$\chi(p, l) = \int_0^1 K_u(x_l, t, \hat{u}^{(\nu)}(t))h_p(t)\mathrm{d}t \tag{15}$$

are introduced.

If we interpret $a^{(\nu)}, \Delta a^{(\nu)}, u^{(\nu)}, x, f, \varphi$ as $2M$-dimensional row vectors, H, χ as $2M \times 2M$ matrices, and introduce the notations

$$S = H - \chi, \quad F = -u^{(\nu)} + \varphi + f, \tag{16}$$

then (11) obtains the form

$$\Delta a^{(\nu)} S = F, \tag{17}$$

whose solution is $\Delta a^{(\nu)} = F S^{-1}$.

It is often convenient to start from the one collocation point solution for which $u_1^{(0)} = a_1^{(0)} h_1(t) = a_1^{(0)}$; this constant is evaluated from the equation

$$a_1^{(0)} = \int_0^1 K(0.5, t, a_1^{(0)})\mathrm{d}t + f(0.5). \tag{18}$$

The estimates for the next step are $\hat{a}_1^{(1)} = a_1^{(0)},\ \hat{a}_2^{(1)} = 0$, and

$$\hat{u}^{(1)}(t) = \hat{a}_1^{(1)} h_1(t) + \hat{a}_2^{(1)} h_2(t) = \hat{a}_1^{(1)}.$$

These estimates are corrected by solving (11) for $M = 1$.

4. EVALUATION OF THE INTEGRALS

The main effort in applying the proposed method is evaluation of the integrals (14), (15). Let us consider the integral (14). It can be put into the form

$$\varphi(l) = \sum_{s=1}^{2M} \int_{\tau_s}^{\tau_{s+1}} K(x_l, t, \hat{u}^{(\nu)}(t))\mathrm{d}t, \tag{19}$$

where τ_s are the grid points defined by (4). It follows from (1) and (7) that $u^{(\nu)}(t) = u^{(\nu)}(t_s) = \text{const}$ in each segment $t \in [\tau_s, \tau_{s+1}]$. Here t_s denotes the sth collocation point.

Let G be an indefinite integral

$$G(x_l, t, u(t_s)) = \int K(x_l, t, u(t_s))\mathrm{d}t + \text{const}. \tag{20}$$

Equation (19) can be rewritten in the form

$$\varphi(l) = \sum_{s=1}^{2M} [G(x_l, \tau_{s+1}, \hat{u}^{(\nu)}(t_s)) - G(x_l, \tau_s, \hat{u}^{(\nu)}(t_s))]. \tag{21}$$

The integral (15) can be evaluated in a similar way. If we introduce the notation

$$G_u(x_l, t, u(t_s)) = \int K_u(x_l, t, u(t_s))\mathrm{d}t + \text{const}, \tag{22}$$

we obtain

$$\chi(p, l) = \sum_{s=1}^{2M} [G_u(x_l, \tau_{s+1}, \hat{u}^{(\nu)}(t_s)) - G_u(x_l, \tau_s, \hat{u}^{(\nu)}(t_s))]h_p(t_s). \tag{23}$$

To demonstrate the efficiency of the proposed solution, in the next section some numerical examples are solved. All calculations were carried out with the aid of MATLAB programs, which are very convenient in matrix representation.

To estimate the exactness of the achieved results, integral equations for which the exact solution u_{ex} is known are considered. The error function ε is taken in the form

$$\varepsilon = \max_{1 \le l \le 2M} (\mid u^{(\nu)}(x_l) - u_{\text{ex}}(x_l) \mid). \tag{24}$$

5. ILLUSTRATING EXAMPLES

Example 1. Solve the equation

$$u(x) = \frac{x}{20}\int_0^1 tu^2(t)\mathrm{d}t + 3 + 0.6625x, \quad (0 < x < 1). \tag{25}$$

The exact solution is $u_{\mathrm{ex}}(x) = 3 + x$.

It follows from (20) and (22):

$$G(x_l, t, u(t_s)) = \frac{1}{40}x_l t^2 u^2(t_s) + \mathrm{const},$$

$$G_u(x_l, t, u(t_s)) = \frac{1}{20}x_l t^2 u(t_s) + \mathrm{const}.$$

In view of (21) we obtain

$$\varphi(l) = \frac{1}{40}x_l\sum_{s=1}^{2M}(\tau_{s+1}^2-\tau_s^2)[\hat{u}^{(\nu)}(t_s)]^2 = \frac{1}{20}x_l\sum_{s=1}^{2M}\frac{\tau_s+\tau_{s+1}}{2}(\tau_{s+1}-\tau_s)[\hat{u}^{(\nu)}(t_s)]^2.$$

Since $t_s = 0.5(\tau_s + \tau_{s+1})$ and $\Delta t = \tau_{s+1} - \tau_s$, this result can be put into the form

$$\varphi(l) = \frac{1}{20}x_l\Delta t\sum_{s=1}^{2M} t_s[\hat{u}^{(\nu)}(t_s)]^2. \tag{26}$$

In a similar way we obtain

$$\chi(p, l) = \frac{1}{10}x_l\Delta t\sum_{s=1}^{2M} t_s\hat{u}^{(\nu)}(t_s)h_p(t_s). \tag{27}$$

According to (18) the one collocation point solution is $u_1^{(0)} = a_1^{(0)} = 3.483$. Estimates for the next approximation are $\hat{a}^{(1)} = \left(a_1^{(0)}, 0\right)$, $\hat{u}^{(1)} = \left(u_1^{(0)}, u_1^{(0)}\right)$. Solution of (17) gives $\Delta a_1^{(1)} = 0.0127$, $\Delta a_2^{(1)} = -0.248$, consequently, $a_1^{(1)} = 3.496$, $a_2^{(1)} = -0.248$. It follows from (7) that $u_1^{(1)} = 3.248$, $u_2^{(1)} = 3.743$. The error estimate (24) for this approximation is $\varepsilon = 0.006$. Errors of the subsequent approximations are given in Table 1. With the purpose of following the convergence speed the quantity $\rho = \varepsilon_{J-1}/\varepsilon_J$, where $J = 1, 2, ...$, is introduced (the value ε_0 corresponds to the one collocation point solution).

Table 1. Error estimates (24) and convergence speed ρ for Eq. (25)

J	$2M$	ϵ	ρ
1	4	2.3E-3	2.9
2	8	6.3E-4	3.7
3	16	1.6E-4	3.9
4	32	4.1E-5	3.9
5	64	1.0E-5	4.0
6	128	2.6E-6	4.0

Example 2. Let us consider the following weakly singular equation

$$u(x) = \int_0^1 \mid x - t \mid^{-1/2} u^2(t)\mathrm{d}t + f(x), \tag{28}$$

where $f(x) = [x(1-x)]^{1/2} + \frac{16}{15}x^{5/2} + 2x^2(1-x)^{1/2} + \frac{4}{3}x(1-x)^{3/2} + \frac{2}{5}(1-x)^{5/2} - \frac{4}{3}x^{3/2} - 2x(1-x)^{1/2} - \frac{2}{3}(1-x)^{3/2}$, $0 < x < 1$.

This problem was solved by Pedas and Vainikko [4], making use of the piecewise linear collocation method. The exact solution is

$$u_{\mathrm{ex}}(x) = \sqrt{x(1-x)}.$$

Let us make use of the Haar wavelet method. It follows from (20) that

$$G(x_l, t, u(t_s)) = 2[u(t_s)]^2 \begin{cases} -\sqrt{x_l - t} & \text{for} \quad x_l > t, \\ \sqrt{t - x_l} & \text{for} \quad x_l < t. \end{cases}$$

It is expedient to introduce the function

$$g(s,l) = \begin{cases} \sqrt{\tau_{s+1} - x_l} - \sqrt{\tau_s - x_l} & \text{for} \quad x_l \le \tau_s, \\ \sqrt{\tau_{s+1} - x_l} + \sqrt{x_l - \tau_s} & \text{for} \quad \tau_s < x_l < \tau_{s+1}, \\ \sqrt{x_l - \tau_s} - \sqrt{x_l - \tau_{s+1}} & \text{for} \quad \tau_{s+1} \le x_l. \end{cases} \tag{29}$$

Now (21) gets the form

$$\varphi(l) = 2\sum_{s=1}^{2M} [\hat{u}^{(\nu)}(t_s)]^2 g(s,l). \tag{30}$$

Carrying out analogical calculations with (23), we obtain

$$\chi(p,l) = 4\sum_{s=1}^{2M} \hat{u}^{(\nu)}(t_s) g(s,l) h_p(t_s). \tag{31}$$

Table 2. Error estimates $\varepsilon,\ \varepsilon_N$ and convergence rates $\rho,\ \rho_N$ for Eq. (28)

J	$2M$	ε	ρ	ε_N	ρ_N
1	4	2.0E-2	1.2	–	–
2	8	1.5E-2	1.3	6.9E-3	–
3	16	4.8E-3	3.1	1.6E-3	4.3
4	32	1.5E-3	3.2	2.3E-4	7.0
5	64	4.6E-4	3.3	2.9E-5	7.9
6	128	1.7E-4	2.7	7.1E-6	4.1

Starting the numerical calculations with one collocation point, we find $u_1^{(0)} = a_1^{(0)} = 0.4102$. The estimate for the second approximation is $\hat{a}^{(1)} = (a_1^{(0)}, 0), \hat{u}^{(1)} = (a_1^{(0)}, a_1^{(0)})$. It follows from (17) that $\Delta a^{(1)} = (-0.0015, 0)$ and, consequently, $a^{(1)} = (0.4087, 0),\ u^{(1)} = (0.4087, 0.4087)$. Since the exact values are $u_{\text{ex}}(0.25) = u_{\text{ex}}(0.75) = 0.4330$, the error is $\epsilon = 0.024$. The quantities ϵ and ρ for different values of J are indicated in Table 2. The symbols $\varepsilon_N,\ \rho_N$ denote the error and convergence speed found in [4]. It should be noted that the error functions ε and ε_N are not strictly comparable, since in [4] the collocation points are not uniformly distributed and in each subinterval linear approximation has been used.

Example 3. Solve the equation

$$u(x) = x\int_0^1 t\sqrt{u(t)}\mathrm{d}t + 2 - \frac{x}{3}(2\sqrt{2} - 1) - x^2, \tag{32}$$

which has the exact solution $u_{\text{ex}} = 2 - x^2$.

Evaluating the functions φ and χ, we find according to (21) and (23):

$$\varphi(l) = x_l\Delta t\sum_{s=1}^{2M} t_s\sqrt{\hat{u}^{(\nu)}(t_s)}, \tag{33}$$

$$\chi(p, l) = 0.5x_l\Delta t\sum_{s=1}^{2M}\frac{t_s h_p(t_s)}{\sqrt{\hat{u}^{(\nu)}(t_s)}}. \tag{34}$$

Satisfying (18), we get two solutions $a^{(0)} = 0.756,\ a_*^{(0)} = 1.253$. Let us consider the first solution. The approximation for $M = 1$ gives $a^{(1)} = (1.695, 0.246)$ and $u^{(1)} = (1.941, 1.448)$ with the error $\varepsilon = 0.01$. Error estimates for the following approximations are shown in Table 3. Practically the same results are achieved if we start with the value $a_*^{(0)} = 1.253$. Correcting it with the aid of (17), we get again $u_*^{(1)} = (1.941, 1.448)$ and, of course, the following approximations also coincide.

Table 3. Error estimates ε and convergence rates ρ for Eq. (32)

J	$2M$	ε	ρ
1	4	3.3E-3	3.0
2	8	2.7E-3	1.2
3	16	1.1E-3	2.5
4	32	3.7E-4	3.3
5	64	1.1E-4	3.3
6	128	3.1E-5	3.6

Example 4. Here the equation

$$u(x) = x^2 \int_0^1 \frac{t^2 \mathrm{d}t}{1+u^2(t)} + \left(\frac{1}{2} - \ln 2\right) x^2 + \sqrt{x}, \quad 0 < x < 1, \tag{35}$$

is solved. The exact solution is $u_{\mathrm{ex}} = \sqrt{x}$. The functions φ and χ take the form

$$\varphi(l) = -\frac{2}{3} x_l^2 \Delta t \sum_{s=1}^{2M} (3t_s^2 + 0.25\Delta t^2) \hat{u}^{(\nu)}(t_s) U_s^2 h_p(t_s), \tag{36}$$

$$\chi(p,l) = \frac{1}{3} x_l^2 \Delta t \sum_{s=1}^{2M} (3t_s^2 + 0.25\Delta t^2) U_s, \tag{37}$$

where $U_s = \left\{1 + [\hat{u}^{(\nu)}(t_s)]^2\right\}^{-1}$.

Satisfying (18), we get the equation

$$4a^{(0)} - \frac{1}{3+3[a^{(0)}]^2} + \ln 2 - \frac{1}{2} - 2\sqrt{2} = 0,$$

which has the solution $a^{(0)} = 0.714$.

Starting with the estimate $\hat{a}^{(1)} = (0.714, 0),\ \hat{u}^{(1)} = (0.714, 0.714)$, we find that the corrected values are $a^{(1)} = (0.684, -0.184),\ u^{(1)} = (0.500, 0.869)$ with the error $\varepsilon = 0.003$. Exactness of the approximations follows from Table 4.

Table 4. Error estimates ε and convergence rates ρ for Eq. (35)

J	$2M$	ε	ρ
1	4	9.6E-4	2.9
2	8	2.7E-4	3.5
3	16	7.2E-5	3.8
4	32	1.8E-5	3.9
5	64	4.7E-6	3.9
6	128	1.2E-6	4.0

6. CONCLUSIONS

The main benefits of the Haar wavelet method are sparse representation, fast transformation, and possibility of implementation of fast algorithms (especially if matrix representation is used). For this reason the accuracy of the obtained solutions is quite high even if the number of calculation points is small. This circumstance follows also from Tables $1-4$: even a four collocation point solution gives in most cases satisfactory precision. By increasing the number of collocation points the error of the solution rapidly decreases.

Approximation with the Haar wavelets is equivalent to the approximation with piecewise constant functions. If the functions K and f in (6) are sufficiently smooth, then the convergence rate for piecewise constant functions is $O(M^{-2})$; this result can be transferred also to the Haar wavelet approach. If the initial values are sufficiently good, the Newton method has also quadratic convergence. So it could be expected that in the case of our solution, by doubling the number of collocation points, the error function roughly decreases four times. This theoretical estimation in general holds also for the data in Tables $1-4$. For small values of J some "adaption" takes place due to the small number of calculation points, but for bigger values of J the convergence rate ρ is quite near to the theoretical value $\rho = 4$. Exceptional is only Example 5.2, where Eq. (28) is weakly singular (this fact reduces the convergence rate). Comparing the results of this example with the data of [4], we see that the results of [4] are somewhat more exact, but the fact that it is simpler supports our solution.

Solving nonlinear integral equations by the Haar wavelet method is favoured by the fact that we can take the initial solution for the Newton method in the form (10). Such a simple way is not applicable in piecewise constant approximation.

ACKNOWLEDGEMENT

Support from the Estonian Science Foundation under Grant ETF-5240 is gratefully acknowledged.

REFERENCES

1. Atkinson, K. E. A survey of numerical methods for solving nonlinear integral equations. *J. Integral Equations Appl.*, 1992, **4**, 15–40.
2. Atkinson, K. and Flores, J. The discrete collocation method for nonlinear integral equations. *IMA J. Numer. Anal.*, 1993, **13**, 195–213.
3. Kaneko, H., Noren, R. D. and Padilla, P. A. Superconvergence of the iterated collocation methods for Hammerstein equations. *J. Comput. Appl. Math.*, 1997, **80**, 335–349.
4. Pedas, A. and Vainikko, G. Superconvergence of piecewise polynomial collocations for nonlinear weakly singular integral equations. *J. Integral Equations Appl.*, 1997, **9**, 379–405.
5. Vainikko, G., Kivinukk, A. and Lippus, J. Fast solvers of integral equations of the second kind: wavelet methods. *J. Complexity*, 2005, **21**, 243–273.

6. Chen, W.-S. and Lin, W. Galerkin trigonometric wavelet methods for the natural boundary integral equations. *Appl. Math. Comput.*, 2005, **121**, 79–92.
7. Anioutine, A. P. and Kyurkchan, A. G. Application of wavelets technique to the integral equations of the method of auxiliary currents. *J. Quant. Spectrosc. Radiat. Transf.*, 2003, **79–80**, 495–508.
8. Sánchez-Ávila, C. Wavelet domain signal deconvolution with singularity-preserving regularization. *Math. Comput. Simul.*, 2003, **61**, 165–176.
9. Liang, X.-Z., Liu, M.-C. and Che, X.-J. Solving second kind integral equations by Galerkin methods with continuous orthogonal wavelets. *J. Comput. Appl. Math.*, 2001, **136**, 149–161.
10. Kaneko, H., Noren, R. D. and Novaprateep, B. Wavelet applications of the Petrov–Galerkin method for Hammerstein equations. *Appl. Numer. Math.*, 2003, **45**, 255–273.
11. Lepik, Ü. and Tamme, E. Application of the Haar wavelets for solution of linear integral equations. In *Dynamical Systems and Applications, Proceedings, 5–10 July 2004, Antalya, Turkey*. 2005, 395–407.
12. Maleknejad, K. and Mirzaee, F. Using rationalized Haar wavelet for solving linear integral equations. *Appl. Math. Comput.*, 2005, **160**, 579–587.
13. Chen, C. F. and Hsiao, C. H. Haar wavelet method for solving lumped and distributed-parameter systems. *IEE Proc. Control Theory Appl.*, 1997, **144**, 87–94.

Mittelineaarsete integraalvõrrandite lahendamine Haari lainikute meetodil

Ülo Lepik ja Enn Tamme

On välja töötatud Haari lainikutel baseeruv numbriline meetod mittelineaarsete Fredholmi integraalvõrrandite lahendamiseks. Meetodi efektiivsust on kontrollitud nelja konkreetse näite varal, mille puhul on täpne lahendus teada. Viimane asjaolu võimaldab hinnata saadud numbriliste resultaatide viga. Selgub, et vajalik täpsus on tagatud juba väikese võrgupunktide arvu korral.