

Generalization of superconnection in noncommutative geometry

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Abstract. We propose the notion of a \mathbb{Z}_N -connection, where $N \geq 2$, which can be viewed as a generalization of the notion of a \mathbb{Z}_2 -connection or superconnection. We use the algebraic approach to the theory of connections to give the definition of a \mathbb{Z}_N -connection and to explore its structure. It is well known that one of the basic structures of the algebraic approach to the theory of connections is a graded differential algebra with differential d satisfying $d^2 = 0$. In order to construct a \mathbb{Z}_N -generalization of a superconnection for any $N > 2$, we make use of a \mathbb{Z}_N -graded q -differential algebra, where q is a primitive N th root of unity, with N -differential d satisfying $d^N = 0$. The concept of a graded q -differential algebra arises naturally within the framework of noncommutative geometry and the use of this algebra in our construction involves the appearance of q -deformed structures such as graded q -commutator, graded q -Leibniz rule, and q -binomial coefficients. Particularly, if $N = 2$, $q = -1$, then the notion of a \mathbb{Z}_N -connection coincides with the notion of a superconnection. We define the curvature of a \mathbb{Z}_N -connection and prove that it satisfies the Bianchi identity.

Key words: superconnection, covariant derivative, graded differential algebra, graded q -differential algebra.

1. INTRODUCTION

The concept of a superconnection was proposed by Mathai and Quillen [1] (see also [2]) in the 1980s to represent the Thom class of a vector bundle by a differential form having a Gaussian shape. Later, Atiyah and Jeffrey [3] proposed the geometric approach to a topological quantum field theory on a four-dimensional manifold [4] based on the superconnection formalism. Assuming that a vector bundle $\pi : E \rightarrow M$ has a \mathbb{Z}_2 -graded structure, i.e. it is a superbundle, the total grading of an E -valued differential form can be defined as the sum of two

gradings, one of which comes from the \mathbb{Z}_2 -graded structure of the algebra of differential forms on a base manifold M and the other from a \mathbb{Z}_2 -graded structure of a superbundle E . A superconnection is a linear mapping of odd degree with respect to this total grading, behaving like a graded differentiation with respect to the multiplication by differential forms. Consequently, if we wish to generalize the notion of a superconnection to any integer $N > 2$, we must have a \mathbb{Z}_N -graded analogue of an algebra of differential forms, and assuming that a vector bundle has also a \mathbb{Z}_N -graded structure, we can elaborate a generalization of a superconnection following the scheme proposed by Mathai and Quillen. In the present paper we introduce the notion of a \mathbb{Z}_N -connection, where N is any integer satisfying $N \geq 2$, within the framework of an algebraic approach to the theory of connections. The first component of our construction is a \mathbb{Z}_N -graded q -differential algebra [5–8], where q is a primitive N th root of unity, denoted by \mathcal{B} . This algebra plays the role of an analogue of an algebra of differential forms. It should be mentioned that a differential d of \mathcal{B} satisfies $d^N = 0$. The second component is a \mathbb{Z}_N -graded left module \mathcal{E} over the subalgebra $\mathcal{A} \subset \mathcal{B}$ of the elements of grading zero of \mathcal{B} . From a geometric point of view, a module \mathcal{E} can be considered as an analogue of the space of sections of a \mathbb{Z}_N -graded vector bundle. Taking the tensor product $\mathcal{E}_{\mathcal{B}} = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}$, which can be viewed as an analogue of a space of \mathbb{Z}_N -graded vector bundle valued differential forms, and defining the \mathbb{Z}_N -graded structure on this product, we give the definition of a \mathbb{Z}_N -connection D in the spirit of Mathai and Quillen. We show that the N th power of a \mathbb{Z}_N -connection is the grading zero endomorphism of the left \mathcal{B} -module $\mathcal{E}_{\mathcal{B}}$, and we define the curvature F_D of a \mathbb{Z}_N -connection by $F_D = D^N$. It is proved that the curvature of a \mathbb{Z}_N -connection satisfies the Bianchi identity.

2. GRADED q -DIFFERENTIAL ALGEBRAS

In this section we describe a generalization of a graded differential algebra, which naturally arises in the framework of q -deformed structures. This generalization is called a graded q -differential algebra, where q is a primitive N th root of unity. We show that given a graded unital associative algebra over \mathbb{C} with element v satisfying $v^N = e$, where e is the identity element of this algebra, one can construct the graded q -differential algebra by means of a q -commutator.

Let $\mathcal{B} = \bigoplus_{k \in \mathbb{Z}} \mathcal{B}^k$ be an associative unital \mathbb{Z} -graded algebra over \mathbb{C} . We shall denote the identity element of \mathcal{B} by e and the grading of a homogeneous element $\omega \in \mathcal{B}$ by $|\omega|$, i.e. if $\omega \in \mathcal{B}^k$, then $|\omega| = k$. An algebra \mathcal{B} is said to be a graded q -differential algebra ([5,6]), where q is a primitive N th root of unity ($N \geq 2$), if it is endowed with a linear mapping $d : \mathcal{B}^k \rightarrow \mathcal{B}^{k+1}$ of degree 1 satisfying the graded q -Leibniz rule $d(\omega \omega') = d(\omega) \omega' + q^{|\omega|} \omega d(\omega')$, where $\omega, \omega' \in \mathcal{B}$, and $d^N(\omega) = 0$ for any $\omega \in \mathcal{B}$. A mapping d is called an N -differential of a graded q -differential algebra. It is easy to see that a graded q -differential algebra is a generalization of

the notion of a graded differential algebra, since a graded differential algebra is a particular case of a graded q -differential algebra if $N = 2$ and $q = -1$.

From the graded structure of an algebra \mathcal{B} it follows that the subspace $\mathcal{B}^0 \subset \mathcal{B}$ of elements of grading zero is the subalgebra of an algebra \mathcal{B} . The pair (\mathcal{B}, d) is said to be an N -differential calculus on a unital associative algebra \mathcal{A} if \mathcal{B} is a graded q -differential algebra with N -differential d and $\mathcal{A} = \mathcal{B}^0$. For any $k \in \mathbb{Z}$ the subspace \mathcal{B}^k of elements of grading k has the structure of a bimodule over the subalgebra \mathcal{B}^0 and a graded q -differential algebra can be viewed as an N -differential complex ([6])

$$\dots \xrightarrow{d} \mathcal{B}^{k-1} \xrightarrow{d} \mathcal{B}^k \xrightarrow{d} \mathcal{B}^{k+1} \xrightarrow{d} \dots,$$

with differential d satisfying the graded q -Leibniz rule. If \mathcal{B} is a \mathbb{Z} -graded q -differential algebra, then we can define the \mathbb{Z}_N -graded structure on an algebra \mathcal{B} by putting $\mathcal{B}^{\bar{p}} = \bigoplus_{i \in \mathbb{Z}} \mathcal{B}^{N i + p}$, where $p = 0, 1, 2, \dots, N-1$, and \bar{p} is the residue class of an integer p modulo N . Then $\mathcal{B} = \bigoplus_{p \in \mathbb{Z}_N} \mathcal{B}^p$. In what follows, if a graded structure of an algebra \mathcal{B} is concerned, we shall always mean the above-described \mathbb{Z}_N -graded structure of \mathcal{B} . Since all graded structures considered in this paper are \mathbb{Z}_N -graded structures, we always assume that the values of each index related to a graded structure are elements of \mathbb{Z}_N . If there is no confusion, we shall denote the values of indices by $0, 1, 2, \dots, N-1$ meaning the residue classes modulo N .

Let us now show that if a graded unital associative algebra contains an element v satisfying $v^N = e$, where e is the identity element of this algebra, then one equips this algebra with the N -differential satisfying the graded q -Leibniz rule, turning this algebra into a graded q -differential algebra. Let \mathcal{A} be an associative unital \mathbb{Z}_N -graded algebra over the complex numbers \mathbb{C} and $\mathcal{A}^k \subset \mathcal{A}$ be the subspace of homogeneous elements of a grading k . Given a complex number $q \neq 1$, one defines a q -commutator of two homogeneous elements $w, w' \in \mathcal{A}$ by the formula

$$[w, w']_q = ww' - q^{|w||w'|} w'w.$$

Using the associativity of an algebra \mathcal{A} and the property $|ww'| = |w| + |w'|$ of its graded structure, it is easy to show that for any homogeneous elements $w, w', w'' \in \mathcal{A}$ it holds that

$$[w, w'w'']_q = [w, w']_q w'' + q^{|w||w'|} w' [w, w'']_q. \quad (1)$$

Given an element v of grading 1, i.e. $v \in \mathcal{A}^1$, one can define the mapping $d_v : \mathcal{A}^k \rightarrow \mathcal{A}^{k+1}$ by the formula $d_v w = [v, w]_q$, $w \in \mathcal{A}^k$. It follows from the property of q -commutator (1) that d_v is the linear mapping of degree 1 satisfying the graded q -Leibniz rule $d_v(ww') = d_v(w)w' + q^{|w|} w d_v(w')$, where w, w' are homogeneous elements of \mathcal{A} . Let $[k]_q = 1 + q + q^2 + \dots + q^{k-1}$ and $[k]_q! = [1]_q [2]_q \dots [k]_q$.

Lemma 1. For any integer $k \geq 2$ the k th power of the mapping d_v can be written as follows:

$$d_v^k w = \sum_{i=0}^k p_i^{(k)} v^{k-i} w v^i,$$

where w is a homogeneous element of \mathcal{A} and

$$p_i^{(k)} = (-1)^i q^{|w|_i} \frac{[k]_q!}{[i]_q! [k-i]_q!} = (-1)^i q^{|w|_i} \begin{bmatrix} k \\ i \end{bmatrix}_q,$$

$$|w|_i = i|w| + \frac{i(i-1)}{2}.$$

The proof of this lemma is based on the following identities:

$$p_0^{(k)} = p_0^{(k+1)} = 1, \quad p_{k+1}^{(k+1)} = -q^{|w|+k} p_k^{(k)},$$

$$p_i^{(k+1)} = p_i^{(k)} - q^{|w|+k} p_{i-1}^{(k)}, \quad 1 \leq i \leq k.$$

Theorem 1. If N is an integer such that $N \geq 2$, q is a primitive N th root of unity, \mathcal{A} is a \mathbb{Z}_N -graded algebra containing an element v satisfying $v^N = e$, where e is the identity element of an algebra \mathcal{A} , then \mathcal{A} equipped with the linear mapping $d_v = [v, \]_q$ is a graded q -differential algebra with N -differential d_v , i.e. d_v satisfies the graded q -Leibniz rule and $d_v^N w = 0$ for any $w \in \mathcal{A}$.

Proof. It follows from Lemma 1 that if q is a primitive N th root of unity, then for any integer $l = 1, 2, \dots, N-1$ the coefficient $p_l^{(N)}$ contains the factor $[N]_q$ which vanishes in the case of q being a primitive N th root of unity. This implies $p_l^{(N)} = 0$. Thus $d_v^N(w) = v^N w + (-1)^N q^{|w|_N} w v^N$. Taking into account that $v^N = e$, we obtain $d_v^N(w) = (1 + (-1)^N q^{|w|_N}) w = \lambda w$. The coefficient $\lambda = 1 + (-1)^N q^{|w|_N}$ vanishes if q is a primitive N th root of unity. Indeed, if N is an odd number, then $1 - (q^N)^{(N-1)/2} = 0$. In the case of an even integer N we have $1 + (q^{N/2})^{N-1} = 1 + (-1)^{N-1} = 0$, and this ends the proof of the theorem.

For applications in differential geometry it is important to have a realization of a graded q -differential algebra as an algebra of analogues of differential forms on a geometric space. The proved theorem allows us to construct a graded q -differential algebra taking as a starting point a generalized Clifford algebra. The structure of a generalized Clifford algebra suggests that we shall get an analogue of an algebra of differential forms with an N -differential on a noncommutative space. Indeed, let us remind that a generalized Clifford algebra $\mathcal{C}_{p,N}$ is a unital associative algebra over \mathbb{C} generated by $\gamma_1, \gamma_2, \dots, \gamma_p$ which are subjected to the relations

$$\gamma_i \gamma_j = q^{\text{sg}(j-i)} \gamma_j \gamma_i, \quad \gamma_i^N = 1, \quad i, j = 1, 2, \dots, p, \quad (2)$$

where q is a primitive N th root of unity and $\text{sg}(x)$ is the usual sign function. The structure of a graded q -differential algebra in the case of the generalized Clifford algebra with two generators is studied in [9]. In this case the corresponding generalized Clifford algebra $\mathcal{C}_{2,N}$ can be interpreted as an algebra of polynomial functions on a reduced quantum plane. Let us denote by x, y the generators of the algebra in this case. The relations (2) take on the form $xy = qyx, x^N = y^N = 1$. The algebra $\mathcal{C}_{2,N}$ becomes a \mathbb{Z}_N -graded algebra if we assign the grading zero to the generator x , the grading 1 to the generator y and define the grading of any monomial made up of generators x, y as the sum of gradings of its factors. The differential d is defined by $dw = [y, w]_q, w \in \mathcal{C}_{2,N}$. Since $y^N = 1$, it follows from Theorem 1 that the algebra $\mathcal{C}_{2,N}$ is a graded q -differential algebra and d is its N -differential. We give this graded q -differential algebra and its N -differential d the following geometric interpretation: the subalgebra of polynomials of grading zero is the algebra of functions on a one-dimensional space with ‘‘coordinate’’ x , and the elements of higher gradings expressed in terms of ‘‘coordinate’’ x and its ‘‘differential’’ dx are the analogues of differential forms with exterior differential d . We have $dx = y\Delta_q x = y(x - qx)$. Since $d^k \neq 0$ for $k < N$, a differential k -form w may be expressed either by means of $(dx)^k$ or by means of $d^k x$, where

$$d^k x = \frac{[k]_q}{q^{k(k-1)/2}} (dx)^k x^{1-k}.$$

If $w = (dx)^k f(x)$, where $f(x)$ is a polynomial of grading zero, and $dw = (dx)^{k+1} \delta_x^{(k)}(f)$, then

$$\delta_x^{(k)}(f) = (\Delta_q x)^{-1} (q^{-k} f - q^k A(f)),$$

where A is the homomorphism of the algebra of polynomials of grading zero determined by $A(x) = qx$. The higher-order derivatives $\delta_x^{(k)}$ have the property

$$\delta_x^{(k)}(fg) = \delta_x^{(k)}(f)g + q^k A(f) \delta_x^{(0)}(g), \quad k = 0, 1, 2, \dots, N-1,$$

where $\delta_x^{(0)}(g) = \frac{\partial g}{\partial x} = (\Delta_q x)^{-1} (g - A(g))$ is the A -twisted derivative. A higher-order derivative $\delta_x^{(k)}$ can be expressed in terms of the derivative $\frac{\partial}{\partial x}$ as follows:

$$\delta_x^{(k)} = q^k \frac{\partial}{\partial x} + \frac{q^{-k} - q^k}{1 - q} x^{-1}.$$

The realization of a graded q -differential algebra as an algebra of analogues of differential forms on an ordinary (commutative) space is constructed in [10]. Let x_1, x_2, \dots, x_n be the coordinates of an n -dimensional space \mathbb{R}^n , $C^\infty(\mathbb{R}^n)$ be the algebra of smooth \mathbb{C} -valued functions, and dx_1, dx_2, \dots, dx_n be the differentials of the coordinates. Let $\mathcal{N} = \{1, 2, \dots, n\}$ be the set of integers, I be a subset of \mathcal{N} , and $|I|$ be the number of elements in I . Given any subset I of \mathcal{N} , i.e.

$I = \{i_1, i_2, \dots, i_k\} \subset \mathcal{N}$, $1 \leq i_1 < i_2 < \dots < i_k \leq n$, we associate to I the formal monomial dx_I , where $dx_I = dx_{i_1} dx_{i_2} \dots dx_{i_k}$ and $dx_\emptyset = 1$. Let $\Omega(\mathbb{R}^n)$ be the free left $C^\infty(\mathbb{R}^n)$ -module generated by all formal monomials dx_I . It is evident that $\Omega(\mathbb{R}^n)$ has a natural \mathbb{Z} -graded structure $\Omega(\mathbb{R}^n) = \bigoplus_k \Omega^k(\mathbb{R}^n)$, where $\Omega^k(\mathbb{R}^n)$ is the left $C^\infty(\mathbb{R}^n)$ -module freely generated by all dx_I , where I contains k elements. An element of the module $\Omega^k(\mathbb{R}^n)$ has the form

$$\omega = \sum_{I, |I|=k} f_I dx_I = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} f_{i_1 i_2 \dots i_k} dx_{i_1} dx_{i_2} \dots dx_{i_k}, \quad (3)$$

where $f_I = f_{i_1 i_2 \dots i_k} \in C^\infty(\mathbb{R}^n)$. Let us define the degree 1 linear operator $d : \Omega^k(\mathbb{R}^n) \rightarrow \Omega^{k+1}(\mathbb{R}^n)$ by the formula

$$d\omega = \sum_{J, |J|=k+1} g_J dx_J, \quad g_{j_1 j_2 \dots j_{k+1}} = \sum_{m=1}^{k+1} q^{m-1} \frac{\partial f_{j_1 j_2 \dots j_m \dots j_{k+1}}}{\partial x_{j_m}}, \quad (4)$$

where ω has the form (3) and $J = \{j_1, j_2, \dots, j_{k+1}\}$. It can be shown that $d^N \omega = 0$ for any $\omega \in \Omega(\mathbb{R}^n)$. Thus we have the N -differential complex

$$\dots \xrightarrow{d} \Omega^{k-1}(\mathbb{R}^n) \xrightarrow{d} \Omega^k(\mathbb{R}^n) \xrightarrow{d} \Omega^{k+1}(\mathbb{R}^n) \xrightarrow{d} \dots \quad (5)$$

We shall call (5) an N -differential de Rham complex. In order to define the structure of an algebra on the N -differential de Rham complex (5), we introduce the following notations: if I, J are two subsets of \mathcal{N} satisfying $I \cap J = \emptyset$, then we denote by $b(I, J)$ ($a(I, J)$) the number of pairs $(i, j) \in I \times J$ such that $i > j$ ($i < j$), and $c(I, J) = b(I, J) - a(I, J)$. It is easy to check that for any subsets I, J we have $b(I, J) = a(J, I)$, $a(I, J) + b(I, J) = |I||J|$ and $c(I, J) = -c(J, I)$. Let us define the multiplication on the left $C^\infty(\mathbb{R}^n)$ -module $\Omega(\mathbb{R}^n)$ by the following rules:

$$f dx_I = dx_I f, \quad dx_I dx_J = \begin{cases} 0 & \text{if } I \cap J \neq \emptyset, \\ q^{b(I, J)} dx_{I \cup J} & \text{if } I \cap J = \emptyset. \end{cases} \quad (6)$$

The left module $\Omega(\mathbb{R}^n)$ with the product defined by the rules (6) is a graded associative algebra, and it can be shown that the N -differential d defined by (4) satisfies the graded q -Leibniz rule with respect to this product, which implies that $\Omega(\mathbb{R}^n)$ is a graded q -differential algebra with N -differential d . We shall call an element of this algebra a differential form and d the N -exterior differential. It is evident that taking $q = -1$ in (4), (6), we get the classical algebra of differential forms with exterior differential d satisfying $d^2 = 0$. It follows from (6) that $dx_I dx_J = q^{c(I, J)} dx_J dx_I = q^{|I||J| - 2a(I, J)} dx_J dx_I$, and in the special case of $q = -1$ this commutation relation depends only on the gradings $|I|, |J|$. This leads to the supercommutativity of the algebra of differential forms in the classical case.

3. \mathbb{Z}_N -CONNECTION AND ITS CURVATURE

In this section we give the definition of a \mathbb{Z}_N -connection, the curvature of a \mathbb{Z}_N -connection, and prove the Bianchi identity. We also study the structure of a \mathbb{Z}_N -connection and show that a superconnection is a particular case of a \mathbb{Z}_N -connection for $N = 2$.

Let \mathcal{A} be a unital associative \mathbb{C} -algebra, (\mathcal{B}, d) be an N -differential calculus over \mathcal{A} , and $\mathcal{E} = \bigoplus_{k \in \mathbb{Z}_N} \mathcal{E}^k$ be a \mathbb{Z}_N -graded left \mathcal{A} -module. It is evident that \mathcal{E} has the structure of \mathbb{Z}_N -graded \mathbb{C} -vector space induced by a left \mathcal{A} -module structure if one defines $\alpha\xi = (\alpha e)\xi$, where $\alpha \in \mathbb{C}, \xi \in \mathcal{E}$, e is the identity element of \mathcal{A} . Let $\mathcal{E}_{\mathcal{B}} = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}$ be the tensor product $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}$ of the right \mathcal{A} -module \mathcal{B} and the left \mathcal{A} -module \mathcal{E} . A graded q -differential algebra \mathcal{B} can be viewed as a $(\mathcal{B}, \mathcal{B})$ -bimodule, which implies that the tensor product $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}$ has the structure of the left \mathcal{B} -module. Since an algebra \mathcal{B} can also be viewed as an $(\mathcal{A}, \mathcal{A})$ -bimodule, the tensor product $\mathcal{E}_{\mathcal{B}} = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}$ has also the left \mathcal{A} -module structure. It should be mentioned that $\mathcal{E}_{\mathcal{B}}$ has also the structure of \mathbb{C} -vector space which is the tensor product of \mathbb{C} -vector space structures of \mathcal{B} and \mathcal{E} .

Each factor in the tensor product $\mathcal{E}_{\mathcal{B}} = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}$ has the \mathbb{Z}_N -graded structure. Using these \mathbb{Z}_N -graded structures, one can construct a \mathbb{Z}_N -graded structure on the tensor product $\mathcal{E}_{\mathcal{B}}$ as follows: given two homogeneous elements $\omega \in \mathcal{B}, \xi \in \mathcal{E}$, one defines the total grading of the element $\omega \otimes_{\mathcal{A}} \xi \in \mathcal{E}_{\mathcal{B}}$ by $|\omega \otimes_{\mathcal{A}} \xi| = |\omega| + |\xi|$. Then

$$\mathcal{E}_{\mathcal{B}} = \bigoplus_{k \in \mathbb{Z}_N} \mathcal{E}_{\mathcal{B}}^k, \quad \mathcal{E}_{\mathcal{B}}^k = \bigoplus_{m+l=k} \mathcal{E}_{\mathcal{B}}^{m,l} = \bigoplus_{m+l=k} \mathcal{B}^m \otimes_{\mathcal{A}} \mathcal{E}^l,$$

where $k, l, m \in \mathbb{Z}_N$. If we consider the tensor product $\mathcal{E}_{\mathcal{B}}$ as the left \mathcal{B} -module, then multiplication by a homogeneous element $\omega \in \mathcal{B}$ of grading k maps an element $\xi \in \mathcal{E}_{\mathcal{B}}^{m,l}$ into the element $\omega\xi \in \mathcal{E}_{\mathcal{B}}^{m+k,l}$, i.e. $\mathcal{E}_{\mathcal{B}}^n \xrightarrow{\omega} \mathcal{E}_{\mathcal{B}}^{n+k}$. If we consider the tensor product $\mathcal{E}_{\mathcal{B}}$ as the left \mathcal{A} -module, then multiplication by any element $u \in \mathcal{A}$ preserves the \mathbb{Z}_N -graded structure of $\mathcal{E}_{\mathcal{B}}$. Consequently, if $m + l = k$, then $\mathcal{E}_{\mathcal{B}}^{m,l}$ is the left \mathcal{A} -submodule of a left \mathcal{A} -module $\mathcal{E}_{\mathcal{B}}^k$. Let us denote

$$\Gamma_{\mathcal{B}}(\mathcal{E}) = \bigoplus_l \mathcal{E}_{\mathcal{B}}^{0,l}, \quad \Omega_{\mathcal{B}}^k(\mathcal{E}) = \bigoplus_l \mathcal{E}_{\mathcal{B}}^{k,l}, \quad k \geq 1.$$

The \mathbb{Z}_N -graded left \mathcal{A} -module $\Gamma_{\mathcal{B}}(\mathcal{E})$ is isomorphic to a left \mathcal{A} -module \mathcal{E} . The corresponding isomorphism $\varrho : \mathcal{E} \rightarrow \Gamma_{\mathcal{B}}(\mathcal{E})$ is defined for any $\xi \in \mathcal{E}$ by $\varrho(\xi) = e \otimes_{\mathcal{A}} \xi \in \Gamma_{\mathcal{B}}(\mathcal{E})$, where e is the identity element of \mathcal{A} . It is worth mentioning that the isomorphism ϱ preserves the graded structures of the \mathcal{A} -modules \mathcal{E} and $\Gamma_{\mathcal{B}}(\mathcal{E})$, i.e. $\varrho : \mathcal{E}^k \rightarrow \mathcal{E}_{\mathcal{B}}^{0,k}$.

Let $\text{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$ be the space of endomorphisms of the vector space $\mathcal{E}_{\mathcal{B}}$, $\text{End}_{\mathcal{A}}(\mathcal{E}_{\mathcal{B}})$ be the space of endomorphisms of the left \mathcal{A} -module $\mathcal{E}_{\mathcal{B}}$, and $\text{End}_{\mathcal{B}}(\mathcal{E}_{\mathcal{B}})$ be the space of endomorphisms of the left \mathcal{B} -module $\mathcal{E}_{\mathcal{B}}$. Obviously, $\text{End}_{\mathcal{B}}(\mathcal{E}_{\mathcal{B}}) \subset \text{End}_{\mathcal{A}}(\mathcal{E}_{\mathcal{B}}) \subset \text{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$. The space $\text{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$ is a \mathbb{Z}_N -graded unital associative algebra if one takes the product $A \circ B$ of two endomorphisms of the space $\mathcal{E}_{\mathcal{B}}$

as an algebra multiplication. The \mathbb{Z}_N -graded structure of this algebra as well as the \mathbb{Z}_N -graded structures of the spaces $\text{End}_{\mathcal{A}}(\mathcal{E}_{\mathcal{B}}), \text{End}_{\mathcal{B}}(\mathcal{E}_{\mathcal{B}})$ are induced by the total \mathbb{Z}_N -graded structure of $\mathcal{E}_{\mathcal{B}}$. Thus we have the decomposition $\text{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}}) = \bigoplus_k \text{End}_{\mathbb{C}}^k(\mathcal{E}_{\mathcal{B}})$, where $\text{End}_{\mathbb{C}}^k(\mathcal{E}_{\mathcal{B}})$ is the space of homogeneous endomorphisms of grading k of the space $\mathcal{E}_{\mathcal{B}}$, and the similar decompositions for the spaces $\text{End}_{\mathcal{A}}(\mathcal{E}_{\mathcal{B}}), \text{End}_{\mathcal{B}}(\mathcal{E}_{\mathcal{B}})$. The structure of a graded associative algebra of $\text{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$ allows us to use the q -commutator $[A, B]_q = A \circ B - q^{|A||B|} B \circ A$, where $A, B \in \text{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$, and $|A|, |B|$ are the corresponding gradings.

Definition 1. A \mathbb{Z}_N -graded \mathcal{B} -connection on the \mathbb{Z}_N -graded left \mathcal{B} -module $\mathcal{E}_{\mathcal{B}}$ is an endomorphism D of degree 1 of the vector space $\mathcal{E}_{\mathcal{B}}$ satisfying the condition

$$D(\omega \xi) = d(\omega) \xi + q^{|\omega|} \omega D(\xi), \quad (7)$$

where $\omega \in \mathcal{B}$, $\xi \in \mathcal{E}_{\mathcal{B}}$, and d is the N -differential of a \mathbb{Z}_N -graded q -differential algebra \mathcal{B} .

Since we use the same graded q -differential algebra \mathcal{B} in many of our constructions, and in order to simplify the terminology, we shall call D a \mathbb{Z}_N -connection on the module $\mathcal{E}_{\mathcal{B}}$. Hence, a \mathbb{Z}_N -connection D can be viewed as an element of grading 1 of the \mathbb{Z}_N -graded algebra $\text{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$, i.e. $D \in \text{End}_{\mathbb{C}}^1(\mathcal{E}_{\mathcal{B}})$, and the behaviour of this element with respect to the structure of the left \mathcal{B} -module of $\mathcal{E}_{\mathcal{B}}$ is fixed by the condition (7).

We can extend a \mathbb{Z}_N -connection D to act on the \mathbb{Z}_N -graded algebra $\text{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$ in a way consistent with the graded q -Leibniz rule if we put

$$D(A) = [D, A]_q = D \circ A - q^{|A|} A \circ D, \quad A \in \text{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}}).$$

It is evident that $D : \text{End}_{\mathbb{C}}^k(\mathcal{E}_{\mathcal{B}}) \rightarrow \text{End}_{\mathbb{C}}^{k+1}(\mathcal{E}_{\mathcal{B}})$ and

$$D(AB) = D(A) \circ B + q^{|A|} A \circ D(B).$$

Proposition 1. For any homogeneous endomorphism A of the left \mathcal{B} -module $\mathcal{E}_{\mathcal{B}}$, homogeneous element ω of the algebra \mathcal{B} , and $\xi \in \mathcal{E}_{\mathcal{B}}$ it holds that

$$D(A)(\omega \xi) = (1 - q^{|A|}) d\omega A(\xi) + q^{|\omega|} \omega D(A)(\xi).$$

It follows from Proposition 1 that if A is a grading zero endomorphism of the left \mathcal{A} -module $\mathcal{E}_{\mathcal{B}}$, i.e. $A \in \text{End}_{\mathcal{A}}^0(\mathcal{E}_{\mathcal{B}})$, then $D(A)(u\xi) = u D(A)(\xi)$ for any $u \in \mathcal{A}$ and $\xi \in \mathcal{E}_{\mathcal{B}}$. Consequently, $D(A) \in \text{End}_{\mathcal{A}}^1(\mathcal{E}_{\mathcal{B}})$. Particularly, if $A \in \text{End}_{\mathcal{B}}^0(\mathcal{E}_{\mathcal{B}})$, then $D(A) \in \text{End}_{\mathcal{A}}^1(\mathcal{E}_{\mathcal{B}})$.

Proposition 2. For any \mathbb{Z}_N -connection D the N th power of an endomorphism $D \in \text{End}_{\mathbb{C}}^1(\mathcal{E}_{\mathcal{B}})$ is the grading zero endomorphism of the left \mathcal{B} -module $\mathcal{E}_{\mathcal{B}}$, i.e. $D^N \in \text{End}_{\mathbb{C}}^0(\mathcal{E}_{\mathcal{B}})$.

Proof. It suffices to show that for any homogeneous $\omega \in \mathcal{B}$ an endomorphism $D \in \text{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$ satisfies $D^N(\omega \xi) = \omega D^N(\xi)$, where $\xi \in \mathcal{E}_{\mathcal{B}}$. We can expand the k th power of an endomorphism D as follows:

$$D^k(\omega \xi) = \sum_{m=0}^k q^{m|\omega|} \left[\begin{matrix} k \\ m \end{matrix} \right]_q d^{k-m}(\omega) D^m(\xi). \quad (8)$$

Since d is the N -differential of an algebra \mathcal{B} which implies $d^N \omega = 0$, and $\left[\begin{matrix} N \\ m \end{matrix} \right]_q = 0$ for q being a primitive N th root of unity, where $1 \leq m \leq N-1$, the expansion (8) takes on the form

$$D^N(\omega \xi) = q^{N|\omega|} \omega D^N(\xi) = \omega D^N(\xi).$$

Definition 2. The curvature F_D of a \mathbb{Z}_N -connection D is the endomorphism D^N of grading zero of the left \mathbb{Z}_N -graded \mathcal{B} -module $\mathcal{E}_{\mathcal{B}}$, i.e. $F_D = D^N \in \text{End}_{\mathcal{B}}^0(\mathcal{E}_{\mathcal{B}})$.

Proposition 3. For any \mathbb{Z}_N -connection D the curvature F_D of this connection satisfies the Bianchi identity $D(F_D) = 0$.

Proof. We have $D(F_D) = [D, F_D]_q = D \circ F_D - F_D \circ D = D^{N+1} - D^{N+1} = 0$.

In order to understand better the structure of a \mathbb{Z}_N -connection, we shall need a notion of a covariant derivative. Let $(\mathfrak{M}, \mathfrak{d})$ be a differential calculus over an algebra \mathfrak{A} , i.e. \mathfrak{M} is a \mathfrak{A} -bimodule and $\mathfrak{d} : \mathfrak{A} \rightarrow \mathfrak{M}$ is a linear mapping satisfying the Leibniz rule $\mathfrak{d}(ab) = \mathfrak{d}(a)b + a\mathfrak{d}(b)$ for any $a, b \in \mathfrak{A}$. Let \mathfrak{F} be a left \mathfrak{A} -module. A linear mapping $\nabla : \mathfrak{F} \rightarrow \mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{F}$ is said to be a covariant derivative on \mathfrak{F} with respect to a differential calculus $(\mathfrak{M}, \mathfrak{d})$ if it satisfies

$$\nabla(\mathfrak{a}f) = \mathfrak{d}(\mathfrak{a}) \otimes_{\mathfrak{A}} f + \mathfrak{a}\nabla(f), \quad (9)$$

for any $\mathfrak{a} \in \mathfrak{A}, f \in \mathfrak{F}$.

Let us show that a \mathbb{Z}_N -connection D induces the covariant derivative. According to the definition of a \mathbb{Z}_N -connection, we have $D : \mathcal{E}_{\mathcal{B}}^k \rightarrow \mathcal{E}_{\mathcal{B}}^{k+1}$. The left \mathcal{A} -modules $\mathcal{E}_{\mathcal{B}}^k, \mathcal{E}_{\mathcal{B}}^{k+1}$ split into the direct sums

$$\begin{aligned} \mathcal{E}_{\mathcal{B}}^k &= \oplus_{m+l=k} \mathcal{E}_{\mathcal{B}}^{m,l} = \mathcal{E}_{\mathcal{B}}^{0,k} \oplus \mathcal{E}_{\mathcal{B}}^{1,k-1} \oplus \mathcal{E}_{\mathcal{B}}^{2,k-2} \oplus \dots \oplus \mathcal{E}_{\mathcal{B}}^{N-1,k+1}, \\ \mathcal{E}_{\mathcal{B}}^{k+1} &= \oplus_{m+l=k+1} \mathcal{E}_{\mathcal{B}}^{m,l} = \mathcal{E}_{\mathcal{B}}^{0,k+1} \oplus \mathcal{E}_{\mathcal{B}}^{1,k} \oplus \mathcal{E}_{\mathcal{B}}^{2,k-1} \oplus \dots \oplus \mathcal{E}_{\mathcal{B}}^{N-1,k+2}. \end{aligned}$$

Let $p_{i,j} : \mathcal{E}_{\mathcal{B}} \rightarrow \mathcal{E}_{\mathcal{B}}^{i,j}, p_i : \mathcal{E}_{\mathcal{B}} \rightarrow \Omega_{\mathcal{B}}^i(\mathcal{E}), \pi_k : \mathcal{B} \rightarrow \mathcal{B}^k, \rho_l : \mathcal{E} \rightarrow \mathcal{E}^l$ be the projections of the left \mathcal{A} -modules onto their \mathcal{A} -submodules. It is evident that each projection is the homomorphism of the corresponding left \mathcal{A} -modules, $p_{k,l} = \pi_k \otimes_{\mathcal{A}} \rho_l$ and

$$p_k = \sum_l p_{k,l}, \quad p_{k,l}(\omega \otimes_{\mathcal{A}} \xi) = \pi_k(\omega) \otimes_{\mathcal{A}} \rho_l(\xi), \quad \forall \omega \in \mathcal{B}, \xi \in \mathcal{E}.$$

The pair (\mathcal{B}^1, d) is the differential calculus over an algebra \mathcal{A} and \mathcal{E} is a left \mathcal{A} -module. Let us consider the linear mapping $\nabla_D : \mathcal{E} \rightarrow \Omega_{\mathcal{B}}^1(\mathcal{E})$ defined by the formula $\nabla_D = p_1 \circ D \circ \varrho$.

Proposition 4. *The linear mapping ∇_D is the covariant derivative on a left \mathcal{A} -module \mathcal{E} with respect to the differential calculus (\mathcal{B}^1, d) . The covariant derivative ∇_D preserves the \mathbb{Z}_N -graded structures of the left \mathcal{A} -modules \mathcal{E} and $\Omega_{\mathcal{B}}^1(\mathcal{E})$, i.e. $\nabla_D : \mathcal{E}^k \rightarrow \mathcal{E}_{\mathcal{B}}^{1,k}$.*

Proof. In order to show that ∇_D is the covariant derivative on a left \mathcal{A} -module \mathcal{E} , we check the covariant derivative condition (9). For any $u \in \mathcal{A}, \xi \in \mathcal{E}$ we have

$$\begin{aligned} \nabla_D(u\xi) &= p_1(D(\varrho(u\xi))) = p_1(D(u\varrho(\xi))) = p_1(du \varrho(\xi) + u D\varrho(\xi)) \\ &= p_1(du (e \otimes_{\mathcal{A}} \xi)) + u p_1(D\varrho(\xi)) = p_1(du \otimes_{\mathcal{A}} \xi) + u \nabla_D(\xi) \\ &= \sum_l p_{1,l}(du \otimes_{\mathcal{A}} \xi) + u \nabla_D(\xi) = \pi_1(du) \otimes_{\mathcal{A}} \sum_l \rho_l(\xi) + u \nabla_D(\xi) \\ &= du \otimes_{\mathcal{A}} \xi + u \nabla_D(\xi). \end{aligned}$$

The algebraic approach to a \mathbb{Z}_N -connection described in this section has a geometric realization on a vector bundle in the case of $N = 2, q = -1$ leading to the known notion of a superconnection. Let us consider a superbundle $E = E^+ \oplus E^-$ over a smooth n -dimensional manifold M^n . In this case let \mathcal{B} be the algebra of differential forms on a manifold M^n and d be the exterior differential of this algebra. It is evident that \mathcal{B} is a \mathbb{Z}_2 -graded algebra, where the grading of a homogeneous differential form is equal to its degree modulo 2. The exterior differential satisfies $d^2 = 0$ and we see that the algebra of differential forms on a finite-dimensional manifold is a special case of a graded q -differential algebra for $N = 2$ and $q = -1$. Let $\mathcal{B}^0 = \mathcal{A}$ be the algebra of smooth functions on a manifold M^n , and $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be the left \mathbb{Z}_2 -graded \mathcal{A} -module of smooth sections of a superbundle E . Then the tensor product $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}$ is the space of smooth differential forms on a manifold M^n with values in a superbundle E . The total grading of a homogeneous differential form with values in E is the sum of two gradings, where the first is determined by the graded structure of the algebra of differential forms and the second is determined by the graded structure of a superbundle E . The space $\text{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$ is the space of differential forms on a manifold M with the values in the superbundle $\text{Hom}(E, E)$. The q -commutator becomes the supercommutator if we take $q = -1$. Finally, the definition of a \mathbb{Z}_N -connection coincides in this special case with the definition of a superconnection as it is given in [2].

Let us construct an example of a \mathbb{Z}_N -connection. One can extend the N -differential of an algebra \mathcal{B} to act on the \mathbb{Z}_N -graded left \mathcal{B} -module $\mathcal{E}_{\mathcal{B}}$ in a way consistent with the graded q -Leibniz rule by putting $d(\omega \otimes_{\mathcal{A}} \xi) = d(\omega) \otimes_{\mathcal{A}} \xi$, where $\omega \in \mathcal{B}, \xi \in \mathcal{E}$. It is evident that $d \in \text{End}_{\mathbb{C}}^1(\mathcal{E}_{\mathcal{B}})$. Let L be an endomorphism of grading 1 of a left \mathcal{A} -module \mathcal{E} , i.e. $L \in \text{End}_{\mathcal{A}}^1(\mathcal{E})$. This endomorphism can be extended to the \mathcal{B} -endomorphism of the left \mathcal{B} -module $\mathcal{E}_{\mathcal{B}}$ in a way consistent with

the \mathbb{Z}_N -graded structure of \mathcal{E}_B by means of $L(\omega \otimes_{\mathcal{A}} \xi) = q^{|\omega|} \omega \otimes_{\mathcal{A}} L(\xi)$. Indeed, if $\zeta = \omega \otimes_{\mathcal{A}} \xi \in \mathcal{E}_B$, then

$$\begin{aligned} L(\theta\zeta) &= L(\theta(\omega \otimes_{\mathcal{A}} \xi)) = L((\theta\omega) \otimes_{\mathcal{A}} \xi) = q^{|\theta\omega|+|\xi|} (\theta\omega) \otimes_{\mathcal{A}} L(\xi) \\ &= q^{|\theta|} \theta (q^{|\omega|} \omega \otimes_{\mathcal{A}} L(\xi)) = q^{|\theta|} \theta L(\zeta). \end{aligned}$$

Obviously, $L \in \text{End}_{\mathbb{B}}^1(\mathcal{E}_B) \subset \text{End}_{\mathbb{C}}^1(\mathcal{E}_B)$. The endomorphism $D = d + L$ of grading 1 of the vector space \mathcal{E}_B is a \mathbb{Z}_N -connection. Indeed, for any $\omega \in \mathcal{B}, \zeta \in \mathcal{E}_B$ we have

$$\begin{aligned} D(\omega\zeta) &= (d + L)(\omega\zeta) = d(\omega\zeta) + L(\omega\zeta) \\ &= d(\omega)\zeta + q^{|\omega|} \omega d(\zeta) + q^{|\omega|} \omega L(\zeta) \\ &= d(\omega)\zeta + q^{|\omega|} \omega D(\zeta). \end{aligned}$$

If A is an endomorphism of a left \mathcal{A} -module \mathcal{E} , then we can decompose it into the homogeneous parts $A_{ij}, i, j \in \mathbb{Z}_N$ with respect to the \mathbb{Z}_N -grading of \mathcal{E} , where $A_{ij} : \mathcal{E}^j \rightarrow \mathcal{E}^i$. Then, for any element $\xi = \xi_0 + \xi_1 + \dots + \xi_{N-1} \in \mathcal{E}$ we have $A(\xi)_i = \sum_{j \in \mathbb{Z}_N} A_{ij}(\xi_j)$, where $A(\xi)_i$ is the component of grading $i \in \mathbb{Z}_N$ of the element $A(\xi)$. We can extend the action of homogeneous parts A_{ij} to the \mathbb{Z}_N -graded left \mathcal{B} -module \mathcal{E}_B . If we associate the $N \times N$ -matrix $(A_{ij}), i, j \in \mathbb{Z}_N$ to an endomorphism A , where the entries of the matrix are the homogeneous parts of A , then in the case of L we get

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & L_{0,N-1} \\ L_{1,0} & 0 & 0 & \dots & 0 & 0 \\ 0 & L_{2,1} & 0 & \dots & 0 & 0 \\ 0 & 0 & L_{3,2} & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & L_{N-1,N-2} & 0 \end{pmatrix}. \quad (10)$$

Let us denote by $\{d^m, L_1 L_2 \dots L_k\}$, where m, k are non-negative integers, the sum of all possible products made up of the mappings d, L_1, L_2, \dots, L_k , where each product contains m -times the differential d and k mappings L_1, L_2, \dots, L_k succeeding in the same order. For instance, for $k = 0$ we have $\{d^m, L_1 L_2 \dots L_k\} = d^m$, and for $m = 2, k = 1$ we have $\{d^2, L\} = d^2 L + d L d + L d^2$. The curvature of the \mathbb{Z}_N -connection $D = d + L$ can be written as follows:

$$F_D = D^N = \sum_{m+k=N} \{d^m, L^k\}.$$

Using the matrix associated to L , we obtain the $N \times N$ -matrix corresponding to the curvature F_D , where the entry $F_{D,ij}$ of this matrix can be written as follows:

$$F_{D,ij} = \sum_{m+k=N, i, j \in \mathbb{Z}_N} \{d^m, L_{i,i-1} L_{i-1,i-2} \dots L_{j+1,j}\}, \quad (11)$$

where m, k are non-negative integers running $m, k = 0, 1, \dots, N$, and each product in $\{d^m, L_{i,i-1} L_{i-1,i-2} \dots L_{j+1,j}\}$ contains k entries of the matrix associated to L , which means that $i - j = k$. For instance, if $N = 2$, then from (10) and (11) we obtain the matrix of a superconnection $D = d + L$ and the matrix of its curvature, which can be written in the standard notations of supergeometry $\mathcal{E}_{\bar{0}} = \mathcal{E}^+, \mathcal{E}_{\bar{1}} = \mathcal{E}^-, L^+ = L_{\bar{0}\bar{1}}, L^- = L_{\bar{1}\bar{0}}$ as follows:

$$L \rightarrow \begin{pmatrix} 0 & L^- \\ L^+ & 0 \end{pmatrix}, \quad F_D \rightarrow \begin{pmatrix} L^- L^+ & dL^- \\ dL^+ & L^+ L^- \end{pmatrix}.$$

The appearance of the terms generated by an endomorphism L is a peculiar property of the curvature of a \mathbb{Z}_N -connection. Just this property of the curvature of a superconnection makes it possible to construct the representative of the Thom class of a fibre bundle rapidly decreasing in a fibre direction. This property plays also an essential part in the Atiyah–Jeffrey geometric approach [3] to the Lagrangian of a topological field theory on a four-dimensional manifold [4]. If, following the algebraic scheme described in this paper, we construct a \mathbb{Z}_N -connection on a noncommutative space with the help of analogues of differential forms with exterior differential d satisfying $d^N = 0$, then the extra terms (11) of the curvature will enable us to construct an analogue of the representative of a Thom class rapidly decreasing along a fibre in the case of a noncommutative space. In turn, this representative could be taken as a starting point for an analogue of a topological field theory on a noncommutative space with BRST-like symmetry δ_{BRST} satisfying $\delta_{\text{BRST}}^N = 0$ up to gauge transformation. Since a topological field theory on a four-dimensional manifold is related to a supersymmetric Yang-Mills theory [11], we may expect that a noncommutative analogue of a topological field theory could be related to a field theory with a fractional supersymmetry [12].

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Superseostuse üldistus mittekommutatiivses geometrias

Viktor Abramov

On sisse toodud \mathbb{Z}_N -seostuse mõiste, mida võib vaadelda superseostuse ehk \mathbb{Z}_2 -seostuse üldistusena. Superseostuse mõiste on antud V. Mathai ja D. Quilteni artiklis [1], kus autorid on superseostust kasutanud vektorkonna topoloogiliste invariantide konstrueerimiseks. Hiljem on M. F. Atiyah ja L. Jeffrey [3] kasutanud superseostuste formalismi E. Witteni [4] topoloogilise kvantväljateooria geomeetrilise interpretatsiooni kirjeldamiseks. Superseostus tugineb oluliselt diferentsiaalvormide algebra ja supervektorkonna \mathbb{Z}_2 -gradeeringutele. \mathbb{Z}_N -seostus defineeritakse seostuste teooria algebraalise formalismi raames. Konstruktsioon baseerub \mathbb{Z}_N -gradeeringuga q -diferentsiaalalgebral \mathcal{B} N -diferentsiaaliga d , mis rahuldab tingimust $d^N = 0$ ([5,6]), ja \mathbb{Z}_N -gradeeringuga vasakpoolsel moodulil üle algebra \mathcal{B} gradeeringuga 0 elementide alamalgebra. On defineeritud \mathbb{Z}_N -seostuse kõverus ja tõestatud, et kõverus rahuldab Bianchi samasust.