# Languages recognized by two-sided automata of graphs 

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#### Abstract

We introduce two-sided automata defined by directed graphs and describe all languages recognized by these automata.


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Throughout the paper the word graph means a finite directed graph without multiple edges but possibly with loops. Graphs and various objects derived from them are essential tools that have been actively used in different branches of modern mathematics and theoretical computer science. It makes sense to consider new ways of defining classical finite state acceptors using graph labellings, and determine how properties of the acceptors depend on the properties of the original graph labelling. The present paper continues the investigation initiated in $\left[{ }^{1-3}\right]$, where the concept of a graph algebra has been used in order to define finite state automata. We introduce a common generalization of two earlier constructions of automata considered in the papers mentioned above and investigate properties of languages accepted by these automata.

We use standard concepts of automata and languages theory following [ ${ }^{4-7}$ ]. A language over an alphabet $X$ is a subset of the free monoid $X^{*}$ generated by $X$. Let $D=(V, E)$ be a graph, $\ell: X \rightarrow\{+,-\}$ and $f: X \rightarrow V$ any mappings, and let $T$ be a subset of $V \cup\{1\}$. The two-sided automaton $\operatorname{Atm}(D)=\operatorname{Atm}(D, T, f, \ell)$ of the graph $D$ is the finite state acceptor with
(DA1) the set of states $V \cup\{1\}$;
(DA2) the initial state 1 ;
(DA3) the set of terminal states $T$;
(DA4) the next-state function given by

$$
a \cdot x= \begin{cases}f(x) & \text { if } \ell(x)=+ \text { and }(a, f(x)) \in E, \text { or if } a=1, \\ a & \text { if } \ell(x)=-\operatorname{and}(f(x), a) \in E,\end{cases}
$$

for a state $a$ and $x \in X$.
Thus the edges of the transition diagram of $\operatorname{Atm}(D)$ are also edges or reversed edges of the graph $D$.

Our first main theorem describes all languages recognized by two-sided automata of graphs in terms of combinatorial properties satisfied by these languages.

Theorem 1. For every language $L$ over an alphabet $X$, the following conditions are equivalent:
(i) there exists a directed graph $D$ such that $L$ is recognized by a two-sided automaton of $D$;
(ii) there exist two disjoint subsets $X_{-}$and $X_{+}$of $X$ such that $X=X_{-} \dot{\cup} X_{+}$ and, for all $x \in X_{+}, y \in X_{-}, z \in X$, and $u, v \in X^{*}$, the following implications hold:
(a) zxu L implies $x u \in L$,
(b) $x u, z x v \in L$ implies $z x u \in L$,
(c) $z y u \in L$ implies $z y^{*} u \in L$,
(d) $z v, z y u \in L$ implies $z y v \in L$,
(a) $x y u \in L$ if and only if $y x u \in L$.

The second main theorem describes all languages recognized by two-sided automata of graphs in terms of regular expressions for their complements. Given a relation $G \subseteq X \times X$, put

$$
G^{-1}=\{(x, y) \mid(y, x) \in G\} .
$$

Theorem 2. For every language $L$ over an alphabet $X$, the following conditions are equivalent:
(i) there exists a directed graph $D$ such that $L$ is recognized by a two-sided automaton of $D$;
(ii) there exist a subset $X_{T}$ of $X$, disjoint subsets $X_{-}$and $X_{+}$of $X$, and relations $G_{1} \subseteq X_{+} \times X_{+}, G_{2} \subseteq X_{-} \times X_{-}$, and $G_{3} \subseteq X_{-} \times X_{+}$such that $X=X_{-} \dot{\cup} X_{+}$and the language $X^{+} \backslash L$ has the following regular expression:

$$
\begin{gather*}
\left(X_{N} \cap X_{-}\right) X_{-}^{*}+X^{*}\left(X_{N} \cap X_{+}\right) X_{-}^{*} \\
+\sum_{\left(x_{i}, x_{j}\right) \in G_{1} \cup G_{3}^{-1}} X^{*} x_{i} X_{-}^{*} x_{j} X^{*}+\sum_{\left(x_{i}, x_{j}\right) \in G_{2} \cup G_{3}} x_{i} X_{-}^{*} x_{j} X^{*}, \tag{1}
\end{gather*}
$$

where $X_{N}$ stands for $X \backslash X_{T}$.
The concept of two-sided automata of graphs provides a common generalization of two earlier constructions of automata introduced in $\left[{ }^{1-3}\right]$ and motivated by the study of graph algebras and their applications. Graph algebras have been investigated by many authors (see, for example, $\left[{ }^{8-15}\right]$ ) in relation to various problems of discrete mathematics and computer science. Let $D=(V, E)$ be a graph. The graph algebra $\operatorname{Alg}(D)$ associated with $D$ is the set $V \cup\{0\}$ equipped with multiplication defined by the rule

$$
x y= \begin{cases}x & \text { if } x, y \in V \text { and }(x, y) \in E, \\ 0 & \text { otherwise } .\end{cases}
$$

The languages recognized by these automata have interesting combinatorial properties (see $\left[{ }^{1,2}\right]$ ). Our main theorem strengthens and generalizes several earlier results of the papers cited above. A routine verification shows that the next-state function of the automaton $\operatorname{Atm}(D, T, f, \ell)$ can be defined by the graph algebra $\operatorname{Alg}\left(D^{\prime}\right)$, where $D^{\prime}=\left(V, E^{-1}\right)$, if we use the following condition equivalent to (DA4):
(DA5) for $a \in \operatorname{Alg}(D) \cup\{1\}, x \in X$,

$$
a \cdot x= \begin{cases}f(x) a & \text { if } \ell(x)=+ \\ a f(x) & \text { if } \ell(x)=-\end{cases}
$$

This means that $x$ acts as a left multiplication by the element $f(x)$ in $\operatorname{Alg}\left(D^{\prime}\right)$ if $\ell(x)=+$, and as a right multiplication by $f(x)$ if $\ell(x)=-$. It follows that our new definition embraces the automata involving only right multiplications (see [ ${ }^{1}$ ]) and the ones using only left multiplications (see $\left[{ }^{2}\right]$ ).

Note that a language $L$ is accepted by $\operatorname{Atm}(D, T, f, \ell)$ if and only if $L \backslash\{1\}$ is accepted by

$$
\operatorname{Atm}(D, T \backslash\{1\}, f, \ell) .
$$

As usual, $X^{+}=X^{*} \backslash\{1\}$ stands for the free semigroup generated by $X$. Since conditions (ii) of our theorems remain unchanged if we replace $L$ by $L \backslash\{1\}$, in both proofs we may assume that $L \subseteq X^{+}$.

Proof of Theorem 1. (i) $\Rightarrow$ (ii): Suppose that the language $L$ is recognized by the automaton $\operatorname{Atm}(D, T, f, \ell)$ of some graph $D=(V, E)$. Define

$$
\begin{aligned}
X_{+} & =\{x \in X \mid \ell(x)=+\}, \\
X_{-} & =\{x \in X \mid \ell(x)=-\},
\end{aligned}
$$

and consider arbitrary elements $x, y \in X$ with $\ell(x)=+$ and $\ell(y)=-$.
To verify implication (a), take any word $z x u$ in $L$, where $u \in X^{*}$ and $z \in X$. Since $1 \cdot z x u$ is defined and belongs to $T$, it follows that $(f(z), f(x)) \in E$. Besides, $f(z) \cdot x=f(x)$, because $\ell(x)=+$. Therefore $1 \cdot x u=f(x) \cdot u=(f(z) \cdot x) \cdot u=$ $1 \cdot z x u \in T$. Hence $x u \in L$, as required.

Now, pick arbitrary elements $x u$ and $z x v$ in $L$ such that $u, v \in X^{*}$ and $z \in X$. Since $1 \cdot z x v$ is defined, we get $(f(z), f(x)) \in E$. Therefore $1 \cdot z x u=$ $(f(z) \cdot f(x)) \cdot u=f(x) \cdot u=1 \cdot x u \in T$, and so $z x u \in L$. Thus implication (b) holds, too.

Next, we verify implication (c). To this end, take a word $z y u$ in $L$, where $u \in X^{*}$ and $z \in X$. Since $1 \cdot z y u$ is defined and $\ell(y)=-$, we get $(f(y), f(z)) \in E$. Hence $1 \cdot z y^{k} u=f(z) \cdot u=(f(z) \cdot f(y)) \cdot u=1 \cdot z y u \in T$, for all $k \in \mathbf{N}_{0}$. Therefore $z y^{*} u \in L$, as required.

Consider any elements $z v$ and $z y u$ in $L$, where $u, v \in X^{*}$ and $z \in X$. Clearly, zyu $\in L$ implies $(f(y), f(z)) \in E$, because $\ell(y)=-$. Hence $1 \cdot z y v=(f(z) f(y)) \cdot v=f(z) \cdot v=1 \cdot z v \in T$, and so $z y v \in L$. We see that (d) holds.

Finally, let us prove the equivalence (e). Take an arbitrary $u \in X^{*}$ such that $x y u \in L$. It follows that $(f(y), f(x)) \in E$. Implication (c) yields us $x u \in L$. Therefore $1 \cdot y x u=(f(y) \cdot f(x)) \cdot u=f(x) \cdot u=1 \cdot x u \in T$; whence $y x u \in L$. To verify the converse implication, suppose that $y x u \in L$. Hence $(f(y), f(x)) \in E$. Besides, $x u \in L$ in view of (a). Therefore $1 \cdot x y u=(f(x) \cdot f(y)) \cdot u=f(x) \cdot u=$ $1 \cdot x u \in T$. This means that $y x u \in L$. Thus the whole condition (ii) holds.
(ii) $\Rightarrow$ (i): Condition (ii) involves the sets $X_{+}$and $X_{-}$. Let us label $x$ by $\ell(x)=+$ if $x \in X_{+}$, and by $\ell(x)=-$ otherwise. Introduce a graph $G=(V, E)$ with the set $V=X$ of vertices, and the set $E=E_{1} \cup E_{2}$ of edges where

$$
\begin{aligned}
& E_{1}=\left\{(x, y) \in X \times X_{+} \mid x y w \in L \text { for some } w \in X^{*}\right\} \\
& E_{2}=\left\{(x, y) \in X_{-} \times X \mid y x w \in L \text { for some } w \in X^{*}\right\} .
\end{aligned}
$$

Let $f$ be the mapping from $X$ to $V$ defined by $f(x)=x$. Put $T=X \cap L$.
By induction on the length $n$ of a word $u$ we show that $u \in L$ if and only if $u$ is recognized by the automaton $\operatorname{Atm}(D, T, f, \ell)$. If $n=1$, then this follows immediately from the definition of $T$. Suppose that the claim is true for all words $v$ of length less than $n$, where $n>1$. Take an arbitrary word $u=x_{1} \cdots x_{n}$ in $L$. We claim that then $\operatorname{Atm}(D, T, f, \ell)$ accepts $u$.

First, consider the case where $x_{2} \in X_{+}$. Implication (a) shows that $x_{2} \cdots x_{n} \in L$, and by the induction hypothesis $x_{2} \cdots x_{n}$ is accepted by $\operatorname{Atm}(D)$.

Note that $\left(f\left(x_{1}\right), f\left(x_{2}\right)\right) \in E_{1}$ by the definition of $E_{1} \subseteq E$. It follows that $1 \cdot u$ $=f\left(x_{1}\right) \cdot x_{2} \cdots x_{n}=f\left(x_{2}\right) \cdots f\left(x_{n}\right)=1 \cdot x_{2} \cdots x_{n} \in T$. This means that the $\operatorname{Atm}(D)$ recognizes $u$.

Second, consider the case where $x_{2} \in X_{-}$. Then (c) shows that $x_{1} x_{3} \cdots x_{n} \in L$. By the induction hypothesis $x_{1} x_{3} \cdots x_{n}$ is recognized by $\operatorname{Atm}(D)$. By the definition of $E_{2}$, we get $\left(f\left(x_{2}\right), f\left(x_{1}\right)\right) \in E_{2}$. Therefore $1 \cdot u=$ $f\left(x_{1}\right) \cdot x_{2} \cdots x_{n}=f\left(x_{1}\right) \cdot x_{3} \cdots f\left(x_{n}\right)=1 \cdot x_{1} x_{3} \cdots x_{n} \in T$. Hence $\operatorname{Atm}(D)$ recognizes $u$, in all cases. Thus the language accepted by $\operatorname{Atm}(D)$ contains $L$.

Conversely, suppose that our word $u=x_{1} \cdots x_{n}$ is accepted by the automaton $\operatorname{Atm}(D, T, f, \ell)$. We are going to show that $u$ belongs to $L$.

First, consider the case where $\ell\left(x_{2}\right)=+$. Clearly, $\left(f\left(x_{1}\right), f\left(x_{2}\right)\right) \in E$. Hence we get

$$
\begin{aligned}
1 \cdot x_{2} \cdots x_{n} & =\left(f\left(x_{2}\right) \cdot f\left(x_{3}\right)\right) \cdots f\left(x_{n}\right) \\
& =\left(f\left(x_{1}\right) \cdot f\left(x_{2}\right)\right) \cdots f\left(x_{n}\right) \\
& =1 \cdot u \in T .
\end{aligned}
$$

By the induction hypothesis $x_{2} \cdots x_{n} \in L$. Since $x_{2} \in X_{+}$, we get $\left(f\left(x_{1}\right), f\left(x_{2}\right)\right) \in E_{1} \cup E_{2}$.

If $\left(f\left(x_{1}\right), f\left(x_{2}\right)\right) \in E_{1}$, then $x_{1} x_{2} w \in L$, for some $w \in X^{*}$. Therefore implication (b) gives us $x_{1} x_{2} \cdots x_{n} \in L$.

If, however, $\left(f\left(x_{1}\right), f\left(x_{2}\right)\right) \in E_{2}$, then $x_{1} \in X_{-}$and $x_{2} x_{1} w \in L$ for some $w \in X^{*}$. By the equivalence (e) we get $x_{1} x_{2} w \in L$, and (d) implies $x_{1} x_{2} \cdots x_{n} \in L$, again.

Second, suppose that $\ell\left(x_{2}\right)=-$. Then $\left(f\left(x_{2}\right), f\left(x_{1}\right)\right) \in E=E_{1} \cup E_{2}$. We get $1 \cdot x_{1} x_{3} \cdots x_{n}=\left(f\left(x_{1}\right) \cdot f\left(x_{3}\right)\right) \cdots f\left(x_{n}\right)=\left(f\left(x_{1}\right) \cdot f\left(x_{2}\right)\right) \cdots f\left(x_{n}\right)=1 \cdot u \in T$. By the induction hypothesis $x_{1} x_{3} \cdots x_{n} \in L$.

If $\left(f\left(x_{2}\right), f\left(x_{1}\right)\right) \in E_{2}$, then $x_{1} x_{2} w \in L$, for some $w \in X^{*}$. Hence implication (b) yields $x_{1} x_{2} \cdots x_{n} \in L$.

If, however, $\left(f\left(x_{2}\right), f\left(x_{1}\right)\right) \notin E_{2}$, then $\left(f\left(x_{2}\right), f\left(x_{1}\right)\right) \in E_{1}$. Therefore $x_{1} \in X_{-}$and we get $x_{2} x_{1} w \in L$, for some $w \in X^{*}$. Hence $x_{1} x_{2} w \in L$, in view of (e). Implication (d) gives us $x_{1} x_{2} \cdots x_{n} \in L$, again. Thus we have proved that $L$ is precisely the language recognized by $\operatorname{Atm}(D)$. This completes the proof.

Proof of Theorem 2. (i) $\Rightarrow$ (ii): Suppose that $L$ is recognized by the automaton $\operatorname{Atm}(D)$ induced by a graph $D=(V, E)$ with functions $\ell$ and $f$. Let us introduce the sets

$$
\begin{aligned}
X_{N} & =\{x \in X \mid f(x) \notin T\}, \\
X_{+} & =\{x \in X \mid \ell(x)=+\}, \\
X_{-} & =\{x \in X \mid \ell(x)=-\},
\end{aligned}
$$

and the relations

$$
\begin{aligned}
& G_{1}=\left\{(x, y) \in X_{+} \times X_{+} \mid(f(x), f(y)) \notin E\right\}, \\
& G_{2}=\left\{(x, y) \in X_{-} \times X_{-} \mid(f(y), f(x)) \notin E\right\}, \\
& G_{3}=\left\{(x, y) \in X_{-} \times X_{+} \mid(f(x), f(y)) \notin E\right\} .
\end{aligned}
$$

We claim that $X^{+} \backslash L$ is equal to the language defined by the regular expression (1). To prove one inclusion, take any element $u \in X^{+} \backslash L$, say $u=x_{1} \cdots x_{n}$, where $n \geq 1$ and $x_{1}, \ldots, x_{n} \in X$, and consider two possible cases.

Case 1: $1 \cdot u$ is defined. If $u \in X_{-}^{*}$, then $1 \cdot u=f\left(x_{1}\right) \notin T$. We have $u \in x_{1} X_{-}^{*} \subseteq\left(X_{-} \cap X_{N}\right) X_{-}^{*}$, and so $u$ belongs to the language (1), as required. Further, we may assume that some letters in $u$ are labelled by + . Let $x_{k}$ be the last letter in $u$ of this kind. We get $1 \cdot u=f\left(x_{k}\right) \notin T$, and hence $u \in X^{*} x_{k} X_{-}^{*} \subseteq X^{*}\left(X_{+} \cap X_{N}\right) X_{-}^{*}$. Therefore $u$ lies in the language defined by (1), again.

Case 2: $1 \cdot u$ is undefined. Let $j$ be the largest integer such that $1 \leq i \leq n$ and $1 \cdot x_{1} \cdots x_{j}$ is defined. This index $j$ always exists because $1 \cdot x_{1}=f\left(x_{1}\right)$.

Subcase 2.1: $x_{1} \cdots x_{j} \in X_{-}^{*}$. Then $1 \cdot x_{1} \cdots x_{j}=f\left(x_{1}\right)$. Since $1 \cdot u$ is undefined, it follows that $\left(f\left(x_{k}\right), f\left(x_{1}\right)\right) \notin E$ with $\ell\left(x_{k}\right)=-$, or $\left(f\left(x_{1}\right), f\left(x_{k}\right)\right) \notin$ $E$ with $\ell\left(x_{k}\right)=+$, for some $j<k \leq n$. Therefore $u \in x_{1} X_{-}^{*} x_{k} X^{*}$ with $\left(x_{1}, x_{k}\right) \in G_{2} \cup G_{3}$.

Subcase 2.2: Some letters in $x_{1} \cdots x_{j}$ are labelled by + . Let $x_{i}$ be the last letter in $x_{1} \cdots x_{j}$ labelled by + . We get $1 \cdot x_{1} \cdots x_{j}=1 \cdot x_{1} \cdots x_{i}=f\left(x_{i}\right)$. Since $1 \cdot u$ is undefined, it follows that there exists $j<k \leq n$ such that $\left(f\left(x_{k}\right), f\left(x_{i}\right)\right) \notin E$ with $\ell\left(x_{k}\right)=-$, or $\left(f\left(x_{i}\right), f\left(x_{k}\right)\right) \notin E$ with $\ell\left(x_{k}\right)=+$. Hence $u \in X^{*} x_{i} X_{-}^{*} x_{k} X^{*}$ with $\left(x_{i}, x_{k}\right) \in G_{3}^{-1} \cup G_{1}$.

Thus in all cases $u$ belongs to the language (1). This completes the proof of one inclusion.

To prove the reverse inclusion, consider an arbitrary element $u=x_{1} \cdots x_{n}$ of the language defined by (1). Obviously, $u \neq 1$. If $1 \cdot u$ is undefined, then $u \in X^{+} \backslash L$, and we are done. Further, we may assume that $1 \cdot u$ is defined.

If $u \in x X_{-}^{*} y X^{*}$, for some $(x, y) \in G_{2} \cup G_{3}$, then $1 \cdot u$ is undefined. Similarly, if $u \in X^{*} x X_{-}^{*} y X^{*}$, for $(x, y) \in G_{1} \cup G_{3}^{-1}$, then $1 \cdot u$ is undefined, too. Therefore we may exclude these two summands of (1) from further consideration: $u$ does not belong to them.

If $u \in\left(X_{-} \cap X_{N}\right) X_{-}^{*}$, or $u \in X^{*}\left(X_{+} \cap X_{N}\right) X_{-}^{*}$, then $1 \cdot u \notin T$, by the definition of $X_{-}$. Therefore $u \in X^{+} \backslash L$, again. Thus $X^{+} \backslash L$ coincides with the language given by the regular expression (1).
(ii) $\Rightarrow$ (i): Let $L$ be a language with the complement $X^{+} \backslash L$ defined by the regular expression (1). Introduce a graph $G=(V, E)$ with the set $V=X$ of vertices, and the set $E$ of edges consisting of all pairs $(x, y)$ which are not in $G_{1} \cup G_{2}^{-1} \cup G_{3}$. Let $f$ be the mapping from $X$ to $V$ defined by $f(x)=x$. Put $\ell(x)=+$ for all $x \in X_{+}$, and $\ell(x)=-$ for all $x \in X_{+}$. Set $T=V \backslash X_{N}$. Denote
by $R$ the language recognized by $\operatorname{Atm}(G, T, f, \ell)$. Since $1 \notin T$, we get $1 \notin R$, and so further we have to deal only with words in $X^{+}$. In order to show that $L=R$, we verify the equality $X^{+} \backslash L=X^{+} \backslash R$.

Take an arbitrary element $u=x_{1} \ldots x_{n} \in X^{+} \backslash R$, where $n \geq 1$ and $x_{1}, \ldots, x_{n} \in X$. We are going to show that $u \in X^{+} \backslash L$.

First, suppose that $1 \cdot u$ is defined. If $u \in X_{-}^{*}$, then $1 \cdot u=x_{1}$, and $x_{1} \notin T$ because $u \notin R$. We have $u \in x_{1} X_{-}^{*} \subseteq\left(X_{-} \cap X_{N}\right) X_{-}^{*}$, and so $u$ belongs to the language (1), as required. Further, we may assume that some letters in $u$ are labelled by + . Let $x_{k}$ be a + -labelled letter in $u$ of this kind. We get $1 \cdot u=x_{k} \notin T$, and so $u \in X^{*} x_{k} X_{-}^{*} \subseteq X^{*}\left(X_{+} \cap X_{N}\right) X_{-}^{*}$. Thus we see that $u \in X^{+} \backslash L$.

Second, suppose that $1 \cdot u$ is undefined. Let $i$ be the last index such that $1 \cdot\left(x_{1} \cdots x_{i}\right)=x_{i}$. If some letters on the right of $x_{i}$ in $u$ are labelled by + , then we consider the nearest letter $x_{i+k}$ be of this kind. Since $1 \cdot\left(x_{1} \cdots x_{i+k}\right)$ is undefined, we get $\left(x_{i}, x_{i+k}\right) \notin E$ or $\left(x_{i+j}, x_{i}\right) \notin E$, for some $j<k$. If, however, all letters on the right of $x_{i}$ in $u$ are labelled by - , we get $\left(x_{i+j}, x_{i}\right) \notin E$, for some $j>0$, because $1 \cdot u$ is undefined. Therefore, we always have $\left(x_{i}, x_{i^{\prime}}\right) \in G_{1} \cup G_{3}$, for some $i^{\prime}>i$, or $\left(x_{i^{\prime \prime}}, x_{i}\right) \in G_{2}^{-1} \cup G_{3}$, for some $i^{\prime \prime}>i$.

Assume that $\ell\left(x_{i}\right)=-$. Let $j$ be the index of the first occurrence of $x_{i}$ in $u$. We have $x_{j}=x_{i}$ and all letters between $x_{j}$ and $x_{i}$ are labelled by - , because $1 \cdot x_{1} \cdots x_{i}=x_{i}$. It easily follows that $j=1$, and so $\left(x_{1}, x_{i^{\prime}}\right) \in G_{3}$, for some $i^{\prime}>1$ or $\left(x_{i^{\prime \prime}}, x_{1}\right) \in G_{2}^{-1}$, for some $i^{\prime \prime}>1$. Hence $u \in x_{1} X_{-}^{*} y X^{*}$, for some $\left(x_{1}, y\right) \in G_{2} \cup G_{3}$.

Assume now that $\ell\left(x_{i}\right)=+$, then $\left(x_{i}, x_{i^{\prime}}\right) \in G_{1}$, for some $i^{\prime}>i$ or $\left(x_{i^{\prime \prime}}, x_{i}\right) \in G_{3}$, for some $i^{\prime \prime}>i$. Therefore we get $u \in X^{*} x_{i} X_{-}^{*} y X^{*}$, for some $(x, y) \in G_{1} \cup G_{3}^{-1}$. Thus $u$ is given by the regular expression (1) and so $u \in X^{+} \backslash L$. Thus we see that $u \in X^{+} \backslash L$ in any case.

We have proved that $X^{+} \backslash L \supseteq X^{+} \backslash R$. To verify the reverse inclusion, consider an arbitrary element $u=x_{1} \cdots x_{n}$ of the language $X^{+} \backslash L$ defined by (1).

First, suppose that $u \in x X_{-}^{*} y X^{*}$ with $(x, y) \in G_{2} \cup G_{3}$. If $(x, y) \in G_{2}$, then $\ell(y)=+$ and there is no edge $(x, y)$. If $(x, y) \in G_{3}$, then $\ell(y)=-$ and there is no edge $(y, x)$. In both cases $1 \cdot u$ is undefined.

Second, suppose that $u \in X^{*} x X_{-}^{*} y X^{*}$ with $(x, y) \in G_{1} \cup G_{3}^{-1}$. If $(x, y) \in G_{1}$, then $\ell(y)=+$ and there is no edge $(x, y)$. If $(x, y) \in G_{3}^{-1}$, then $\ell(y)=-$ and there is no edge $(y, x)$. It follows that $1 \cdot u$ is undefined in any case, and so $u$ is not accepted by $\operatorname{Atm}(D, T)$.

Third, suppose that $u \in\left(X_{-} \cap X_{N}\right) X_{-}^{*}$. Then $1 \cdot u=f\left(x_{1}\right) \notin T$, and therefore $u \notin R$.

Finally, assume that $u \in X^{*}\left(X_{+} \cap X_{N}\right) X_{-}^{*}$. If $1 \cdot u$ is defined, then $1 \cdot u=f\left(x_{k}\right) \notin T$, and so $u$ is not accepted by $\operatorname{Atm}(D, T)$. If, however, $1 \cdot u$ is undefined, then $u$ does not belong to $R$, either.

Thus $X^{+} \backslash L \subseteq X^{+} \backslash R$. This completes the proof.
Remark 3. Condition (ii) of Theorem 2 can be expressed in the equivalent form using a grammar generating the language $X^{+} \backslash L$. Indeed, the standard
method gives us a grammar which generates the language described by the regular expression of the form (1) (see, for example, the proof of Proposition 6.2.3 in $\left[{ }^{16}\right]$ ).

Remark 4. If one of the sets $X_{+}$or $X_{-}$is empty, then our proof shows that condition (ii) of Theorem 2 remains in force, and all summands of (1) involving empty sets vanish.

Corollary 5. It is decidable whether a regular language is recognizable by twosided automata of graphs.
Proof follows from condition (ii) of Theorem 2. Indeed, given a regular language $L$, we can find a finite automaton accepting it, and then find a regular expression for the language $X^{+} \backslash L$. After that, it remains to consider all subsets $X_{T}$ of $X$, partitions $X=X_{-} \dot{\cup} X_{+}$, and relations $G_{1} \subseteq X_{+} \times X, G_{2} \subseteq X_{-} \times X$, and use well-known algorithms to verify whether $X^{+} \backslash L$ is equal to the regular language defined by the regular expression (1).

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# Graafide abil defineeritud kahepoolsete automaatide poolt äratuntavad keeled 

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Käesolevas artiklis on väljas uuritud kahepoolseid automaate, mis defineeritakse orienteeritud graafide abil. Kirjeldatakse kõiki keeli, mis on nende automaatide abil äratuntavad.

