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Trichotomous noise: applications to stochastic transport

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Abstract. A three-level Markovian noise as a model of nonequilibrium fluctuations is presented and the effect of flatness of fluctuations on the noise-driven nonequilibrium dynamics of overdamped Brownian particles in nonlinear systems is considered. Examples of exactly soluble models of stochastic transport are given and the conditions of current reversals in ratchet systems are discussed.

Key words: nonequilibrium fluctuations, stochastic transport, current reversals.

1. INTRODUCTION

Over the past two decades, the behaviour of open systems depending on environmental fluctuations (noise) has received considerable attention. Some recently discovered phenomena, such as stochastic resonance [1], noise-induced transitions [2], noise-induced phase transitions in spatially extended systems [3,4], and stochastic transport in ratchets [5−8], have raised the idea that noise is able to induce order in nonlinear nonequilibrium systems.

Theoretical investigations indicate that noise-induced nonequilibrium effects are sensitive to noise flatness, that is, to the ratio of the fourth moment to the square of the second moment (see [9−16]). Although the flatness of fluctuations has obvious significance, its role has not received due attention. Therefore, the purpose of the present paper is twofold: first, to provide a compact review of a series of our earlier papers [13−18] in which coloured noises were modelled as trichotomous fluctuations and a number of exact results were obtained, and second, to discuss

– on the basis of the above-mentioned exact results – some novel phenomena in stochastic systems where the role of noise flatness as a control parameter is crucial. Here we report for the first time (i) that triple and quadruple current reversals, i.e. changes in the sign of the current while one control parameter is varied, found in [16–18] at the large flatness limit, are also present in the case of moderate flatnesses and (ii) that even as many as six current reversals versus correlation time occur in ratchets with a simple sawtooth potential.

A major virtue of the models with trichotomous noise is that they constitute a case admitting exact analytical solutions for some nonlinear stochastic problems, such as coloured-noise-induced transitions [13,15] and reversals of noise-induced flow [14,16]. Furthermore, it is remarkable that for trichotomous noises the flatness parameter φ, contrary to the cases of the Gaussian coloured noise ($\varphi = 3$) and the symmetric dichotomous noise ($\varphi = 1$), can have any value from 1 to ∞. This extra degree of freedom proves to be useful when modelling actual fluctuations.

2. MODEL WITH TRICHOTOMOUS NOISE

We extend the idea of dichotomous noise further to a symmetric three-level random telegraph process $Z(t)$ called the trichotomous process [13]. This is a random stationary Markovian process that consists of jumps between three values $z = a_0, 0, -a_0$. The jumps follow in time according to a Poisson process, while the values occur with the stationary probabilities

$$P_s(a_0) = P_s(-a_0) = q, \qquad P_s(0) = 1 - 2q, \tag{1}$$

so that the trichotomous process is a special case of the kangaroo process [9] with the switching rate ν. The transition probabilities between the states $z(t) = a_0, 0, -a_0$ can be obtained as follows:

$$\begin{gathered} P(\pm a_0, t+\tau \mid 0, t) = P(\pm a_0, t+\tau \mid \mp a_0, t) = q(1 - e^{-\nu\tau}), \\ P(0, t+\tau \mid \pm a_0, t) = (1-2q)(1 - e^{-\nu\tau}), \\ \tau > 0, \quad 0 < q < 1/2, \quad \nu > 0. \end{gathered} \tag{2}$$

One can also calculate the mean value $\langle Z(t)\rangle = 0$ and the correlation function $\langle Z(t), Z(t')\rangle = 2qa_0^2 e^{-\nu|t-t'|}$. The flatness parameter φ, indicating how long the noise level dwells on the state $z = 0$, is defined as $\varphi := \langle Z^4(t)\rangle / \langle Z^2(t)\rangle^2 = 1/2q$.

We will apply the trichotomous noise to systems described by one variable, i.e. to phenomenological kinetic equations of the type

$$\frac{dX}{dt} = h(x) + Z(t) + \xi(t), \tag{3}$$

which are stochastic differential equations, where h is a deterministic function. The thermal fluctuations $\xi(t)$ are modelled by a zero-mean Gaussian white noise with the correlation function $\langle \xi(t_1), \xi(t_2) \rangle = 2D\delta(t_1 - t_2)$, where $D = \text{const}$. For brevity, in what follows, we shall call D temperature. The joint probability density $P_n(x,t)$ for the position variable $x(t)$ and the fluctuation variable $z(t)$ satisfies the Fokker–Planck master equation

$$\frac{\partial}{\partial t} P_n(x,t) = -\frac{\partial}{\partial x}\left[\left(h(x) + z_n - D\frac{\partial}{\partial x}\right) P_n(x,t)\right] + \sum_m U_{nm} P_m(x,t), \quad (4)$$

where $P_n(x,t)$ denotes the probability density for the combined process (x, z_n, t); $n, m = 1, 2, 3;\ z_1 \equiv -a_0,\ z_2 \equiv 0,\ z_3 \equiv a_0$ and $U_{nm} = \nu[q + (1-3q)\delta_{n2} - \delta_{nm}]$.

We consider overdamped motion of Brownian particles in a one-dimensional spatially periodic potential $\tilde{V}(\tilde{x}) = \tilde{V}(\tilde{x} + \tilde{L})$ of the period $\tilde{L}$ and the barrier height $\tilde{V}_0 = \tilde{V}_{\max} - \tilde{V}_{\min}$. In what follows, we will use dimensionless units with $L = 1$ and $V_0 = 1$, and suppose that the potential $V(x)$ in the dynamical equation (3) is piecewise linear (simple sawtoothlike), so that the corresponding force is $h(x) = b := 1/d$ for $x \in (0, d)$ and $h(x) = -c := -1/(1-d)$ for $x \in (d, 1)$. Here the parameter $d \in (0, 1)$ determines the asymmetry of the potential, which is symmetric if $d = \frac{1}{2}$. So we can confine ourselves to the case $d \leq \frac{1}{2}$. The Fokker–Planck equation (4) has a unique solution if on the stationary probability density $P_n^s(x)$ are imposed the conditions of periodicity $P_n^s(x) = P_n^s(x+1)$ and of normalization $\int_0^1 P_2^s(x)dx = 1-2q$ with $\int_0^1 P_i^s(x)dx = q$, $i = 2, 3$, over the rescaled period interval $L = 1$. If we denote $P(x) = \sum_n P_n^s(x)$, Eqs. (3) and (4) with the imposed conditions will yield the following relation between the average of the particle velocity $\langle dX/dt \rangle$ and the current J:

$$\langle dX/dt \rangle = \int_0^1 h(x)P(x)dx = J. \quad (5)$$

3. THE CASE OF ZERO TEMPERATURE

In the case of zero temperature ($D = 0$) the following characteristic regions can be discerned for the noise amplitude a_0:

(i) There is no current if $0 < a_0 < c$, as there is a stationary stable point for any state n.

(ii) In the case of $c < a_0 < b$, there exists one stationary stable point for $z(t) = -a_0$, the motion to the left is switched off, and the current is positive.

(iii) In the case of $a_0 > b$ the stochastic process $Z(t)$ can, though need not, induce a reversal of the current (CR).

Let us consider the last case in some detail. An exact yet complex formula for the current J was derived by our group in [14] in which its properties were also investigated. In the phase space of the parameters φ, a_0 one can distinguish between four domains of qualitatively different shapes of the current $J(\nu)$, characterized also by sign reversals (see Fig. 1). The following three facts must be pointed out:

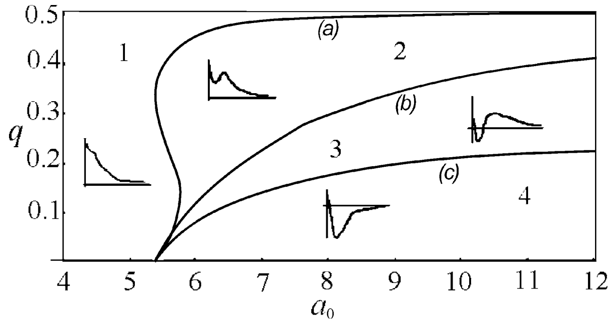

Fig. 1. The (q, a_0) phase diagram for the dependence of the stationary current J on ν in the case of $d = 0.25$. The shape of the function $J(\nu)$ for the different domains formed by curves (a)–(c) is sketched. Current reversals occur in domains Nos. 3 and 4. Curve (c) is determined by Eq. (8). Curves (a) and (b) are found by numerical methods.

(i) There is a lower limit for the noise amplitude, $a_0 = b + c$, below which no CR at any ν or φ occur.

(ii) The correlation time $\tau_c = 1/\nu$ has an upper limit $\tau_{c0} = 1/\nu_0$, over which there cannot be more than one CR.

(iii) The flatness parameter φ has a critical value $\varphi = 2$. If $\varphi < 2$, then, as the correlation time grows from 0 to ∞, there can be either two reversals or none, and if $\varphi > 2$, one reversal may but need not occur. In the general case, the critical switching rate ν_0 can be found from the following transcendental equation:

$$(\nu_0 + b^2)e^{-\nu_0/b^2} = (b^2 - c^2) + (\nu_0 + c^2)e^{-\nu_0/c^2}. \tag{6}$$

For the calculation of the current reversal points ν^*, $J(\nu^*) = 0$, the following transcendental equation can be applied:

$$F(\nu, a_0) := \gamma\eta(\alpha_1, -\beta_1) - qa_0(\eta\beta_2 - \gamma\alpha_2) = 0, \tag{7}$$

where

$$\alpha_1 = e^{\lambda_{1+}} + e^{\lambda_{1-}} - 2, \quad \alpha_2 = e^{\lambda_{1+}} - e^{\lambda_{1-}}, \quad \beta_1 = e^{\lambda_{2+}} + e^{\lambda_{2-}} - 2,$$

$$\beta_2 = e^{\lambda_{2+}} - e^{\lambda_{2-}}, \ \ \eta := \sqrt{(1-2q)b^2 + q^2a_0^2}, \ \ \gamma := \sqrt{(1-2q)c^2 + q^2a_0^2},$$

$$\lambda_{1\pm} = -\tfrac{\nu}{b(a_0^2 - b^2)}(qa_0^2 - b^2 \pm a_0\eta), \quad \lambda_{2\pm} = \tfrac{\nu}{c(a_0^2 - c^2)}(qa_0^2 - c^2 \pm a_0\gamma).$$

It is remarkable that the phase boundary line (c) (see Fig. 1) can be described by an exact analytical formula ($0 < q < \frac{1}{4}$)

$$a_0^2 = a_{0c}^2 = \frac{(1-2q)(b^2+c^2)}{1-4q}\left[1 + \sqrt{1 - \frac{(b^2-c^2)^2(1-4q)}{(1-2q)^2(b^2+c^2)^2}}\right]. \tag{8}$$

Curve (b), where $a_0 = a_2(q)$, is determined by the system of equations

$$F(\nu, a_2(q)) = 0, \qquad \frac{\partial}{\partial \nu} F(\nu, a_2(q)) = 0. \tag{9}$$

It should be noted that the ratchet model with dichotomous noise belongs to domain No. 1 as a limit case of $q = \frac{1}{2}$. In this case no CR occurs.

4. THE CASE OF NONZERO TEMPERATURE

Starting from a simple sawtooth potential, a complex exact formula as a quotient of two 11th-order determinants can be derived from Eq. (4) for the probability current J in the case of $D \neq 0$. The method of finding the current J in this case is presented in detail in [[16]], where a full analysis of the asymptotic regimes of J is performed. Here only the most interesting features will be outlined. It is remarkable that at $D \neq 0$ new cooperative effects occur between the statistically independent white and symmetric trichotomous noises, namely, multiple (more than two) current reversals and disjunct "windows" for the control parameters.

From the asymptotic expressions of J at $D \neq 0$ one can infer the following facts:

(i) If we vary the amplitude a_0, odd numbers of CRs occur.

(ii) If we vary the correlation time τ_c, the number of CRs is even or zero.

(iii) By changing the temperature D, we have to distinguish between two cases. First, if $a_0 < a_2(q)$, or if $a_0 > a_2(q)$ and $\nu < \nu_1^*$ or $\nu > \nu_2^*$, there can occur either an even number of CRs or none (see also Fig. 1); here the critical switching rates $\nu_2^* > \nu_1^*$ are the solutions of Eq. (6). Second, if $a_0 > a_2(q)$ and $\nu_1^* < \nu < \nu_2^*$, there are always odd numbers of CRs. Notably, at $q < \frac{1}{4}$ and $a_0 > a_{0c}$ the critical switching rate $\nu_2^* = \infty$. Numerical analysis of an exact expression for the current J has revealed that variation of the system parameters can bring forth more than two CRs. In particular, a change of τ_c can cause up to six, change of D up to four, and that of a_0 up to three CRs.

As has been said above, in some cases the current exhibits characteristic disjunct zones ("windows") of temperature and switching rate where the direction of the current is opposite to that in the surroundings (see Fig. 2). In the case of large flatness, $\varphi \gg 1$, the presence of the disjunct "windows" (DW) has been considered in [[17]] with some discussion on possible applications of the effect in particle separation techniques. Figure 2 exhibits zeros of the current, $J = J(D, \nu) = 0$, at the large-flatness limit, $\varphi \to \infty$, for $d = 0.005$ at different values of a_0. For certain values of (d, a_0) closed curves appear on the plane (D, ν) on which CRs occur. Inside these curves the direction of the current is negative, whereas outside them it is positive. The DWs occur if and only if the surface $J(D, a_0, \nu) = 0$ (with $d = \mathrm{const}$) has a local extremum and a saddle point. If the saddle and the extremum merge, the region of the existence of DWs shrinks to a critical point with

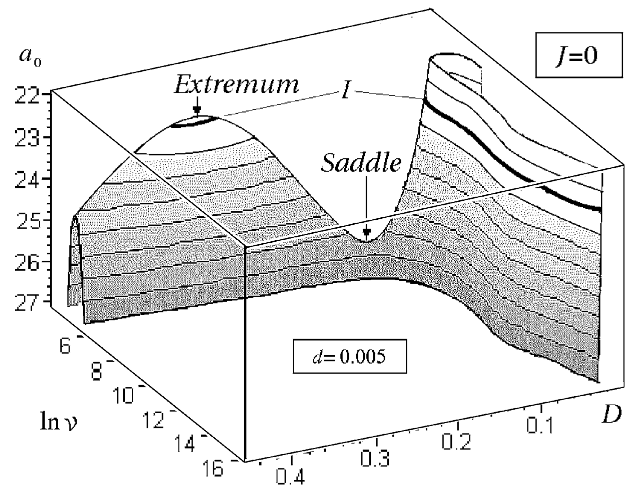

Fig. 2. The surface of current reversals, $J(D, a_0, \nu) = 0$, for a fixed asymmetry parameter $d = 0.005$ in the case of large flatness $\varphi \gg 1$. The extremum lies at $a_0 = 22.95$ and the saddle point at $a_0 = 26.223$. By the choice of a_0 a "window", as small as necessary, can be formed on the (D, ν)-plane around the extremum where the current is reversed as compared to the surrounding region.

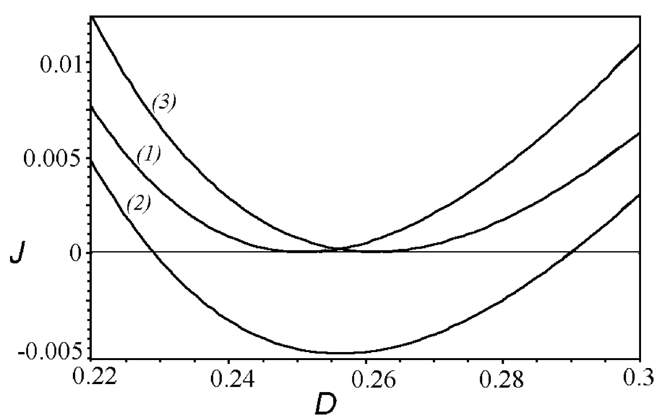

Fig. 3. Current vs temperature: two current reversals in the region of the disjunct "windows" on the plane (D, ν). For curves (1)–(3) $d = 0.007, q = 0.1, a_0 = 168.9$ and the switching rates are: $(1)\, \nu = 130$, $(2)\, \nu = 190$, $(3)\, \nu = 250$.

the following coordinates: $d_c \approx 0.009$, $a_c \approx 19.40$, $D_c \approx 0.250$ and $\ln \nu_c \approx 5.25$. The occurrence of DWs is possible if $d \in (0, d_c)$ and $a_0 \in (a_c, \infty)$. Attention should be called to the fact that in Fig. 2 the shapes of the plane curves reveal the occurrence of four CRs for J vs ν at certain values of the temperature D. Namely, for the curves I there are five critical temperatures: $D_1 \approx 0.334$, $D_2 \approx 0.280$, $D_3 \approx 0.062$, $D_4 \approx 0.058$, $D_5 \approx 0.050$; within the values $D > D_1$ and

$D_3 < D < D_2$ there is no CR; within $D_2 < D < D_1$, $D_4 < D < D_3$ and $D < D_5$ there are two CRs; within $D_5 < D < D_4$ there are just four CRs.

It is of importance that two effects, DWs and four CRs, both appear also at moderate values of the flatness parameter, $\varphi > 2$, as was ascertained by numerical analysis of an exact expression of J (the method of finding the current J is presented in [16]). Some curves $J = J(D)$ for the flatness $\varphi = 5$ illustrating the 2-CR effect in the region of the DWs are shown in Fig. 3. We believe that DWs will be useful for particle separation techniques. Comparing two possible techniques of particle separation – one with two CRs (with no use of DWs) [5,17–23] and the other with DWs – one can see certain advantages of the latter, because it enables one to obtain a sharp negative extremum of $J(\nu)$ with a relatively large absolute value. Another advantage of this model is that the control parameter is temperature, which is convenient for technology.

5. DISCUSSION

A three-level Markovian noise different from ours has been applied to investigate the reversals of noise-induced flow in [10–12], where most of the results have been obtained for the limits of slow and fast noises. The three-level process applied in [10,11] coincides with our trichotomous process at the limit of large flatness, $\varphi \gg 1$. However, variation of the flatness of the trichotomous noise can cause systems to behave in unexpected ways. The authors of [22,24] have investigated a correlation ratchet in which directed transport is subjected to both thermal equilibrium noise and zero-mean asymmetric dichotomous fluctuations. They show that the transport direction of Brownian particles can be controlled by thermal noise, but not in the case of symmetric dichotomous fluctuations. It is remarkable that such control is feasible in the case of a symmetric trichotomous noise. Moreover, for certain system parameters more than two CRs occur. For example, in the vicinity of the system parameters' phase space point $(d = 10^{-5},\ D = 0.0927,\ a_0 = 17240,\ \varphi = 757.6)$ six current reversals vs correlation time τ_c can be seen. To our knowledge, so many CRs have never been reported yet for correlation ratchets of simple sawtooth potentials. However, in the case of deterministic rocking ratchets the occurrence of infinitely many CRs is known [25]. Though we are not aware of any simple physical explanation for the above-mentioned effects, the distinct behaviours of the currents induced by dichotomous and trichotomous noises are not surprising if we remember that there is a so-called flashing barrier effect at $\varphi > 1$, which generates a counter current induced by dichotomous noise. An excellent explanation for the flashing barrier effect can be found in [10].

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Trihhotoomne müra: rakendused stohhastilise transpordi juures

Romi Mankin, Ain Ainsaar ja Risto Tammelo

Käsitletud on kolmetasemelist Markovi müra kui mittetasakaaluliste fluktuatsioonide mudelit. Mittelineaarsetes süsteemides on vaadeldud müra tasasusparameetri mõju mittetasakaaluliste fluktuatsioonide poolt käitatud Browni osakeste dünaamikale. On vaadeldud, millistel tingimustel esinevad hammaslatt-mudelites voolupöörded.