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Haar wavelets based technique in evolution problems

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Abstract. The compression property of wavelets in the analysis of an evolution problem (with unsmooth initial conditions) is investigated. The effectiveness of wavelets both in the reduction of complexity (number of coefficients) and in better approximation is shown. Haar wavelets, having the simplest interpretation of the wavelet coefficients, are used for defining the wavelet solution of an evolution (parabolic-hyperbolic) problem. The approximate solution, at a given fixed scale (resolution), results from the superimposition of (a small set of) fundamental wavelets, thus giving (also) a physical interpretation to wavelets. Since Haar wavelets are not smooth enough, a numerical derivative algorithm, which allows the scale approximation of partial differential evolution operators, is also defined. As application, the heat propagation (of an initial square wave) is explicitly given in terms of wavelets.

Key words: wavelet, Haar, Haar transform, interpolation, differential operator, discrete operator.

1. INTRODUCTION

Let us consider the (one-dimensional) evolution problem, in the unknown function $u(x,t) : (\mathbb{R} \subset L^2(\mathbb{R})) \times [0,T] \to \mathbb{R}$, defined by the partial differential equation

$$\begin{cases} \left(a\dfrac{\partial^2}{\partial t^2} + b\dfrac{\partial}{\partial t} + c\right) u(x,t) = Lu(x,t), \\ \qquad\qquad (x \in \mathbb{R}, t \in (0,T], T \leq \infty,\ a,b,c \in \mathbb{R}) \\ L \equiv \displaystyle\sum_{j=1}^{p} \lambda_j \frac{\partial^j}{\partial x^j}, \\ \qquad\qquad (p < \infty,\ \lambda_j = \text{const.}, j = 1, \ldots, p)\ , \end{cases} \tag{1}$$

and by the initial conditions

$$\left\{\begin{array}{lcl} u(x,0) & = & u_0(x) , \\ \dfrac{\partial}{\partial t}\, u(x,t)\Big|_{t=0} & = & u_0'(x) , \qquad (\text{only if } a \neq 0) \ . \end{array}\right. \tag{2}$$

The solution of the problem (1)–(2) might be obtained by a series expansion in terms of evolving coefficients $\beta_j(t)$ with respect to an orthonormal (fixed) basis of functions $\{p_j(x)\}$:

$$u(x,t) = \sum_{j=0}^{\infty} \beta_j(t) p_j(x) \ . \tag{3}$$

At a given approximation the corresponding truncated series, inserted in (1), gives rise, by using the Petrov–Galerkin method, to a finite set of ordinary differential equations in the unknown coefficients $\beta_j(t)$. The number of the unvanishing functions $\beta_j(t)$ depends on the approximation method, on the basis functions, and namely on the function to be reconstructed. Usually, trigonometric functions (such as for the Fourier series expansion) or some other orthogonal functions are taken as bases $\{p_j(x)\}$, but in order to give a sufficiently “good" approximation of the initial function $u_0(x)$, the number of significant coefficients might increase considerably.

Quite recently many successful attempts to solve differential equations have been made by using regular bases of wavelets [1−3], mostly Daubechies wavelets or interpolating wavelets [4,5]. However, one of the main properties of wavelets, the compression, was not significantly emphasized in the construction of the wavelet solutions. The compression, indeed, features the smallest number of coefficients, which, according to the wavelet theory [4,5], are needed to reconstruct completely any $L^2(\mathbb{R})$ function.

In the following we shall consider the Haar wavelets [5] which are piecewise constant finitely defined functions, each one with compact support on a finite interval. Since they are not differentiable functions, in order to compute explicitly the connection coefficients, the derivative of the Haar wavelets is found by a numerical algorithm [6,7], through a suitable smooth interpolation of points. Namely, the accuracy would improve by choosing more appropriate interpolating functions, but, even if some rough estimates of the error are given, the main purpose of this paper is not the investigation of the error. It will be shown, instead, that also the simple Haar wavelets can be used to describe, at a given approximation, evolution problems, and, moreover, their use might strongly reduce the number of unknown functions. It follows that the differential operator L is transformed into a discrete operator $\mathcal{L}$ which maps piecewise constant functions (Haar series) into piecewise constant functions.

Haar wavelets will be used for the definition of the wavelet solution [8] of the evolution problem (1)–(2), but their use, instead of some other smooth wavelets [5], is justified not only by the simplest and closed definition. In fact, dealing with experimental problems, only an initial discrete set of numerical data, observed at some fixed time intervals (giving histograms), is known. Thus the initial function $u_0(x)$ is a histogram that can be completely represented by a Haar series having the smallest number of coefficients (in comparison with other wavelet families). In general, other wavelets, such as the interpolating wavelets and the Daubechies families (except for D2), are smooth functions which allow an easy representation of smooth functions but are unsuitable, at least in principle, for the representation (at a fixed scale approximation $M < \infty$) of functions with finite jumps.

With the segmentation of the discrete Haar wavelet transform algorithm, shortly called *reduced Haar transform* [6], it is possible also to reduce further the number of basis functions and to keep unchanged the piecewise constant interpolation of the Haar series, thus, to reduce the complexity of the Haar wavelet transform. An application in the heat propagation theory is given in order to compute explicitly the Haar wavelet solution and to show the very low number of coefficients due to compression.

2. DISCRETE HAAR TRANSFORM

Let $\boldsymbol{Y} \equiv \{Y_i\}$, $(i = 0, \ldots, 2^M - 1,\ 2^M = N < \infty,\ M \in \mathbb{N})$, be a real and square summable time series $\boldsymbol{Y} \in \mathbb{K}^N \subset l^2$, with $\mathbb{K}$ real field; $x_i = i/(2^M - 1)$, the regular equispaced grid of *dyadic points* on the interval $\Omega = [0, 1]$. The *Haar scaling function* $\varphi(x)$ is the characteristic function on $[0, 1]$; its family of translated and dilated scaling functions is defined (in $[0, 1]$) as

$$\begin{cases} \varphi_k^n(x) \equiv 2^{n/2}\varphi(2^n x - k), & (0 \le n,\ 0 \le k \le 2^n - 1), \\ \varphi(2^n x - k) = \begin{cases} 1, & x \in \Omega_k^n, \qquad \Omega_k^n \equiv \left[\dfrac{k}{2^n}, \dfrac{k+1}{2^n}\right), \\ 0, & x \notin \Omega_k^n. \end{cases} \end{cases} \tag{4}$$

The *Haar wavelet* family $\{\psi_k^n(x)\}$ is the orthonormal basis for the $L^2([0, 1])$ functions [4,5]:

$$\begin{cases} \psi_k^n(x) \equiv 2^{n/2}\psi(2^n x - k), \qquad ||\psi_k^n(x)||_{L^2} = 1, \\ \psi_k^n(x) = \begin{cases} -2^{-n/2}, & x \in \left[\dfrac{k}{2^n}, \dfrac{k+1/2}{2^n}\right), \\ 2^{-n/2}, & x \in \left[\dfrac{k+1/2}{2^n}, \dfrac{k+1}{2^n}\right),\ (0 \le n\ ,\ 0 \le k \le 2^n - 1), \\ 0, & \text{elsewhere}\ , \end{cases} \end{cases} \tag{5}$$

which fulfills the recursive equations

$$\begin{cases} 2^{n+1/2}\,\varphi_k^n(x) &= \varphi_k^{n+1}(x)+\varphi_{k+1}^{n+1}(x), \\ 2^{n+1/2}\,\psi_k^n(x) &= \varphi_k^{n+1}(x)-\varphi_{k+1}^{n+1}(x). \end{cases} \tag{6}$$

Although, without loss of generality, we restrict ourselves to $0 \le n,\ 0 \le k \le 2^n - 1 \Longrightarrow \Omega_k^n \subseteq [0,1]$, the family of the Haar scaling functions and wavelets is defined also outside $[0,1]$, for other integer values of k, making it possible to extend the following considerations to any interval of $\mathbb{R}$.

The *discrete Haar wavelet transform* is the operator $\mathcal{W}^N : \mathbb{K}^N \subset l^2 \to \mathbb{K}^N \subset l^2$ which associates with a given finite energy vector $\boldsymbol{Y}$ the finite energy vector of the *wavelet coefficients* $\{\alpha\ ,\ \beta_k^n\}$:

$$\mathcal{W}^N\boldsymbol{Y} = \{\alpha, \beta_0^0, \ldots, \beta_{2^{M-1}-1}^{M-1}\}\ , \quad \boldsymbol{Y} = \{Y_0,\ Y_1,\ \ldots,\ Y_{N-1}\}\ , \quad (2^M = N)\ . \tag{7}$$

The $N \times N$ matrix $\mathcal{W}^N$ can be computed by the recursive formula [6,7]

$$\mathcal{W}^N\boldsymbol{Y} \equiv \left[\prod_{k=1}^{M}\Big((P_{2^k} \oplus I_{2^M-2^k})(H_{2^k} \oplus I_{2^M-2^k})\Big)\right]\boldsymbol{Y}\ , \tag{8}$$

($\oplus$ being the direct sum), which is based on the kth-order identity matrix I_k, on the kth-order permutation (shuffle) matrix P_k, which moves the odd (place) components of a vector $\boldsymbol{Y}$ into the first half positions and the even (place) components into the second half, and on the *lattice coefficients* kth-order matrix H_k, which follows from a matrix factorization of the recursive equation coefficients in (6):

$$H_2 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \quad H_4 = H_2 \oplus H_2\ , \ldots\ .$$

For example, with $N = 4,\ M = 2$, assuming the empty set $I_0 \equiv \emptyset$ as the neutral term for the direct sum, it is

$$\begin{aligned} \mathcal{W}^4 &= \prod_{k=1,2}\left[(P_{2^k} \oplus I_{4-2^k})(H_{2^k} \oplus I_{4-2^k})\right] \\ &= \left[(P_2 \oplus I_2)(H_2 \oplus I_2)\right]_{k=1}\left[(P_4 \oplus I_0)(H_4 \oplus I_0)\right]_{k=2} \\ &= \left[(P_2 \oplus I_2)(H_2 \oplus I_2)\right][P_4 H_4]\ . \end{aligned}$$

The wavelet coefficients of the discrete Haar transform have a simple interpretation in terms of finite differences. If we define the mean average value $\overline{Y}_{i,i+s} \equiv (s+1)^{-1}\sum_{k=i}^{i+s} Y_k$, it follows, with easy computation [6], that

$$\begin{cases} \alpha &= \overline{Y}_{0,2^M-1}, \\ \beta_k^r &= 2^{(M-2-r)/2}\delta_{(M-1-r)h}\overline{Y}_{k+2^{M-2-r+2k},k+2^{M-2-r+2k}+M-r-2+2^{M-r-2}}\ , \\ \beta_k^{M-1} &= 2^{-1/2}\Delta_h Y_{2k}\ , \end{cases}$$

where $r = 0,\ldots,M-2,\ k = 0,\ldots,2^{M-1}-1,\ h = 2^M$ and the forward and central (finite) difference formulas, as usual, are $\Delta_h Y_i \equiv (Y_{i+hN} - Y_i)$ and $\delta_h Y_i \equiv (Y_{i+hN} - Y_{i-hN})$, respectively. Therefore the wavelet coefficients β, also called details coefficients, express (at least in the Haar wavelet approach) the finite differences, i.e. the first-order approximate derivative.

2.1. Haar series

Let $V_N,\ N \in \mathbb{Z}$, the subspace of $L^2(\mathbb{R})$, of the piecewise constant functions $y(x)$ with compact support on Ω_k^N (N fixed, $k \in \mathbb{Z}$) be

$$V_N \equiv \{y(x) \in L^2(\mathbb{R}) : y(x) = y_k^N = \text{const.},\ \ x \in \Omega_k^N,\ \ y(x) = 0, x \notin \Omega_k^N\},$$

so that any $y(x) \in V_N$ with $y(x)|_{x\in\Omega_k^N} = Y_k(k = 0,\ldots,N-1)$, according to (4), admits the representation

$$y(x) = 2^{-N/2}\sum_{k=0}^{N-1} Y_k\ \varphi_k^N(x). \tag{9}$$

The *Haar wavelet series* is the piecewise constant function $y(x)$, defined in $\Omega = [0,1]$, interpolating the points $\{x_i, Y_i\}$, with $x_i = i/(2^M-1),\ i = 0,\ldots,2^M-1$,

$$y(x) = \alpha\varphi(x) + \sum_{n=0}^{M-1}\sum_{k=0}^{2^n-1}\beta_k^n\psi_k^n(x), \qquad (2^M = N), \tag{10}$$

such that $y(x) = y(x_i) = Y_i$, for all $x \in \Omega_i^N$. Thus, the discrete Haar transform might also be considered as an operator $\mathcal{W}^N : \mathbb{K}^N \subset l^2 \rightarrow V_N$, which, according to (4), (5), (7), makes (10) equivalent to (9).

In the general framework of the wavelet theory [4,5], any function (not only piecewise constant) $F(x)$ in $L^2(\mathbb{R})$ can be completely reconstructed as

$$F(x) = \sum_{n,k\in\mathbb{Z}}\beta_k^n\psi_k^n(x), \tag{11}$$

while, by fixing the *scale* of approximation, or *resolution* $M < \infty$, the (wavelet) approximation of $F(x)$ is $\pi^N F(x)$. The projection of $F(x)$ into V_N, namely

$\pi^N F(x)$, is given by the right-hand side of Eq. (10), where the wavelet coefficients are computed using the *dyadic discretization* of $F(x)$: $\boldsymbol{F} = \{F_k\},\ F_k \equiv F(x_k),\ k = 0, \dots, 2^M - 1$, shortly $\boldsymbol{F} = \nabla^N F(x),\ (N = 2^M)$. According to the wavelet theory [5,9], when $M \to \infty$, it is, at least in the weak sense [5], $\lim_{M\to\infty} \left|\left|F(x) - \pi^{2^M} F(x)\right|\right|_{L^2} = 0$. In general, for a function $F(x)$, belonging to the Sobolev space $W_p^{S+1}(\mathbb{R})$, the approximation is [9]: $\left|\left|F(x) - \pi^{2^M} F(x)\right|\right|_{L^p} = O(2^{-M(S+1)}),\ M \to \infty$, so that, for the Haar wavelet reconstruction (10), the approximation is $O(2^{-M})$ at the resolution M.

The scalar product of two functions $F(x),\ G(x)$, of $L^2(\mathbb{R})$, is $\langle F(x), G(x)\rangle \equiv \int_{-\infty}^{\infty} F(x)\overline{G}(x)\mathrm{d}\, x$, where the bar stands for complex conjugation. Taking into account (11) and the orthonormality conditions

$$\langle \varphi_k^n(x), \varphi_h^m(x)\rangle = \delta^{nm}\delta_{hk}, \quad \langle \varphi(x), \psi_h^m(x)\rangle = 0, \quad \langle \psi_k^n(x), \psi_h^m(x)\rangle = \delta^{nm}\delta_{hk}, \tag{12}$$

and since for the Haar wavelets $\psi_k^n(x) = \overline{\psi}_k^n(x)$, we have

$$\begin{gathered}\langle F(x), G(x)\rangle = \sum_{n,k\in\mathbb{Z}} \overset{F}{\beta}{}_k^n \overset{G}{\beta}{}_k^n, \\ \left(F(x) = \sum_{n,k\in\mathbb{Z}} \overset{F}{\beta}{}_k^n \psi_k^n(x), \quad G(x) = \sum_{n,k\in\mathbb{Z}} \overset{G}{\beta}{}_k^n \psi_k^n(x)\right).\end{gathered} \tag{13}$$

As a consequence, from Eq. (10) it follows that $||\boldsymbol{Y}||_{l^2} = ||\mathcal{W}^N\boldsymbol{Y}||_{l^2}$.

2.2. The p-parameters reduced (or windowed) Haar discrete wavelet transform

The wavelet transform (7) implies the computation of $N = 2^M$ wavelet coefficients, at the resolution M, with N basis functions $\psi_k^n(x)$ involved, and a computation complexity $\mathcal{O}(N^2)$. However, if we consider only $p = 2^m \leq N$ basis functions, the complexity reduces to $\mathcal{O}(pN)$. This corresponds to the slicing of data with a fixed window, as it is usually done, for instance, in the local sine and cosine transforms or in the wavelet packet decomposition [10,11]. With the following reduced Haar transform it is possible both to reduce the number of basis functions and the computational complexity and to keep unchanged the piecewise constant interpolation (9).

Let the set $\boldsymbol{Y} = \{Y_i\}$ of N data, segmented into $\sigma = N/p$ segments of $p = 2^m$ data [6] be

$$\boldsymbol{Y} = \{Y_i\}_{i=0,\dots,N-1} = \bigoplus_{s=0}^{\sigma-1}\{\boldsymbol{Y}^s\}\ , \qquad \boldsymbol{Y}^s \equiv \{Y_{sp},\ Y_{sp+1}, \dots,\ Y_{sp+p-1}\}.$$

The p*-parameters reduced (or windowed) discrete Haar wavelet transform* of $\boldsymbol{Y}$ is $\mathcal{W}^{p,\sigma}\boldsymbol{Y}$, where explicitly

$$\left\{\begin{array}{l}
\mathcal{W}^{p,\sigma} \equiv \bigoplus_{s=0}^{\sigma-1} \mathcal{W}^{p}, \quad \boldsymbol{Y} = \bigoplus_{s=0}^{\sigma-1} \boldsymbol{Y}^{s}, \\
\mathcal{W}^{p,\sigma}\boldsymbol{Y} = \left(\bigoplus_{s=0}^{\sigma-1} \mathcal{W}^{p}\right)\boldsymbol{Y} = \left(\bigoplus_{s=0}^{\sigma-1} \mathcal{W}^{p}\boldsymbol{Y}^{s}\right), \\
\mathcal{W}^{2^m}\boldsymbol{Y}^{s} = \left\{\alpha_0^{0(s)},\ \beta_0^{0(s)},\ \beta_0^{1(s)},\ \beta_1^{1(s)}, \dots,\ \beta_{2^{m-1}-1}^{m-1(s)}\right\},\ (2^m = p).
\end{array}\right.$$

The corresponding Haar series interpolation gives

$$\left\{\begin{array}{lcl}
y(x) & = & \displaystyle\sum_{s=0}^{\sigma-1} y^{s}(x), \\
y^{s}(x) & = & \left\{\begin{array}{ll}
\alpha_0^{0(s)}\varphi^{(s)}(x) + \displaystyle\sum_{n=0}^{m-1}\sum_{k=0}^{2^m-1} \beta_k^{n(s)}\psi_k^{n(s)}(x), & x \in [x_{sp}, x_{sp+p-1}) \\
0, & \text{elsewhere},
\end{array}\right.
\end{array}\right.$$

each function $y^s(x)$ being with compact support in the interval (x_{ps}, x_{ps+p-1}) which corresponds to the data segment $\boldsymbol{Y}^s$. The p basis functions $\{\varphi^{(s)}(x)$, $\psi_0^{0(s)}(x), \dots, \psi_{2^{m-1}-1}^{m-1(s)}(x)\}$, are

$$\varphi^{(s)}(x) = \left\{\begin{array}{ll}
\varphi(x), & x \in [x_{sp}, x_{sp+p-1}) \\
0, & \text{elsewhere},
\end{array}\right.$$

$$\psi_k^{n(s)}(x) = \left\{\begin{array}{ll}
\psi_k^{n}(x), & x \in [x_{sp}, x_{sp+p-1}) \\
0, & \text{elsewhere}.
\end{array}\right.$$

In general, for a vector of 2^M elements, $\boldsymbol{Y} = \{Y_i\}_{i=0,\dots,2^M-1}$, the Haar wavelet transform is the vector $\mathcal{W}^{2^M}\boldsymbol{Y}$, while there are different reduced transforms that can be done with one of the following matrices $\{\mathcal{W}^{2^i,2^j}\}_{i+j=M}$, even if the resulting piecewise interpolation will be the same. Of course, when $\sigma = 1 \to p = N$ and $\mathcal{W}^{p\sigma}\boldsymbol{Y} = \mathcal{W}^{N}\boldsymbol{Y}$.

3. GENERALIZED DERIVATIVE IN V_N

The generalized derivative [6,7] is a derivative algorithm that maps any discrete set of data into discrete sets of data, and, after a dyadic discretization, also any function $F(x)$ into a discrete set (the discrete derivative of $F(x)$).

Through the discrete set of points $\boldsymbol{Y} = \{Y_i\}$, at the dyadic nodes $x_i = i/(N-1),\ (i = 0, \dots, N-1),\ N = 2^M$, the Lagrange interpolation polynomial is defined as

$$\mathcal{P}^N(\boldsymbol{Y}) = \sum_{i=0}^{N-1} l_i(x) Y_i, \qquad l_i(x) \equiv \prod_{k=0,\dots,i-1,i+1,\dots,N-1} \frac{(x - x_k)}{(x_i - x_k)},$$

$l_i(x)$ being the Lagrange coefficients. This polynomial is a differentiable function and, after a dyadic discretization, the following definition of the generalized derivative may be given.

Definition 1 (Generalized qth-order derivative of $\boldsymbol{Y} \in \mathbb{K}^N$ [6,7]). *The generalized qth-order derivative of the N-length vector $\boldsymbol{Y}$ is the vector $\boldsymbol{Y}^{(q)} = \delta^{(q)}\boldsymbol{Y}$, where $\delta^{(q)} : \mathbb{K}^N \to \mathbb{K}^N$ is*

$$\delta^{(q)} \equiv \nabla^N \frac{\mathrm{d}^q}{\mathrm{d}x^q} \mathcal{P}^N. \tag{14}$$

This operator can be extended also to any function $y(x) \in V_N$ assuming $\boldsymbol{Y}^{(q)} = \delta^{(q)}\nabla^N y(x)$, with $y(x)$ given by Eqs. (9), (10).

Definition 2 (Generalized qth-order derivative of $y(x) \in V_N$). *The generalized qth-order derivative of the piecewise constant function $y(x) \in V_N$ is the piecewise function $\delta_N^{(q)} y(x)$, where $\delta_N^{(q)} : V_N \to \mathbb{K}^N$ is*

$$\delta_N^{(q)} \equiv \delta^{(q)}\nabla^N. \tag{15}$$

In particular, this operator can be applied to any function after a projection into V_N, i.e. $F(x) \in L^2([0,1])$, or such that $\max_{i=0,\dots,N-1} \left\{\nabla^N F(x)\right\}_i < \infty$, so that the generalized derivative $\delta_N^{(q)} F(x)$ gives the approximation (projection into V_N) of the derivative $(\mathrm{d}^{\,q}/\mathrm{d}\,x^q)F(x)$.

3.1. Error estimate

Let us first evaluate the approximation error $R_N(x)$ for the L^2-function $F(x)$ after a dyadic discretization and Lagrange interpolation: $F(x) = \mathcal{P}^N(x) + R_N(x)$. Assuming $F(x)$ differentiable up to $N+1$, the error R_N in $x^* \neq x_i,\ x^* \in [0,1]$, is given [12] by

$$R_N(x^*) = \frac{F^{(N+1)}(x^*)}{(N+1)!} \prod_{i=0}^{N-1} |x^* - x_i|, \quad \left(F^{(N+1)} \equiv \frac{\mathrm{d}^{\,N+1}}{\mathrm{d}\,x^{N+1}} F\right),$$

so that it can be estimated by

$$\max_{x\in[0,1]} |R_N(x)| \le \frac{\max_{x\in[0,1]} F^{(N+1)}(x)}{(N+1)!} \max_{x\in[0,1]} \prod_{i=0}^{N-1} |x - x_i|, \tag{16}$$

and, for the first derivative, at the nodes x_i, it is estimated by [12]

$$\max_{i=0,\dots,N-1}\left|\frac{\mathrm{d}}{\mathrm{d}\,x}R_N(x_i)\right| \leq \frac{\max_{i=0,\dots,N-1} F^{(N+2)}(x_i)}{(N+1)!}\left(\frac{1}{M^{N-1}}\right), \qquad N=2^M.$$

In practice, if we take as a rough estimate of the approximation error of the first generalized derivative (at the nodes)

$$E(F(x),N) = \max_{i=0,\dots,N-1}\left|\nabla^N\left(\frac{\mathrm{d}\,F(x)}{\mathrm{d}\,x}\right) - \delta'_N F(x)\right|_i,$$

we see that the error decay depends on the increasing number of nodes N. For instance, for the functions $\sin 2\pi x$ and e^x discretized respectively in 16 and 32 (dyadic) nodes, we obtain

$$\begin{array}{ll} E(\sin 2\pi\, x, 16) = 0.0544876, & E(\sin 2\pi x, 32) = 0.00434216, \\ E(e^x, 16) = 0.000185967, & E(e^x, 32) = 0.0000219477. \end{array}$$

Of course, the above numerical estimates of the error for functions cannot be directly extended to discrete sequences $\boldsymbol{Y}$, but we might assume that when the generalized derivative acts on sequences whose length is 16 (or greater), the error estimate of $\delta^{(q)}$ is substantially the same of $\delta_N^{(q)}$ applied to functions. Haar wavelets give a bad approximation at low scales, but the approximation, both of the function and of its derivatives, improves for increasing resolution levels and gives the complete reconstruction, for $M \to \infty$. Moreover, with the windowed (reduced) Haar transform, also the complexity might be strongly reduced. Thus a sufficiently good reconstruction of the derivative (see Fig. 1) is obtained in a very short computation time. With respect to the segmentation $\boldsymbol{Y} = \bigoplus_{s=0}^{\sigma-1}\boldsymbol{Y}^s$, the generalized derivative is not a linear function, that is

$$\delta^{(q)}\boldsymbol{Y} = \delta^{(q)}\bigoplus_{s=0}^{\sigma-1}\boldsymbol{Y}^s \neq \bigoplus_{s=0}^{\sigma-1}\delta^{(q)}\boldsymbol{Y}^s,$$

because of the Lagrange interpolation acting, in general, on different sets of nodes. However, Eq. (14) can be approximated by

$$\delta^{(q)}\bigoplus_{s=0}^{\sigma-1}\boldsymbol{Y}^s \cong \sigma\bigoplus_{s=0}^{\sigma-1}\delta^{(q)}\boldsymbol{Y}^s.$$

In any case, it is easy to check that the accuracy of the generalized derivative converges more slowly to the derivative when the frequency function is higher (see Fig. 1).

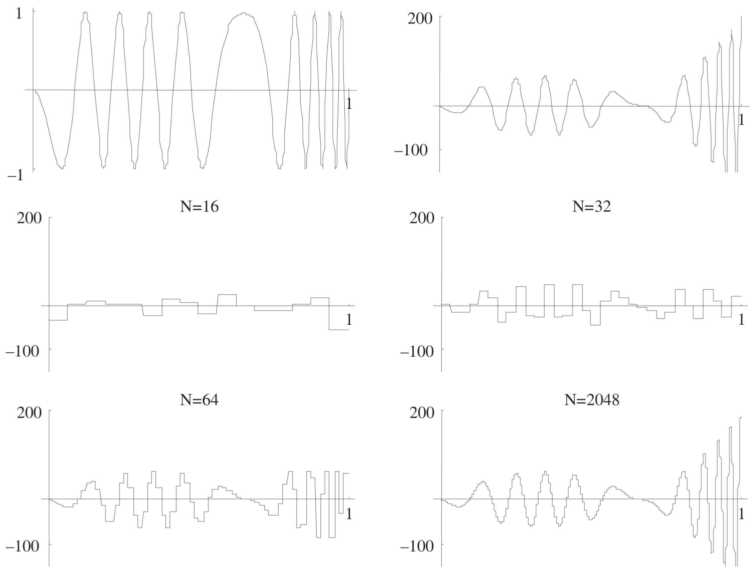

Fig. 1. The function $y(x) = \sin 200\, x^2(x-1)$ (top left) with its derivative (right) and the generalized derivative at various scale resolutions. The generalized derivative at the resolution $N = 2048$ has been obtained using the reduced Haar transform with data slicing into 16-length segments.

3.2. Generalized derivative as a linear operator on $\mathbb{K}^N$

The generalized derivative, according to (14), is a linear operator on $\mathbb{K}^N$, which maps finite vectors into vectors. For instance $\delta' \boldsymbol{Y} \in \mathbb{K}^N$, so that, taking Eq. (9) into account, it is

$$\delta' \boldsymbol{Y} = 2^{-N/2} \sum_{k=0}^{N-1} Y'_k \varphi_k^N(x),$$

where the components Y'_k are related to the components Y_k of $\boldsymbol{Y}$ by the matrix $\boldsymbol{A} \equiv A_{kh}$:

$$Y'_k \equiv \sum_{h=0}^{N-1} A_{kh} Y_h.$$

At the lower scales we have

- $M = 1,\ N = 2$

  $$\boldsymbol{A} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

  so that for a vector $\{Y_1,\ Y_2\}$ the numerical derivative is the vector $\boldsymbol{AY} = \{A_{11}Y_1 + A_{12}Y_2,\ A_{21}Y_1 + A_{22}Y_2\} = \frac{1}{2}\{Y_2 - Y_1,\ Y_1 - Y_2\}$.

- $M = 2, \ N = 4$

$$\boldsymbol{A} = \begin{pmatrix} -5.5 & 9 & -4.5 & 1 \\ -1 & -1.5 & 3 & -0.5 \\ 0.5 & -3 & 1.5 & 1 \\ -1 & 4.5 & -9 & 5.5 \end{pmatrix}.$$

For higher scales, it can be seen that among the components of the matrix $\boldsymbol{A} = \{A_{hk}\}$ the relations $A_{1j} = -A_{2^M, 2^M-j+1}$, $A_{2j} = -A_{2^M-1, 2^M-j+1}$, $(j = 1, \ldots, 2^M)$, for all M, and $A_{kj} = A_{k+1,j+1}$, $(2 < k < 2^M - 2, \ 1 < j < 2^M - 1)$, hold so that, in general, we have

- $M > 2, \ N = 2^M$

$$\boldsymbol{A} = \begin{pmatrix} -11(2^M-1)/6 & 3(2^M-1) & -3(2^M-1)/2 & (2^M-1)/3 & & & \\ -(2^M-1)/3 & -(2^M-1)/2 & (2^M-1) & -(2^M-1)/6 & & \mathbf{0} & \\ (2^M-1)/6 & -(2^M-1) & (2^M-1)/2 & (2^M-1)/3 & & & \\ & (2^M-1)/6 & -(2^M-1) & (2^M-1)/2 & (2^M-1)/3 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ \mathbf{0} & & (2^M-1)/6 & -(2^M-1) & (2^M-1)/2 & (2^M-1)/3 & \\ & & & (2^M-1)/6 & -(2^M-1) & (2^M-1)/2 & (2^M-1)/3 \\ & & & -(2^M-1)/3 & 3(2^M-1)/2 & -3(2^M-1) & 11(2^M-1)/6 \end{pmatrix}$$

3.3. Connection coefficients

In this paragraph, the generalized first derivative of the Haar wavelets $\psi_k^n(x)$ is explicitly given. In a fixed V_N, we have from (9)

$$\psi_k^n(x) = 2^{-N/2} \sum_{h=0}^{N-1} p_{kh}^{nN} \ \varphi_h^N(x),$$

where $p_{kh}^{nN} \equiv \nabla^N \psi_k^n(x), h = 0, \ldots, N-1$, according to (5), is either $\{0, \ 1, \ -1\}$. Thus, taking Eq. (14) into account, the first derivative of the Haar wavelets is

$$\psi'^n_k(x) \equiv \delta' \nabla^N \psi_k^n(x) = 2^{-N/2} \sum_{h=0}^{N-1} g_{kh}^{nN} \ \varphi_h^N(x) \qquad (N \text{ fixed}),$$

$g_{kh}^{nN} = \langle \boldsymbol{A} \nabla^N \psi_k^n(x), \varphi_h^N(x) \rangle$ being the component of $\psi'^n_k(x)$ with respect to $\varphi_h^N(x)$.

In terms of wavelet coefficients, according to (10), the qth-order generalized derivative is

$$\delta^{(q)}\nabla^N \psi^n_k(x) = \alpha^n_k \varphi(x) + \sum_{m=0}^{M-1} \sum_{h=0}^{2^m-1} \gamma^{nm}_{kh}\; \psi^m_h(x), \quad (N = 2^M),$$

where the *connection coefficients* $\alpha^n_k,\ \gamma^{nm}_{kh}$,

$$\alpha^n_k \equiv \langle \delta^{(q)}\nabla^N \psi^n_k(x), \varphi(x)\rangle, \quad \gamma^{nm}_{kh} \equiv \langle \delta^{(q)}\nabla^N \psi^n_k(x), \psi^m_h(x)\rangle, \tag{17}$$

are expressed by

$$\begin{aligned}\left(\alpha^n_k \bigoplus \gamma^{nm}_{kh}\right)_{m=0,\dots,M-1;\; h=0,\dots,2^m-1} &= \mathcal{W}^N g^{nN}_{kh} \\ &= \mathcal{W}^N \left(\langle \boldsymbol{A}^q \nabla^N \psi^n_k(x), \varphi^N_h(x)\rangle\right), \\ &\qquad (N = 2^M),\end{aligned}$$

$\boldsymbol{A}^q$ being the q-power matrix.

At the resolution $N = 2^M$, the dyadic discretization of the function $\psi^n_k(x)$, $k \le 2^{n-1}$, gives only 2^{M-n} unvanishing components. Therefore, in order to have nontrivial generalized derivatives of $\psi^n_k(x)$, with 4 unvanishing components, we should take at least $M = 3$ (see Tables 1, 2 for the first and second derivative, respectively).

Table 1. Wavelet components of the first generalized derivative $\psi'^{n}_{k}(x) = \delta'\nabla^8 \psi^n_k(x)$

	φ	ψ^0_0	ψ^1_0	ψ^1_1	ψ^2_0	ψ^2_1	ψ^2_2	ψ^2_3
ψ'^0_0	–0.145839	–3.7123	2.33335	–12.25	0.0000212132	3.29983	–9.89949	–7.42462
ψ'^1_0	–0.291672	–2.47485	5.83335	2.33333	19.799	–11.5494	4.94975	0
ψ'^1_1	–0.875001	–0.824964	–1.16667	–2.33334	0	–1.64991	8.24957	–19.799
ψ'^2_0	3.20833	–9.07452	–19.8333	0	–24.7487	6.59967	0	0
ψ'^2_1	0.874999	–7.42462	2.33334	3.5	–14.8492	8.24958	6.59967	0
ψ'^2_2	1.45833	5.7747	–1.16667	–2.33334	0	–1.64991	8.24958	14.8492
ψ'^2_3	3.79167	3.79167	0	17.5	0	0	–1.64991	24.7487

Table 2. Wavelet components of the second generalized derivative $\psi''^{n}_{k}(x) = \delta''\nabla^{8}\psi^{n}_{k}(x)$

	φ	ψ_0^0	ψ_0^1	ψ_1^1	ψ_0^2	ψ_1^2	ψ_2^2	ψ_3^2
ψ''^{0}_{0}	−25.3506	−114.05	24.4997	−44.2361	−11.5496	23.0988	−65.4466	−112.607
ψ''^{1}_{0}	39.4722	−55.8221	−287.194	−28.5834	−167.466	−19.2489	7.69954	28.8736
ψ''^{1}_{1}	−44.9167	−105.87	−12.25	−259.972	5.7747	−11.5494	53.8972	−236.763
ψ''^{2}_{0}	−54.7845	143.405	292.638	8.16669	167.465	26.9486	13.4744	0
ψ''^{2}_{1}	−28.243	70.2588	215.055	20.4166	196.34	−53.8973	123.194	34.6483
ψ''^{2}_{2}	52.0625	160.729	−6.80555	157.889	5.7747	−3.84981	−9.62443	369.581
ψ''^{2}_{3}	63.6319	63.6319	0	270.861	0	0	−3.84981	242.538

4. HAAR WAVELET SOLUTIONS

Let us first define the projection of the differential operator L into V_N.

Definition 3 (Discrete operator $\mathcal{L}^N$ associated with the differential operator L). *The projection of the operator L in $(1)_2$ into V_N is the discrete operator $\mathcal{L}^N : V_N \to V_N$,*

$$\mathcal{L}^N \equiv \sum_{j=1}^{q} \lambda_j \delta_N^{(j)}, \tag{18}$$

with $\delta_N^{(j)}$ given by (15), so that $\mathcal{L}^N$ approximates L, at the scale resolution N,

$$\pi^N L \cong \mathcal{L}^N \,. \tag{19}$$

4.1. Approximate wavelet solution

Let us consider, according to (11), the Haar wavelet series

$$u(x,t) = \sum_{n,k=-\infty}^{\infty} \beta_k^n(t)\psi_k^n(x), \qquad (n,k\in\mathbb{Z}), \tag{20}$$

as a solution of Eq. (1). From (1), (2), there follows a system for the wavelet coefficients $\beta_k^n(t)$, $(x\in\mathbb{R}, t\in(0,T],\ T\leq\infty,\ a,\ b,c\in\mathbb{R})$,

$$\sum_{n,k=-\infty}^{\infty}\left(a\frac{\mathrm{d}^2\beta_k^n(t)}{\mathrm{d}t^2} + b\frac{\mathrm{d}\beta_k^n(t)}{\mathrm{d}t} + c\,\beta_k^n(t)\right)\psi_k^n(x) = L\sum_{n,k=-\infty}^{\infty}\beta_k^n(t)\psi_k^n(x), \tag{21}$$

together with the initial conditions,

$$\begin{cases} \displaystyle\sum_{n,k=-\infty}^{\infty} \beta_k^n(0)\psi_k^n(x) = u_0(x), \\ \displaystyle\sum_{n,k=-\infty}^{\infty} \left.\frac{\mathrm{d}\,\beta_k^n(t)}{\mathrm{d}\,t}\right|_{t=0} \psi_k^n(x) = u_0'(x), \quad \text{only if } a \neq 0. \end{cases} \tag{22}$$

Thus, at a given resolution N, we have from (20).

Definition 4. *The approximate Haar wavelet solution, at the resolution* $N = 2^M$ *and for a fixed time* $\bar{t} \in [0, T]$, *is the vector* $\boldsymbol{U}(\bar{t}) = \{\alpha(\bar{t}),\ \beta_0^0(\bar{t}), \ldots, \beta_{2^{M-1}-1}^{M-1}(\bar{t})\}$, *i.e.*

$$\boldsymbol{U}(\bar{t}) \equiv \pi^N u(x, \bar{t}) = \pi^N \sum_{n,k=-\infty}^{\infty} \beta_k^n(\bar{t})\psi_k^n(x) = \alpha(\bar{t})\varphi(x) + \sum_{n=0}^{M-1}\sum_{k=0}^{2^n-1} \beta_k^n(\bar{t})\psi_k^n(x),$$

where the projection π^N *acts on* $u(x,t)$ *by keeping* $t = \bar{t}$ *fixed. It follows that*

$$\pi^N \frac{\mathrm{d}^{\,q}}{\mathrm{d}\,t^q} = \frac{\mathrm{d}^{\,q}}{\mathrm{d}\,t^q}\pi^N, \qquad q = 0,\ 1, \ldots \tag{23}$$

From the above we get

Theorem 1. *The approximate Haar wavelet solution of Eqs.* (1), (2), *at the resolution* N (*projection of* (20) *into* V_N), *is the vector function* $\boldsymbol{U}(t) = \{\alpha(t),\ \beta_0^0(t), \ldots,\ \beta_{2^{M-1}-1}^{M-1}(t)\}$. *The corresponding truncated wavelet series is*

$$\boldsymbol{U}(t) = \alpha(t)\varphi(x) + \sum_{n=0}^{M-1}\sum_{k=0}^{2^n-1} \beta_k^n(t)\psi_k^n(x), \quad (N = 2^M,\ t \in [0, T]), \tag{24}$$

where $\alpha(t)$ *and* $\beta_k^n(t)$ *are the solution of the Cauchy problem* ($n = 0, \ldots, M-1,\ \ k = 0, \ldots, 2^n - 1$)

$$\begin{cases} a\dfrac{\mathrm{d}^2\alpha(t)}{\mathrm{d}t^2} + b\dfrac{\mathrm{d}\alpha(t)}{\mathrm{d}t} + c\,\alpha(t) = \alpha(t) + \displaystyle\sum_{n=0}^{M-1}\sum_{k=0}^{2^n-1} \alpha_k^n \beta_k^n(t), \\ a\dfrac{\mathrm{d}^2\beta_k^n(t)}{\mathrm{d}t^2} + b\dfrac{\mathrm{d}\beta_k^n(t)}{\mathrm{d}t} + c\,\beta_k^n(t) = \displaystyle\sum_{m=0}^{M-1}\sum_{h=0}^{2^m-1} \gamma_{kh}^{nm}\beta_h^m(t), \end{cases} \tag{25}$$

with the initial conditions

$$\begin{cases} \alpha(0) = \langle u_0(x), \varphi(x)\rangle, \quad \beta_k^n(0) = \langle u_0(x), \psi_k^n(x)\rangle. \\ \left.\dfrac{\mathrm{d}\alpha(t)}{\mathrm{d}t}\right|_{t=0} = \langle u_0'(x),\ \varphi(x)\rangle,\ \left.\dfrac{\mathrm{d}\beta_k^n(t)}{\mathrm{d}t}\right|_{t=0} = \langle u_0'(x), \psi_k^n(x)\rangle, \text{ (only if } a \neq 0). \end{cases} \tag{26}$$

The connection coefficients $\alpha_k^n, \gamma_{kh}^{nm}$, *at the resolution* N *and according to* (17), (18), *are*

$$\alpha_k^n \equiv \langle \mathcal{L}^N \psi_k^n(x), \varphi(x) \rangle , \quad \gamma_{kh}^{nm} \equiv \langle \mathcal{L}^N \psi_k^n(x), \psi_h^m(x) \rangle . \tag{27}$$

Proof. From Eqs. (20), (21), the projection into V_N is

$$\pi^N \left(a \frac{\mathrm{d}^2}{\mathrm{d}t^2} + b \frac{\mathrm{d}}{\mathrm{d}t} + c \right) \sum_{n,k=-\infty}^{\infty} \beta_k^n(t)\, \psi_k^n(x) = \pi^N L \sum_{n,k=-\infty}^{\infty} \beta_k^n(t) \psi_k^n(x),$$

and taking Eqs. (19)–(23) into account,

$$\begin{aligned}
&\left(a \frac{\mathrm{d}^2 \alpha(t)}{\mathrm{d}t^2} + b \frac{\mathrm{d}\,\alpha(t)}{\mathrm{d}t} + c\, \alpha(t) \right) \varphi(x) \\
&\qquad + \sum_{n=0}^{M-1} \sum_{k=0}^{2^n-1} \left(a \frac{\mathrm{d}^2 \beta_k^n(t)}{\mathrm{d}t^2} + b \frac{\mathrm{d}\,\beta_k^n(t)}{\mathrm{d}t} + c\, \beta_k^n(t) \right) \psi_k^n(x) \\
&= \mathcal{L}^N \left(\alpha(t)\varphi(x) + \sum_{n=0}^{M-1} \sum_{k=0}^{2^n-1} \beta_k^n(t) \psi_k^n(x) \right) \\
&= \alpha(t)\varphi(x) + \sum_{n=0}^{M-1} \sum_{k=0}^{2^n-1} \beta_k^n(t)\, \mathcal{L}^N \psi_k^n(x) \,.
\end{aligned}$$

By using the orthonormality conditions (12) and the connection coefficients (27), (17), (18), there follows the Haar wavelet solution (24)–(26).

5. HEAT EQUATION

Let us reduce Eq. (1) to the one-dimensional heat equation for an infinite bar $\Omega \to \mathbb{R}$, with normalized physical constants

$$\frac{\partial u}{\partial t} = Lu, \qquad L \equiv \frac{\partial^2}{\partial x^2} \,, \tag{28}$$

and the initial condition

$$u(x,0) = u_0(x), \;\; -\infty \leq x \leq \infty, \;\; t = 0. \tag{29}$$

The box function

$$u_0(x) = \begin{cases} 1, \; x \in \Lambda, & \Lambda \equiv \{ x : 0 < l_0 \leq x \leq l_1 < 1 \} \\ 0, \; x \notin \Lambda \end{cases} \tag{30}$$

is taken as the initial function. It follows that the solution of the problem (28)–(30), in terms of Fourier integrals, is (see Fig. 2, left):

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{l_0}^{l_1} \exp\left\{ -\frac{(x-\xi)^2}{4t} \right\} \mathrm{d}\,\xi, \qquad t \neq 0. \tag{31}$$

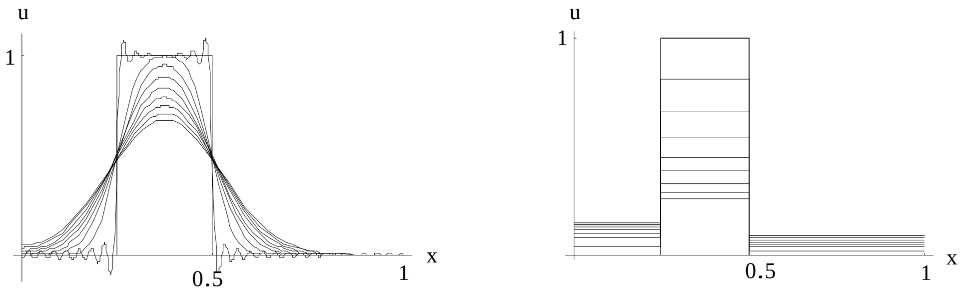

Fig. 2. Fourier solution (left) and Haar wavelet solution of the heat equation with initial rectangle function.

On the other hand, for the projection of the differential operator we have

$$\mathcal{L}^N = \pi^N \frac{\partial^2}{\partial x^2} = \delta'' \nabla^N = \boldsymbol{AA},$$

so that, assuming, e.g., $l_0 = \frac{1}{4}$, $l_1 = \frac{1}{2}$, the Haar wavelet approximate solution (25), at the resolution $N = 8$, is

$$\begin{cases} \dfrac{\mathrm{d}\,\alpha(t)}{\mathrm{d}\,t} = \alpha(t) + \displaystyle\sum_{n=0}^{2} \sum_{k=0}^{2^n-1} \alpha_k^n \beta_k^n(t), \\ \dfrac{\mathrm{d}\,\beta_k^n(t)}{\mathrm{d}\,t} = \displaystyle\sum_{m=0}^{2} \sum_{h=0}^{2^m-1} \gamma_{kh}^{nm} \beta_h^m(t), \quad (n = 0, 1, 2; k = 0, \ldots, 2^n - 1), \end{cases}$$

with the initial conditions ($\beta_k^n(0) = \langle u_0(x), \psi_k^n(x) \rangle$),

$$\alpha(0) = \langle u_0(x), \varphi(x) \rangle = 1/4,\ \beta_0^0(0) = -1/4,\ \beta_0^1(0) = 1, \tag{32}$$

the remaining coefficients β being zero. Thus we have,

$$\begin{cases} \dfrac{\mathrm{d}\,\alpha(t)}{\mathrm{d}\,t} = \alpha(t) + \alpha_0^0 \beta_0^0(t) + \alpha_0^1 \beta_0^1(t), \\ \dfrac{\mathrm{d}\,\beta_0^0(t)}{\mathrm{d}\,t} = \gamma_{00}^{00} \beta_0^0(t) + \gamma_{00}^{01} \beta_0^1(t), \\ \dfrac{\mathrm{d}\,\beta_0^1(t)}{\mathrm{d}\,t} = \gamma_{00}^{10} \beta_0^0(t) + \gamma_{00}^{11} \beta_0^1(t). \end{cases} \tag{33}$$

With the connection coefficients (see Table 2)

$$\alpha_0^0 = -25.3506, \quad \gamma_{00}^{00} = -114.05, \quad \gamma_{00}^{01} = 24.997,$$
$$\alpha_0^1 = -39.4722, \quad \gamma_{00}^{10} = -55.8221, \quad \gamma_{00}^{11} = -287.194,$$

and the initial conditions (32), the system (33) is solved by

$$\begin{aligned}
&u_H(x,t) \equiv \pi^8 u(x,t) = \alpha(t)\phi(x) + \beta_0^0(t)\psi_0^0(x) + \beta_0^1(t)\psi_0^1(x),\\
&\alpha(t) = 0.137146\, e^t + \left(0.122886\, e^{-278.72t} - 0.010032\, e^{-122.524t}\right),\\
&\beta_0^0(t) = \left(-0.146473\, e^{-278.72t} - 0.103527\, e^{-122.524t}\right),\\
&\beta_0^1(t) = \left(0.964905\, e^{-278.72t} + 0.035095\, e^{-122.524t}\right).
\end{aligned} \tag{34}$$

The detailed coefficients, responsible for the jumps of the solution, decay rapidly to zero and the initial function, after a short time, becomes a smooth flat function (see Fig. 3). With the time step 0.001 we obtain the approximate Haar wavelet solution of Fig. 2 (right) after 8 time steps ($T = 0.008$). Figure 2 (left) represents the Fourier series solution

$$\begin{cases} u_F(x,t,N) = \dfrac{1}{4} + \displaystyle\sum_{n=1}^{N} a(n)\, \cos(n\,\pi\,x)\, e^{-n^2\,\pi^2\,t}, \\ a(n) \equiv \dfrac{2}{n\,\pi}\left(\sin\dfrac{n\,\pi}{2} - \sin\dfrac{n\,\pi}{4}\right), \end{cases}$$

where, in order to reconstruct also the initial function, $N = 60$ trigonometric basis functions have been used (against $N = 8$ of the Haar wavelet technique), thus

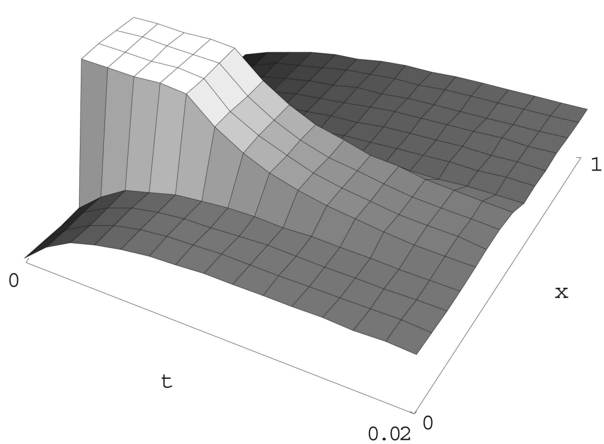

Fig. 3. Haar wavelet solution (34) at the scale resolution $M = 3$.

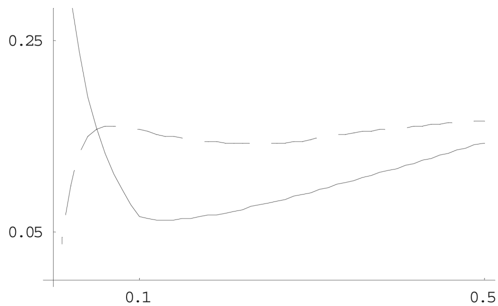

Fig. 4. Error of the Fourier series solution and the Haar wavelet solution (solid line) with respect to the integral solution (31).

showing the undesirable Gibbs phenomenon and the powerful compression of wavelets.

Let us evaluate the approximation error $\mathcal{E}$ at 8 dyadic nodes $x_k = k/7$, $k = 0, \ldots, 7$, with respect to the solution (31), both for the Haar solution (34):

$$\mathcal{E}_H(t) \equiv \max_{k=0,\ldots,7} |u_H(x_k,t) - u(x_k,t)| ,$$

and the 60 terms Fourier series:

$$\mathcal{E}_F(t) \equiv \max_{k=0,\ldots,7} |u_F(x_k,t,60) - u(x_k,t)| ,$$

respectively. Except for the initial time $t = 0$, it is possible to compute the error at the first set of values of $0.01 \le t \le 0.5$ with the time step 0.08. It follows that (see Fig. 4) even with the simple Lagrange interpolation, also from the numerical point of view the error in the Haar wavelet solution (at a very low scale $M = 3$, with only 3 coefficients) is lower in comparison with the Fourier series with 60 coefficients.

6. CONCLUSIONS

In this paper, the physical meaning of wavelets was discussed by expressing the solution of evolution problems in terms of wavelets. In particular, the compression property of wavelets strongly reduces the number of basis functions, as it has been shown from the propagation of heat, described by Haar wavelets.

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Haari lainekestel põhinev tehnika evolutsiooniülesannetes

Carlo Cattani

On uuritud lainekeste kompressiooniomadusi mittesiledate algtingimustega evolutsiooniülesannete analüüsil ning näidatud, et lainekeste kasutamine võimaldab vähendada ülesande keerukust (kordajate arvu) ja annab parema lähenduse. Konkreetselt on vaadeldud Haari lainekesi kui lihtsaimaid võimalikke lainekesi. Et Haari lainekesed ei ole siledad, on kasutatud tuletise lähendamiseks numbrilise diferentseerimise meetodit.