Proc. Estonian Acad. Sci. Phys. Math., 2004, **53**, 1, 3–12
https://doi.org/10.3176/phys.math.2004.1.01

Unary local polynomial functions of $\mathcal{K}_2$-algebras

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Received 21 January 2003, in revised form 5 March 2003

Abstract. The class $\mathcal{K}_2$ is a variety of distributive Ockham algebras that includes the varieties of Kleene and Stone algebras. Polynomial functions of algebras $\mathbf{A} \in \mathcal{K}_2$ were first studied by Haviar (*Acta Math. Univ. Comenianae*, 1993, **2**, 179–190). After that (local) polynomial functions were described for all proper subvarieties of $\mathcal{K}_2$. In this paper we characterize unary (local) polynomial functions of $\mathcal{K}_2$-algebras.

Key words: Ockham algebra, compatible function, (local) polynomial function.

1. INTRODUCTION

Let $\mathbf{A}$ be a universal algebra. A function $f : A^n \to A$ is called *compatible* (or *congruence preserving*), if, for any congruence ρ of $\mathbf{A}$, $(a_i, b_i) \in \rho$, $i = 1, \ldots, n$, implies

$$(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in \rho.$$

The function f is said to be a *local polynomial function* if it can be interpolated by a polynomial on every finite subset of A^n.

Our main goal is to study the clone of compatible functions on a given algebra. We say that an algebra is *affine complete* if the clone of compatible functions coincides with the clone of polynomial functions.

Originally, the problem of characterization of affine complete algebras was formulated in [1]. For various varieties of algebras affine completeness has already been studied. The affine completeness for Stone and Kleene algebras has been investigated in [2–4]. These results, together with many other results on affine completeness, can be found also in a recent monograph [5]. The generalization of these results to the variety $\mathcal{K} \vee \mathcal{S}$ generated by all Stone and Kleene algebras is presented in [6]. The results of [6] show that local polynomials of algebras in $\mathcal{K} \vee \mathcal{S}$ can be characterized fairly similarly to those of Kleene and Stone algebras. The variety $\mathcal{K}_2$ contains $\mathcal{K} \vee \mathcal{S}$ as a unique maximal subvariety. Polynomial functions of algebras in the variety $\mathcal{K}_2$ were first studied in [7]. In the present paper we show that in $\mathcal{K}_2$ the theory of local polynomial functions is more complicated: there exist functions which preserve congruences and the uncertainty order but are not local polynomials. Recently it was proved in [8] that in the variety $\mathcal{K} \vee \mathcal{S}$ the clone of congruence and the uncertainty order preserving functions of a given algebra is generated by polynomial functions together with certain special unary functions. We conjecture that the same statement is valid for $\mathcal{K}_2$. This motivates to study first unary local polynomials. Our main result is Theorem 4.3 saying that unary local polynomial functions $f(x)$ of any algebra $\mathbf{A} \in \mathcal{K}_2$ can be characterized by three properties: (1) they must preserve congruences, (2) they must preserve the uncertainty order, and (3) the function $(f(x) \vee f(x)^*) \wedge x^{**}$ has to satisfy a certain very special interpolation property on the subset $A^{\vee} \subseteq A$. We hope that this result will help us to describe (locally) affine complete members of the variety $\mathcal{K}_2$.

2. PRELIMINARIES

A *distributive Ockham algebra* is an algebra $\langle L; \vee, \wedge, ^*, 0, 1\rangle$, where $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and * is a unary operation such that $0^* = 1$, $1^* = 0$, and for all $x, y \in L$,

$$(x \wedge y)^* = x^* \vee y^*,$$
$$(x \vee y)^* = x^* \wedge y^*.$$

We refer to [9] as the basic source of information about distributive Ockham algebras. The variety $\mathcal{K}_2$ is the subvariety of the variety of all distributive Ockham algebras defined by the following additional identities:

$$\begin{aligned} &x \leq x^{**}, \\ &x \wedge x^* = x^{**} \wedge x^*, \\ &x \wedge x^* \leq y \vee y^*. \end{aligned} \tag{1}$$

Equivalently, $\mathcal{K}_2$ is the variety generated by the four-element chain $\mathbf{N_4} = \{0, a, b, 1\}$, where $0 < a < b < 1$ and $0^* = 1$, $a^* = a$, $b^* = 1^* = 0$. It is known that $\mathbf{N_4}$ and its subalgebras $\mathbf{S_3} = \{0, b, 1\}$, $\mathbf{K_3} = \{0, a, 1\}$, and $\mathbf{B_2} = \{0, 1\}$ are the only subdirectly irreducible algebras in $\mathcal{K}_2$. Thus, given an algebra $\mathbf{A} \in \mathcal{K}_2$, we may write

$$\mathbf{A} \leq_{\text{s.d.}} \prod_{i \in I} \mathbf{A}_i\,,$$

where $\mathbf{A}_i \in \{\mathbf{B_2}, \mathbf{K_3}, \mathbf{S_3}, \mathbf{N_4}\}$. Further, we will often say that $\mathbf{A}$ is a $\mathcal{K}_2$-algebra.

The largest proper subvariety of $\mathcal{K}_2$ is the variety $\mathcal{K} \vee \mathcal{S}$. The variety $\mathcal{K} \vee \mathcal{S}$ is characterized in $\mathcal{K}_2$ by the identity $x \vee y^* \vee y^{**} \geq x^{**}$.

The next lemma will be crucial in the proofs of our main results because it reduces all problems to Kleene algebras and to functions with range contained in $A^\vee$ defined below.

Lemma 2.1. *Every algebra in $\mathcal{K}_2$ satisfies the identity*

$$x = x^{**} \wedge (x \vee x^*) \,. \tag{2}$$

Proof. It is easy to check that the identity holds in $\mathbf{N_4}$. □

For every algebra $\mathbf{A} \in \mathcal{K}_2$ we denote

$$A^\vee = \{x \vee x^* \mid x \in A\} = \{x \in A \mid x \geq x^*\} \,.$$

It is not difficult to see that $A^\vee$ is a filter of the lattice $\mathbf{A}$.

An element $u \in A$ is called *a Kleene element* if $u^{**} = u$. The set of all Kleene elements of $\mathbf{A}$ is a subuniverse of $\mathbf{A}$; in fact it is a Kleene algebra, which we will denote by $\mathbf{A}^{**}$. It is easy to observe that the operation ** is an idempotent endomorphism of $\mathbf{A}$ with range $\mathbf{A}^{**}$. We shall call the kernel Φ of this homomorphism the *Glivenko congruence* of $\mathbf{A}$.

Given an element $u \in A$, we denote by $[u]_\Phi$ the Φ-block containing u. Note that $[u]_\Phi$ is a distributive lattice with the greatest element u^{**} which obviously is a Kleene element.

Throughout the paper we assume that $\mathbf{A} \in \mathcal{K}_2$ and there is an embedding

$$\mathbf{A} \leq \mathbf{N_4}^I, \tag{3}$$

for some index set I. We denote by $\pi_i : A \to A_i$ the projection map to the ith subdirect factor of $\mathbf{A}$. We write the elements of $\mathbf{A}$ in the form $x = (x_i)_{i \in I}$. It is not difficult to see that if $f : A^n \to A$ is a compatible function of $\mathbf{A}$ and $\mathbf{x}, \mathbf{y} \in A^n$, then $\mathbf{x}_i = \mathbf{y}_i$ implies $f(\mathbf{x})_i = f(\mathbf{y})_i$. This means that every compatible function f of $\mathbf{A}$ determines the coordinate functions f_i of $\pi_i(\mathbf{A})$ such that $f_i(\mathbf{x}_i) = f(\mathbf{x})_i$ for all $\mathbf{x} \in A^n$. Obviously, the family $(f_i)_{i \in I}$ completely determines f, so we may identify f with this family.

The *uncertainty order* of the $\mathcal{K}_2$-algebra $\mathbf{A}$ is a binary relation $\sqsubseteq$, defined by

$$x \sqsubseteq y \ \ \Leftrightarrow \ \ x \wedge s \leq y \leq x \vee s^* \text{ for some } s \in A^\vee.$$

This relation generalizes one which for Kleene algebras was introduced by Haviar et al. in [2]. It is always a partial order relation on $\mathcal{K}_2$-algebras. In the algebra $\mathbf{N_4}$ we have $0 \sqsubseteq a$ and $1 \sqsubseteq b \sqsubseteq a$.

We say that function f on an algebra $\mathbf{A} \in \mathcal{K}_2$ is *uncertainty preserving* if it preserves the uncertainty order relation of $\mathbf{A}$.

The next two lemmas list some properties of the uncertainty order relation. Originally similar results were proved and used in [2] for Kleene algebras, but one can easily generalize them to algebras from the variety $\mathcal{K}_2$.

Lemma 2.2. *The restriction of the uncertainty order to $A^{\vee}$ coincides with the reverse order relation of the lattice $\mathbf{A}^{\vee}$.*

Lemma 2.3. *If $\mathbf{A} \leq_{\text{s.d.}} \prod_{i\in I} \mathbf{A}_i$ and $x, y \in A$, then $x \sqsubseteq y$ iff $x_i \sqsubseteq y_i$ for every $i \in I$.*

For distributive lattices we have the following description of local polynomial functions.

Theorem 2.4. ([10]) *A function on a distributive lattice is a local polynomial function iff it is compatible and order preserving.*

The next theorem gives a description of local polynomial functions for algebras in the variety $\mathcal{K} \vee \mathcal{S}$.

Theorem 2.5. ([6]) *A function on $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$ is a local polynomial function iff it preserves the congruences of $\mathbf{A}$ and the uncertainty order.*

Remark 2.6. Let $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$ and assume that $\mathbf{A} \leq_{\text{s.d.}} \prod_{i\in I} \mathbf{A}_i$. Using Lemma 2.3, we can reformulate Theorem 2.5 as follows.

A function on $\mathbf{A}$ is a local polynomial function iff it is compatible and for every $i \in I$ the coordinate function f_i is a polynomial function of $\mathbf{A}_i$.

The following example shows that in general this does not hold for algebras in $\mathcal{K}_2$.

Example 1. Consider algebras $\mathbf{A}_1$ and $\mathbf{A}_2$, $\mathbf{A}_2 \leq \mathbf{A}_1 \leq \mathbf{N_4} \times \mathbf{S_3}$ as shown in Fig. 1 and define a unary polynomial function p on $\mathbf{A}_1$

$$p(x) = ((1, b) \vee x) \wedge x^{**}.$$

It is easy to see that $\mathbf{A}_2$ is closed with respect to p. Since the variety of distributive Ockham algebras has the congruence extension property, the restriction $f = p|_{A_2}$ is a compatible function of $\mathbf{A}_2$.

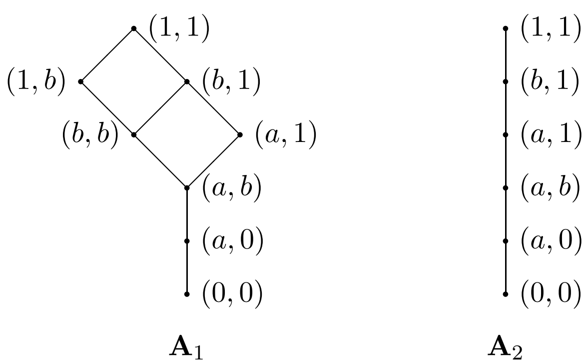

Fig. 1. Algebras $\mathbf{A}_1$ and $\mathbf{A}_2$.

Note that the coordinate functions f_1 and f_2 of f are polynomial functions, namely $f_1(x) = x^{**}$ and $f_2(x) = x$. But, using the canonical form (6) of polynomials in $\mathcal{K}_2$, one can show that f is not a polynomial function of $\mathbf{A}_2$. $\square$

Our main motivation to study unary polynomial functions in $\mathcal{K}_2$ is the following recent result.

Theorem 2.7. ([8]) *Every compatible, uncertainty preserving function on $\mathbf{A} \in \mathcal{K} \vee \mathcal{S}$ is a composition of polynomial functions and unary compatible, uncertainty preserving functions.*

3. COMPATIBLE, UNCERTAINTY PRESERVING FUNCTIONS

In this section we present some properties of compatible, uncertainty preserving functions.

Lemma 3.1. ([9]) *Any lattice congruence $\psi \in \mathrm{Con}_{\mathrm{lat}}(\mathbf{A})$ such that $\psi \leq \Phi$ is a congruence of $\mathbf{A}$.*

Corollary 3.1.1. *Let f be a compatible function on an algebra $\mathbf{A} \in \mathcal{K}_2$. Assume that for some $u \in A$ the function f preserves $[u]_\Phi$. Then the restriction $f|_{[u]_\Phi}$ is a compatible function of the lattice $[u]_\Phi$.*

Proof. If ρ is a congruence of the lattice $[u]_\Phi$, then it can be extended to some congruence σ of the lattice $\mathbf{A}$ (since the variety of distributive lattices has the congruence extension property). Now Lemma 3.1 implies that $\tau = \sigma \wedge \Phi$ is a congruence of the algebra $\mathbf{A}$, and it is easy to see that $\tau|_{[u]_\Phi} = \rho$. $\square$

Let f be a compatible function on an algebra $\mathbf{A} \in \mathcal{K}_2$. By (2) we have

$$f(\mathbf{x}) = f(\mathbf{x})^{**} \wedge (f(\mathbf{x}) \vee f(\mathbf{x})^*) .$$

The next lemma shows that the function $f(\mathbf{x})^{**}$ can be considered as a compatible function of the Kleene algebra $\mathbf{A}^{**}$.

Lemma 3.2. *Let f be a compatible, uncertainty preserving function on $\mathbf{A}$. Then the restriction $g = f^{**}|_{A^{**}}$ is a compatible function of the Kleene algebra $\mathbf{A}^{**}$ which preserves the uncertainty order relation of $\mathbf{A}^{**}$.*

Proof. Since $\mathbf{A}^{**}$ is a subalgebra of $\mathbf{A}$ and the variety of distributive Ockham algebras has the congruence extension property, any congruence of $\mathbf{A}^{**}$ can be extended to a congruence of $\mathbf{A}$. Thus, since f^{**} preserves A^{**}, we have that g is a compatible function of $\mathbf{A}^{**}$.

Since $\mathbf{A}^{**}$ is a subalgebra of $\mathbf{A}$, it follows from Lemma 2.3 that the uncertainty order on $\mathbf{A}^{**}$ is the restriction of the uncertainty order on $\mathbf{A}$. Thus g preserves the uncertainty order relation of $\mathbf{A}^{**}$. $\square$

4. UNARY LOCAL POLYNOMIAL FUNCTIONS

In this section we will describe unary local polynomials of algebras $\mathbf{A} \in \mathcal{K}_2$.

Lemma 4.1. *A unary compatible, uncertainty preserving function f on $\mathbf{A}$ is a local polynomial function iff the function $f \vee f^*$ can be interpolated by a polynomial function on any finite subset of $A^\vee$.*

Proof. Obviously we only have to prove the sufficiency. By (2) we know that

$$f(x) = f(x)^{**} \wedge (f(x) \vee f(x)^*)\,.$$

Since f preserves the Glivenko congruence, we have the identity $f(x)^{**} = f(x^{**})^{**}$. By Lemma 3.2 and Theorem 2.5 this implies that $f(x)^{**}$ is a local polynomial function of $\mathbf{A}$. Hence it remains to prove that $g(x) = f(x) \vee f(x)^*$ is a local polynomial function, too. Let F be a finite subset of A and p be a polynomial function of $\mathbf{A}$ which coincides with g on $F^\vee = \{x \vee x^* \mid x \in F\}$. We shall show that then g coincides with the polynomial $q(x) = (p(x) \vee x^*) \wedge (g(0) \vee x^{**})$ on F. In view of the embedding (3) this means that we must prove the equality

$$g_i(x_i) = (p_i(x_i) \vee x_i^*) \wedge (g_i(0) \vee x_i^{**})\,, \tag{4}$$

for every $i \in I$ and $x \in F$. It is easy to see that if $x_i = 0$, then (4) holds. Assume that $x_i \in \{a, b, 1\}$. Then $x_i = x_i \vee x_i^*$ and since $x \vee x^* \in F^\vee$, we have $g_i(x_i) = p_i(x_i)$. The identity (1) implies $p_i(x_i) \geq x_i^*$. Thus, in order to prove (4), we only have to show that $g_i(x_i) \leq g_i(0) \vee x_i^{**}$. The latter is obvious if $x_i \in \{b, 1\}$. If $x_i = a$, then $x_i = x_i \wedge x_i^*$. Since $0 \sqsubseteq a$, Lemma 2.3 implies $0 \sqsubseteq x \wedge x^*$. Since g preserves the uncertainty order, we have $g(0) \geq g(x \wedge x^*)$. Thus $g_i(x_i) \leq g_i(0) \vee x_i^{**}$. $\square$

Lemma 4.2. *Let g be a unary compatible, uncertainty preserving function on $\mathbf{A}$ such that the range of g is contained in $A^\vee$. Then the following are equivalent*:

(i) *g is a local polynomial function*;

(ii) *$h(x) = g(x) \wedge x^{**}$ is a local polynomial function*;

(iii) *$h(x)$ can be interpolated by a polynomial on any finite subset of the set $\{x \in A^\vee \,|\, x \leq g(1)\}$.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. In order to prove that (iii) implies (i), we first prove the equality

$$g(x) = g(u) \vee h(x)\,, \tag{5}$$

for any $u \in A^\vee$ and every $x \geq u$. In view of the embedding (3) this is equivalent to $g_i(x_i) = g_i(u_i) \vee (g_i(x_i) \wedge x_i^{**})$ for every $i \in I$. If $x_i \in \{b, 1\}$, then the latter is obvious. Assume that $x_i = a$. Then $u_i = x_i$ and $g_i(x_i) = g_i(u_i)$.

Now we show that $g(x) = g(x \wedge g(1))$ for every $x \in A^{\vee}$. We again use the embedding (3). Obviously $g_i(x_i) = g_i(x_i \wedge g_i(1))$ holds if $g_i(1) \in \{a, 1\}$, or $g_i(1) = b$ and $x_i \in \{a, b\}$. It remains to prove that $g_i(1) = b$ implies $g_i(b) = b$. Since g preserves the uncertainty order, $g_i(1) = b$ implies $g_i(b) \in \{a, b\}$ but $g_i(b) = a$ is impossible because g preserves the Glivenko congruence.

Let F be a finite subset of $A^{\vee}$ and $F_1 = \{x \wedge g(1) \mid x \in F\}$. By condition (iii) there exists a polynomial $p(x)$ that interpolates h at F_1. Then obviously the polynomial $p(x \wedge g(1))$ interpolates h at F. By formula (5), if $u = \bigwedge\{x \mid x \in F\}$, then $g(x) = g(u) \vee h(x)$ holds for every $x \in F$. This yields that the polynomial $g(u) \vee p(x \wedge g(1))$ interpolates g at F. We have proved that if condition (iii) holds, then the function g can be interpolated by polynomials at all finite subsets of $A^{\vee}$. By Lemma 4.1 this proves the implication (iii) $\Rightarrow$ (i). $\square$

Now we are ready to prove our main result.

Theorem 4.3. *A unary function f on a $\mathcal{K}_2$-algebra $\mathbf{A}$ is a local polynomial function iff it satisfies the following conditions*:

(i) *f preserves congruences*;

(ii) *f preserves the uncertainty order*;

(iii) *the function $h(x) = (f(x) \vee f(x)^*) \wedge x^{**}$ can be interpolated by a polynomial on every two-element set $\{u, v\} \subseteq A^{\vee}$ such that $u^{**} < v \leq h(1)^{**}$; moreover, there exists $w \in A$ such that*

$$\begin{aligned} h(u) &= ((w \vee u) \wedge h(1)) \wedge u^{**}, \\ h(v) &= ((w \vee v) \wedge h(1)) \wedge v^{**}. \end{aligned}$$

Proof. Let f be a local polynomial function on $\mathbf{A}$. Clearly, f preserves the congruences and the uncertainty order of $\mathbf{A}$. Since f is a local polynomial function, also $h(x)$ is a local polynomial function. Thus there exist constants $k_i \in A$ such that $h(x)$ coincides with the polynomial

$$\begin{aligned} p(x) = (k_1 \vee x) \wedge (k_2 \vee x^*) \wedge (k_3 \vee x^{**}) \\ \wedge (k_4 \vee x \vee x^*) \wedge (k_5 \vee x^* \vee x^{**}) \wedge k_6 \end{aligned} \tag{6}$$

on the set $\{u, v, 1\}$. Choosing $x = 1$, we get $k_2 \wedge k_6 = h(1)$. However, since $h(A^{\vee}) \subseteq {\downarrow} h(1) = \{y \in A \mid y \leq h(1)\}$, we may replace $p(x)$ by the polynomial $p(x) \wedge h(1)$ and then it is easy to see that we may take $k_2 = k_6 = h(1)$. Similarly, since $h(x) \leq x^{**}$ for every $x \in A^{\vee}$, we may replace $p(x)$ by the polynomial $p(x) \wedge x^{**}$ which means that we are free to take $k_3 = 0$. Since $\{u, v, 1\} \subset A^{\vee}$, we may also assume that $k_4 = k_5 = 1$. Thus we have

$$p(x) = (k_1 \vee x) \wedge x^{**} \wedge h(1)\,.$$

Hence the condition (iii) holds with $w = k_1$.

Now suppose that f is a unary compatible, uncertainty preserving function on $\mathbf{A}$ which satisfies condition (iii) of the theorem. We have to prove that f is a local polynomial function. By Lemmas 4.1 and 4.2 we only need to prove that the function $h(x)$ can be interpolated by a polynomial function on every finite subset $F \subseteq A^{\vee} \cap \downarrow h(1)$. Since $\mathbf{A}$ admits the majority term $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$, it follows from the well-known Baker–Pixley Lemma (see [5], Theorem 3.2.2) that it is sufficient to consider only two-element sets F.

First consider the case $F = \{u, v\}$, where $u < v \leq h(1)$. Let $z = v \vee u^{**}$. Then $z \leq h(1)^{**}$ and by condition (iii) there exists $w \in A$ such that

$$h(u) = ((w \vee u) \wedge h(1)) \wedge u^{**} \text{ and } h(z) = ((w \vee z) \wedge h(1)) \wedge z^{**}.$$

Using the embedding (3), we note that $v_i \in \{u_i, z_i\}$, for every $i \in I$. Thus we also have

$$h(v) = ((w \vee v) \wedge h(1)) \wedge v^{**}.$$

To conclude the proof, we consider the general case, with u and v possibly incomparable. Let $z = u \wedge v$. By the above proof, there exist polynomials p_1 and p_2 such that p_1 meets h at $\{z, u\}$ and p_2 at $\{z, v\}$. Now we define

$$p(x) = p_1(x \wedge u) \vee p_2(x \wedge v).$$

Since h preserves the uncertainty order, it is easy to check that p agrees with h at $\{u, v\}$. □

We must say that we are not completely satisfied with Theorem 4.3. We hoped that local polynomial functions of $\mathcal{K}_2$-algebras could be characterized using some preservation properties, that is, there were one or more relations defined uniformly for all $\mathcal{K}_2$-algebras such that f was a local polynomial function iff it preserved the congruences and these relations. Example 1 constructed in Section 2 shows that if such relations exist, they cannot be defined first for subdirectly irreducibles and then extended componentwise to arbitrary algebras. It is still possible that such relation(s) exist, but so far we do not know how to construct them.

In some sense the characterization given by Theorem 4.3 is not much better than what we already know by Lemma 4.2. The advantage of Theorem 4.3 is that it reduces general polynomial interpolation to the interpolation on very special subsets by very special polynomials. That this is of some use is illustrated by the following example.

Example 2. An algebra $\mathbf{A} \in \mathcal{K}_2 \setminus \mathcal{K} \vee \mathcal{S}$ such that every unary compatible, uncertainty preserving function of $\mathbf{A}$ is a local polynomial.

Let $\mathbf{A}_1 = \langle A_1; \vee, \wedge, ^{\circ}, 0, 1\rangle$ be a non-Boolean Stone algebra such that $[0]_{\Phi} = \{0\}$ and $[1]_{\Phi} = A_1^{\vee}$ are the only blocks of the Glivenko congruence of $\mathbf{A}_1$. (For example, we may take $\mathbf{A}_1 = \mathbf{S_3}$.) Let $\mathbf{A}_2 = \langle A_2; \vee, \wedge, ^\bullet, 0, 1\rangle$ be an arbitrary Kleene algebra. Put $A = A_1 \cup A_2$ with $0_{A_1} < x < y$ for any $x \in A_2$ and $y \in A_1^\vee$. Clearly, $\mathbf{A}$ is a bounded distributive lattice with $\mathbf{0} = 0_{A_1}$, $\mathbf{1} = 1_{A_1}$. Define a unary operation * on A by

$$x^* = \begin{cases} x^\circ & \text{if } x \in A_1, \\ x^\bullet & \text{if } x \in A_2\,. \end{cases}$$

Then straightforward calculations show that $\mathbf{A} = \langle A; \vee, \wedge, ^*, \mathbf{0}, \mathbf{1}\rangle$ belongs to $\mathcal{K}_2$. Now, if we take any $x \in A_2^\vee$ and $y \in A_1 \setminus \{0_{A_1}, 1_{A_1}\}$, then $x^{**} = x \le y < y^{**} = 1$. By [6], Lemma 1.2, this implies that $\mathbf{A} \notin \mathcal{K} \vee \mathcal{S}$.

Let f be a unary compatible, uncertainty preserving function of $\mathbf{A}$. Consider $h(x) = (f(x) \vee f(x)^*) \wedge x^{**}$. Take $u, v \in A^\vee$ such that $u^{**} < v \le h(1)^{**}$. We are going to show that the polynomial

$$p(x) = (h(v) \vee x) \wedge h(1) \wedge x^{**}$$

coincides with h on the set $\{u, v\}$. Since $u^{**} = 1$ if $u \in A_1$, we may assume $u \in A_2^\vee$. Hence $u^{**} = u$, implying $p(u) = h(1) \wedge u$, and we have to prove $h(u) = h(1) \wedge u$. In view of the embedding (3) this is equivalent to $h_i(u_i) = h_i(1) \wedge u_i$, for every $i \in I$. The latter holds obviously if $u_i = 1$. Since u is a Kleene element, the only remaining possibility is $u_i = a$. By definition of h we have $h_i(a) \le a^{**} = a$. On the other hand, since $h(A) \subseteq A^\vee$, we also have $h_i(a) \ge a$. Hence, $h_i(a) = a = h_i(1) \wedge a$.

We proved the equality $h(u) = p(u)$. Similarly one can prove $h(v) = p(v)$ if $v \in A_2^\vee$. Thus we only have to prove $h(v) = p(v)$ if $v \in A_1^\vee$. In this case $v^{**} = 1$, hence

$$p(v) = (h(v) \vee v) \wedge h(1) = (h(v) \wedge h(1)) \vee (v \wedge h(1)) = h(v) \vee (v \wedge h(1))\,.$$

Now we shall be done if we prove the inequality $h(v) \ge v \wedge h(1)$, which due to the embedding (3) is equivalent to

$$h_i(v_i) \ge v_i \wedge h_i(1) \tag{7}$$

for every $i \in I$. Since $v^{**} = 1$, we have $v_i \in \{b, 1\}$, for every $i \in I$. The formula (7) obviously holds if $v_i = 1$ or $v_i = b \le h_i(v_i)$. If $v_i = b$ and $h_i(v_i) = a$, then we also have $h_i(1) = a$ because h preserves the Glivenko congruence. Hence (7) is valid also in this case. $\square$

ACKNOWLEDGEMENT

The author was supported by grant No. 4353 from the Estonian Science Foundation.

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$\mathcal{K}_2$-algebrate unaarsed lokaalsed polünoomfunktsioonid

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$\mathcal{K}_2$-algebrate klass on distributiivsete Ockhami algebrate muutkonna alammuutkond, mis sisaldab Kleene ja Stone'i algebrate muutkondi. Klassi $\mathcal{K}_2$ kuuluvate algebrate polünoomfunktsioone uuris esimesena M. Haviar oma 1993. aastal ilmunud artiklis. Hiljem on (lokaalseid) polünoomfunktsioone kirjeldatud muutkonna $\mathcal{K}_2$ kõigis pärisalammuutkondades. Käesolevas artiklis kirjeldatakse $\mathcal{K}_2$-algebrate unaarseid lokaalseid polünoomfunktsioone.